Non-arbitrage condition and thin random times

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6th General AMaMeF and Banach Center Conference

Warszawa 14 June 2013

Problem

Question

Consider a Default-free Model $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P}, X)$ without arbitrage opportunities. Then, incorporate the default event τ to this model.
Is the Defaultable Model $(\Omega, \mathcal{A}, \mathbb{F}^\tau, \mathbb{P}, X)$ arbitrage free?

- Choulli T., Aksamit A., Deng J., and Jeanblanc M., *Non-arbitrage up to Random Horizon and after Honest Times for Semimartingales Models*, Working paper, 2013.
- Aksamit A., Choulli T., Jeanblanc M., *Thin random times*, Working paper, 2013

Random times

Set a filtered probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ and consider a random time τ, i.e. positive $\mathcal{A}$-measurable random variable.

- The process Z defined as $Z_t = \mathbb{P}(\tau > t | \mathcal{F}_t)$ is called the Azéma supermartingale associated with τ.
- The process $\widetilde{Z}$ defined as $\widetilde{Z}_t = \mathbb{P}(\tau \geq t | \mathcal{F}_t)$ is second important supermartingale associated with τ.
- The processes A^o and A^p denote respectively dual optional and predictable projections of the process $A = \mathbb{1}_{[\tau,\infty)}$.
- The Doob Meyer decomposition of Azéma supermartingale Z is $Z = \mu - A^p$, with $\mu_t = \mathbb{E}(A^p_\infty | \mathcal{F}_t)$ and optional decomposition is $Z = m - A^o$, with $m_t = \mathbb{E}(A^o_\infty | \mathcal{F}_t)$.

Enlargement of filtration

Progressively enlarged filtration $\mathbb{F}^\tau$ associated with τ is defined as

$$\mathcal{F}_t^\tau = \bigcap_{s>t} (\mathcal{F}_s \vee \sigma(\tau \wedge s)).$$

Initially enlarged filtration $\mathbb{F}^{\sigma(\tau)}$ associated with τ is defined as

$$\mathcal{F}_t^{\sigma(\tau)} = \bigcap_{s>t} (\mathcal{F}_s \vee \sigma(\tau)).$$

For general enlargement of filtration $\mathbb{F} \subset \mathbb{G}$ we talk about two hypotheses

(H) hypothesis: Each $\mathbb{F}$-martingale remains a $\mathbb{G}$-martingale.

(H') hypothesis: Each $\mathbb{F}$-martingale remains a $\mathbb{G}$-semimartingale.

Honest times

- Random time τ is an honest time if for each $t \geq 0$ there exists $\mathcal{F}_t$-measurable random variable τ_t such that $\tau = \tau_t$ on $(\tau < t)$.
- If τ is an honest time, then each $\mathbb{F}$-martingale X remains an $\mathbb{F}^\tau$-semimartingale with semimartingale decomposition given by

$$X_t = \widehat{X}_t + \int_0^{t\wedge\tau} \frac{1}{Z_{s-}} d\langle X, m\rangle_s - \int_\tau^t \frac{1}{1 - Z_{s-}} d\langle X, m\rangle_s$$

  where $\widehat{X}$ is $\mathbb{F}^\tau$-martingale.
- Let τ be a random time. For each $\mathbb{F}$-martingale X, the process X^τ is $\mathbb{F}^\tau$-semimartingale with semimartingale decomposition

$$X_t = \widehat{X}_t + \int_0^{t\wedge\tau} \frac{1}{Z_{s-}} d\langle X, m\rangle_s$$

  where $\widehat{X}$ is $\mathbb{F}^\tau$-martingale.

Non-arbitrage condition – NUPBR

Let X be an $\mathbb{F}$-semimartingale. We say that X satisfies No Unbounded Profit with Bounded Risk (NUPBR) if

$$\mathcal{K}(X) := \{(H \cdot X)_T : H \in L(X) \text{ and } H \cdot X \geq -1\}$$

is bounded in $L^0(\mathbb{P})$.

Theorem(Takaoka)
The $\mathbb{F}$-semimartingale X satisfies NUPBR if and only if $\mathcal{L}_\sigma(X) \neq \emptyset$, where $\mathcal{L}_\sigma(X)$ is the set of σ-densities given by

$$\mathcal{L}_\sigma(X) := \{L \in \mathcal{M}_{loc}(\mathbb{F}) : L > 0 \text{ and } LX \text{ is an } \sigma\text{-martingale}\}\,.$$

Theorem

1. Let τ be a random time. Then, the following are equivalent:
   - The thin set $\{\widetilde{Z} = 0 \ \& \ Z_- > 0\}$ is evanescent.
   - For any process X satisfying NUPBR($\mathbb{F}$), X^τ satisfies NUPBR($\mathbb{F}^\tau$).
2. Let τ be an honest time satisfying $Z_\tau < 1$ a.s.. Then, the following are equivalent:
   - The thin set $\{\widetilde{Z} = 1 \ \& \ Z_- < 1\}$ is evanescent.
   - For any process X satisfying NUPBR($\mathbb{F}$), $X - X^\tau$ satisfies NUPBR($\mathbb{F}^\tau$).

