# Digital Double Barrier Options: Several Barrier Periods and Structured Floors

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**Objectives:**

- Determine the price of digital double barrier options with an arbitrary number of barrier periods.
- Find the exact value of the structured floor for structured notes whose individual coupons are digital double barrier options.
- Approximate the value of the structured floor by the price of a corridor option.

# Outline

# Setup and Pricing for One Period

- The underlying $(S_t)_{t\geq 0}$ has the risk-neutral dynamics

$$\frac{dS_t}{S_t} = r\,dt + \sigma\,dW_t,$$

  with the initial value $S_0 > 0$, constant interest rate $r \geq 0$, volatility $\sigma > 0$ and a standard Brownian motion $W$.
- At maturity $T_0 + P$, the payoff is one unit of currency if the underlying has stayed between the two barriers:

$$C_1 := \mathbb{I}_{\{B_{\mathrm{low}} < S_t < B_{\mathrm{up}},\ \forall t \in [T_0, T_0 + P]\}}, \tag{1}$$

  where $T_0 > 0$, $P > 0$ and $B_{\mathrm{up}} > B_{\mathrm{low}} > 0$ are barriers.

# Setup and Pricing for One Period

- Let us denote the price of this "one-period double barrier digital" at $t < T_0$ by
$$BD(S_t, t; \{T_0\}, P, B_{\mathrm{low}}, B_{\mathrm{up}}, r) := e^{-r(T_0+P-t)}\mathbf{E}[C_1|\mathcal{F}_t]. \qquad (2)$$
- The value function
$$f(S,t) := BD(S, t; \{T_0\}, P, B_{\mathrm{low}}, B_{\mathrm{up}}, r)$$
satisfies the Black–Scholes PDE
$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} - rf = 0$$
with the terminal condition $f(S, T_0 + P) = 1$, for $S \in (B_{\mathrm{low}}, B_{\mathrm{up}})$, and the boundary conditions $f(B_{\mathrm{low}}, t) = f(B_{\mathrm{up}}, t) = 0$ for $t \in [T_0, T_0 + P]$.

# Setup and Pricing for One Period

- Use the standard transformation $f(S,t) = e^{\alpha x+\beta\tau}U(x,\tau)$, where

$$x := \log(S/B_{\text{low}}), \qquad \tau := \tfrac{1}{2}\sigma^2(T_0 + P - t), \tag{3}$$
$$\alpha := -\frac{1}{2}\left(\frac{2}{\sigma^2}r - 1\right), \qquad \beta := -\frac{2r}{\sigma^2} - \alpha^2,$$

  to transform the Black–Scholes PDE into the heat equation

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial \tau}. \tag{4}$$

- The boundary conditions in the new coordinates are

$$U(0,\tau) = U(L,\tau) = 0, \qquad \tau \in [0,p], \tag{5}$$

  where $L := \log(B_{\text{up}}/B_{\text{low}})$ and $p = \frac{1}{2}\sigma^2 P$. The terminal condition translates to the initial condition

$$U(x,0) = e^{-\alpha x}, \qquad x \in (0,L). \tag{6}$$

## Proposition

*For $0 \leq t < T_0$, the price of a barrier digital with barrier period $[T_0, T_0 + P]$ and payoff $C_1$ at $T_0 + P$ is*

$$BD(S_t, t; \{T_0\}, P, B_{\text{low}}, B_{\text{up}}, r) = \sqrt{2\pi}\left(\frac{S}{B_{\text{low}}}\right)^{\alpha} \sum_{k=1}^{\infty} k \frac{1-(-1)^k e^{-\alpha L}}{\alpha^2 L^2 + k^2 \pi^2} e^{-(\frac{k\pi}{L})^2 p + \beta\tau}$$
$$\times \int_{-\frac{x}{\sqrt{2(\tau-p)}}}^{\frac{L-x}{\sqrt{2(\tau-p)}}} \sin\left(\frac{k\pi}{L}(x + y\sqrt{2(\tau-p)})\right) e^{-y^2/2} dy. \quad (7)$$

- this is a *rear-end* barrier option, because the two barriers are alive only towards the end of the contract (see Hui (1997)) .
- in probabilistic terms, we are integrating the probability to stay between the barriers (see e.g Borodin and Salminen (2002)) , with $S_{T_0}$ viewed as a parameter, against the law of $S_{T_0}$.

**Sketch of the Proof:**

- First consider the rectangle $(0, L) \times (0, p)$, the (unique) solution can be found by separation of variables

$$U(x,\tau) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{L}x\right) e^{-(\frac{k\pi}{L})^2\tau}, \qquad (x,\tau) \in (0,L)\times(0,p), \tag{8}$$

  where

$$b_k := \frac{2}{L}\int_0^L e^{-\alpha x_1} \sin\left(\frac{k\pi}{L}x_1\right) dx_1 = 2k\pi \frac{1-(-1)^k e^{-\alpha L}}{\alpha^2 L^2 + k^2\pi^2}$$

  are the Fourier coefficients of the boundary function $U(x,0) = e^{-\alpha x}$.