Avoidance of $\mathbb{F}$-stopping times

Often in the literature standard assumption on a random time is used:

(A) assumption: τ avoids $\mathbb{F}$ stopping times, i.e. $\mathbb{P}(\tau = T) = 0$, for any $\mathbb{F}$-stopping time T.

Strict and thin random times

Definition

A random time τ is called

1. *a strict random time if $[\![\tau]\!] \cap [\![T]\!] = \emptyset$ for any $\mathbb{F}$-stopping time T.*
2. *a thin random time if its graph $[\![\tau]\!]$ is contained in a thin set. i.e. if there exists a sequence of $\mathbb{F}$ stopping times $(T_n)_{n=1}^{\infty}$ with disjoint graphs such that $[\![\tau]\!] \subset \bigcup_n [\![T_n]\!]$.*
   The sequence $(T_n)_n$ is then exhausting sequence of a thin random time.

A random time τ is strict and thin random time if and only if $\tau = \infty$.

Definition

Pair of random times (τ_1, τ_2) *is the decomposition of a random time* τ *if*

1. τ_1 *is a strict random time and* τ_2 *is a thin random time;*
2. $\tau_1 \wedge \tau_2 = \tau$;
3. $\tau_1 \vee \tau_2 = \infty$.

Theorem

Each random time τ *has a decomposition* (τ_1, τ_2).

Decomposition of a random time

Define

$$\tau_1 := \tau_{\{\Delta A^o_\tau = 0\}} \quad \text{and} \quad \tau_2 := \tau_{\{\Delta A^o_\tau > 0\}}.$$

We see that the time τ_1 is a strict random time as

$$\mathbb{P}(\tau_1 = T < \infty) = \mathbb{E}\Big(\int_0^\infty 1\!\!1_{\{u=T\}} 1\!\!1_{\{\Delta A^o_u = 0\}} dA^o_u\Big) = 0.$$

and the time τ_2 is a thin random time as

$$[\![\tau_2]\!] = [\![\tau]\!] \cap \{\Delta A^o > 0\} = [\![\tau]\!] \cap \bigcup_n [\![T_n]\!] \subset \bigcup_n [\![T_n]\!].$$

Lemma: Alternative characterisation

1. The random time τ is a thin random time if and only if its dual optional projection is a pure jump process.
2. The random time τ is a strict random time if and only if its dual optional projection is a continuous process.

Decomposition of a random time

Remark

1. Decomposition of a stopping time, $\mathcal{P}$ instead of $\mathcal{O}$.
2. $\tau = \tau_1 \wedge \tau_2^i \wedge \tau_2^a$

$$\tau_1 = \tau_{\{\Delta A^o_\tau = 0\}} \quad \text{strict part}$$
$$\tau_2^i = \tau_{\{\Delta A^o_\tau > 0, \Delta A^p_\tau = 0\}} \quad \text{totally inaccessible thin part}$$
$$\tau_2^a = \tau_{\{\Delta A^p_\tau > 0\}} \quad \text{accessible thin part}$$

Poisson filtration example

- Let $\mathbb{F}^X$ be a filtration of CPP $X_t = \sum_{k=1}^{N_t} Y_k$, where N is a Poisson process with parameter η and sequence of jump times $(\theta_n)_{n=1}^{\infty}$ and Y_k are i.i.d. positive random variables, independent from N, with cumulative distribution function F.
- Define the random time $\tau = \sup\{t : \mu t - X_t \leq a\}$ with $a > 0$. Under the condition $\mu > \eta \mathbb{E}(Y_1)$, the random time τ is finite a.s. Since τ is a last passage time, it is an honest time in the filtration $\mathbb{F}$. Furthermore, since the process $\mu t - X_t$ has only negative jumps, one has $\mu\tau - X_\tau = a$
- $A^o = C \sum_{n\geq 1} \mathbb{1}_{[T_n,\infty)}$ with $T_n = \inf\{t > T_{n-1} : \mu t - X_t = a\}$ and $T_0 = 0$.

Brownian filtration example: local time approximation

- Let B be a Brownian motion. For $\varepsilon > 0$, define a double sequence of stopping times by

$$U_0 = 0, \quad V_0 = 0$$

$$U_n = \inf\{t \geq V_{n-1} : B_t = \varepsilon\}, \quad V_n = \inf\{t \geq U_n : B_t = 0\}.$$

  and process $D_t = \max\{n : V_n \leq t\}$ which is the number of downcrossings of B from level ε to level 0 before time t.

- Define a random time

$$\tau^\varepsilon = \sup\{V_n : V_n \leq T_1\}$$

  with $T_1 = \inf\{t : B_t = 1\}$.

- $A^o = \varepsilon D_{t \wedge T_1} + \varepsilon$ and $\{\Delta A^o > 0\} = [0, T_1] \cap \bigcup_{n=0}^{\infty} [\![V_n]\!]$

(H') hypothesis and decomposition formula

Let τ be a thin random time with exhausting sequence $(T_n)_n$. Denote by $C_n = \{\tau = T_n\}$, so $\tau = \sum_n \mathbb{1}_{C_n} T_n$.

Theorem

For each thin random time τ, the hypothesis (H') is satisfied between $\mathbb{F}$ and $\mathbb{F}^\tau$. Any $\mathbb{F}$ martingale X can be decomposed as

$$X_t = \widehat{X}_t + \int_0^{t\wedge\tau} \frac{1}{Z_{s-}} d\langle X, m\rangle_s + \sum_n \mathbb{1}_{C_n} \int_0^t \mathbb{1}_{\{T_n < s\}} \frac{1}{z^n_{s-}} d\langle X, z^n\rangle_s,$$

with $z^n_t = \mathbb{P}(C_n|\mathcal{F}_t)$, where $\widehat{X}$ is $\mathbb{F}^\tau$ local martingale.

(H') hypothesis and decomposition formula

Theorem (Jacod)
Suppose that $\mathbb{F}^{\mathcal{C}}$ is an initial enlargement of the filtration $\mathbb{F}$ with an atomic σ-field generated by $\mathcal{C} = ((C_n)_n)$.
Then, the filtration $\mathbb{F}^{\mathcal{C}}$ satisfies (H') hypothesis and each $\mathbb{F}$ martingale X can be decomposed in $\mathbb{F}^{\mathcal{C}}$ as

$$X_t = \widehat{X}_t + \sum_n \mathbb{1}_{C_n} \int_0^t \frac{1}{z^n_{s-}} d\langle X, z^n \rangle_s$$

with $z^n_t = \mathbb{P}(C_n | \mathcal{F}_t)$, where $\widehat{X}$ is $\mathbb{F}^{\mathcal{C}}$ local martingale.

(H') hypothesis and decomposition formula

$\mathbb{F}^\tau$ predictable process H can be decomposed as

$$H_t = \mathbb{1}_{\{t \le \tau\}} J_t + \mathbb{1}_{\{\tau < t\}} K_t(\tau) \quad t > 0$$

where J is $\mathbb{F}$ predictable process and $K : \mathbb{R}_+ \times \Omega \times \mathbb{R}_+ \to \mathbb{R}$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+)$ measurable.

As τ is thin we can rewrite process H as

$$H_t = J_t \mathbb{1}_{\{t \le \tau\}} + \sum_n \mathbb{1}_{\{T_n < t\}} K_t(T_n) \mathbb{1}_{C_n}$$

Note that each process $\mathbb{1}_{\{T_n < t\}} K_t(T_n)$ is $\mathbb{F}$ predictable.

(H') hypothesis in general case

Corollary

Let τ be a random time and (τ_1, τ_2) its random time decomposition. Then:

1. The filtration $\mathbb{F}^{\tau}$ satisfies (H') hypothesis if and only if the filtration $\mathbb{F}^{\tau_1}$ satisfies (H') hypothesis.
2. The Azéma supermartingale of τ_2 in filtration $\mathbb{F}$ coincides with the Azéma supermartingale of τ_2 in $\mathbb{F}^{\tau_1}$, i.e. $\mathbb{P}(\tau_2 > t | \mathcal{F}_t) = \mathbb{P}(\tau_2 > t | \mathcal{F}_t^{\tau_1})$.

Thin honest times

- Let τ be an honest time and (τ_1, τ_2) its random time decomposition. Then, times τ_1 and τ_2 are honest times.

Theorem

For any honest time τ with decomposition (τ_1, τ_2) its Azéma supermartingale at τ can be written as

$$Z^{\tau}_{\tau} \mathbb{1}_{\{\tau<\infty\}} = \mathbb{1}_{\{\tau=\tau_1<\infty\}} + Z^{\tau}_{\tau_2} \mathbb{1}_{\{\tau=\tau_2<\infty\}}$$

where $Z^{\tau}_{\tau_2} < 1$.

Thin honest times

Remark

For a thin honest time τ, the two following decomposition formulas coincide

$$X_t = \widehat{X}_t + \int_0^{t\wedge\tau} \frac{1}{Z_{s-}} d\langle X, m\rangle_s + \sum_n \mathbb{1}_{C_n} \int_0^t \mathbb{1}_{\{T_n<s\}} \frac{1}{z^n_{s-}} d\langle X, z^n\rangle_s,$$

$$X_t = \widehat{X}_t + \int_0^{t\wedge\tau} \frac{1}{Z_{s-}} d\langle X, m\rangle_s - \int_0^t \mathbb{1}_{\{\tau<s\}} \frac{1}{1-Z_{s-}} d\langle X, m\rangle_s$$

This is due to two possible representations of predictable process

$$\mathbb{1}_{\{\tau<t\}} K(\tau) = \mathbb{1}_{\{\tau<t\}} K(\tau_t) \quad \text{from honest time property,}$$

$$\mathbb{1}_{\{\tau<t\}} K(\tau) = \sum_n \mathbb{1}_{\{T_n<t\}} K(T_n) \mathbb{1}_{C_n} \quad \text{from thin time property.}$$

Local martingale deflator in $\mathbb{F}^{\mathcal{C}}$ and $\mathbb{F}^{\tau}$

Theorem

Assume that all martingales in $\mathbb{F}$ are continuous.

Let X be an $\mathbb{F}$ local martingale. Then:

1. The process $L_t^{\mathcal{C}} := \sum_n \mathbb{1}_{C_n} \frac{1}{Z_t^n}$ is a local martingale deflator in $\mathbb{F}^{\mathcal{C}}$ for X, i.e. it is strictly positive $\mathbb{F}^{\mathcal{C}}$ local martingale with $L_0^{\mathcal{C}} = 1$ and $L_\infty^{\mathcal{C}} > 0$ a.s. such that $XL^{\mathcal{C}}$ is an $\mathbb{F}^{\mathcal{C}}$ local martingale.
2. The process $L_t^{\tau} := \sum_n \mathbb{1}_{C_n} \left(\frac{1}{Z_t^n} - \frac{1}{Z_{t\wedge T_n}^n} \right)$ is a local martingale deflator in $\mathbb{F}^{\tau}$ for $X - X^{\tau}$, i.e. it is strictly positive $\mathbb{F}^{\tau}$ local martingale with $L_0^{\tau} = 1$ and $L_\infty^{\tau} > 0$ a.s. such that $(X - X^{\tau}) L^{\tau}$ is an $\mathbb{F}^{\tau}$ local martingale.

Entropy of the partition

Let $\mathcal{C} = (C_n)_n$ be an $\mathcal{F}_\infty$ measurable partition of Ω. Then, the quantity

$$H(\mathcal{C}) := -\sum_n \mathbb{P}(C_n) \log(\mathbb{P}(C_n))$$

is an entropy of $\mathcal{C}$.

Theorem[Meyer, Yor]
Assume that all $\mathbb{F}$ local martingales are continuous and $H(\mathcal{C}) < \infty$. Let an $\mathbb{F}$ local martingale X be an element of $H^2(\mathbb{F})$. Then, an $\mathbb{F}^{\mathcal{C}}$ semimartingale X is an element of $H^1(\mathbb{F}^{\mathcal{C}})$.

Entropy of the partition

- Meyer P.-A., 1978. *Sur un Théorème de Jacod*
- Yor M., 1985. *Entropie d'une Partition et Grossissement Initial d'une Filtration*
- The author of the first paper posed the question about additional knowledge associated with thin random time:
  Un problème voisin, mais plus intéressant peut être, consiste à mesurer le bouleversement produit, sur un système probabiliste, non pas en forçant des connaissances à l'instant 0*, mais en les forçant progressivement dans le système.*

Entropy of thin random time

In case of progressive enlargement with thin random time $\tau = \sum_n \mathbb{1}_{C_n} T_n$ we suggest measurement of additional knowledge by

$$H(\tau) = -\sum_n \mathbb{E}\left(\mathbb{1}_{C_n} \log z^n_{T_n}\right).$$

Remark

If τ is an $\mathbb{F}$ stopping time then $H(\tau) = 0$.

If for any n the set C_n is already in $\mathcal{F}_{T_n}$ then we do not gain any additional information.

$H(\tau)$ is invariant under different decompositions of τ.

To justify this measurement of additional knowledge we give analogous result to the previous one

Theorem

Assume that all $\mathbb{F}$ local martingales are continuous and $H(\tau) < \infty$. Let an $\mathbb{F}$ local martingale X be an element of $H^2(\mathbb{F})$. Then, an $\mathbb{F}^\tau$ semimartingale X is an element of $H^1(\mathbb{F}^\tau)$.

Thank you for your attention!