- Next, at $\tau = p$, the solution is given by (8) for $0 < x < L$ and vanishes otherwise.

$$U(x,p) = \begin{cases} \sum_{k=1}^{\infty} 2k\pi \frac{1-(-1)^k e^{-\alpha L}}{\alpha^2 L^2 + k^2\pi^2} \sin(\frac{k\pi}{L}x) e^{-(\frac{k\pi}{L})^2 p}, & 0 < x < L, \\ 0, & x \leq 0 \text{ or } x \geq L. \end{cases} \tag{9}$$

- Finally solve for $U$ in the region $\mathbb{R} \times (p, \frac{1}{2}\sigma^2(T_0 + P))$. A solution is found by convolving the initial condition (9) with the heat kernel (see Evans [4, p. 47]).

# Double Barrier Digitals with Finitely Many Periods

- $n$ tenor dates

  $$0 < T_0 < \cdots < T_{n-1}$$

  and fixed period length $P > 0$ satisfying $T_i + P \leq T_{i+1}$ for $i = 0, \ldots, n-2$.
- Consider a contract that pays one unit of currency at time $T_{n-1} + P$, if the underlying has remained between the two barriers $B_{\text{low}}$ and $B_{\text{up}}$ during each of the time intervals $[T_i, T_i + P]$, $i = 0, \ldots, n-1$.
- The price of this "multi-period double barrier digital" unless it is not knocked out before $t$ is given by

  $$BD(S_t, t; \{T_0, \ldots, T_{n-1}\}, P, B_{\text{low}}, B_{\text{up}}, r) := e^{-r(T_{n-1}+P-t)} \mathbf{E}\left[ \prod_{i=1}^{n} C_i \middle| \mathcal{F}_t \right], \tag{10}$$

  where

  $$C_i := \mathbb{I}_{\{B_{\text{low}} < S_t < B_{\text{up}},\ \forall t \in [T_{i-1}, T_{i-1}+P]\}}.$$

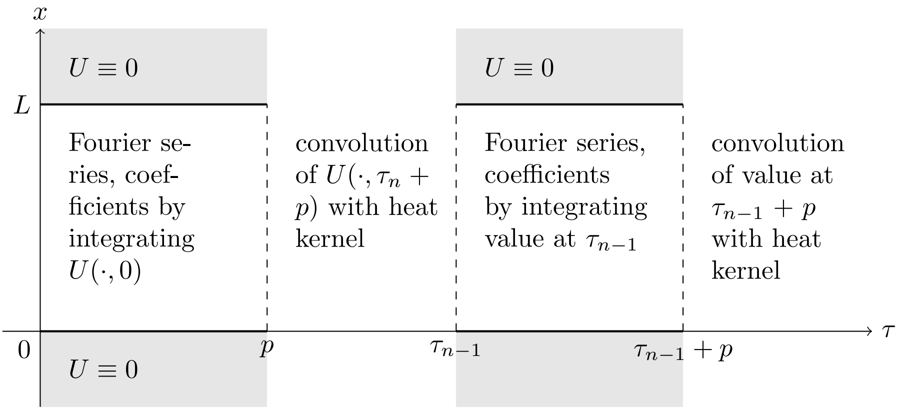


Figure: Solving the boundary value problem for an arbitrary number of barrier periods.

The $n$ barrier periods $[T_i, T_i + P]$ are mapped to $[\tau_i, \tau_i + p]$, where

$$\tau_i := \frac{1}{2}\sigma^2(T_{n-1} - T_{i-1}), \qquad i = n, \ldots, 1,$$

are the images of the barrier period endpoints under the coordinate change.

## Theorem

*The value function* (10) *equals* $e^{\alpha x+\beta\tau}U(x,\tau)$*, where for* $j \in \{0,\dots,n-1\}$*,* $\tau_{n-j} \le \tau \le \tau_{n-j}+p$*,* $0 < x < L$*, we have*

$$U(x,\tau) = \underbrace{\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}}_{j}\underbrace{\int_{0}^{L}\dots\int_{0}^{L}}_{j+1}\sum_{k_1=0}^{\infty}\cdots\sum_{k_{j+1}=0}^{\infty}$$

$$g_j(k_1,\dots,k_{j+1};x_1,\dots,x_{j+1};y_1,\dots,y_j;x,\tau)\,dx_1\dots dx_{j+1}dy_1\dots dy_j, \quad (11)$$

*whereas for* $j \in \{0,\dots,n-1\}$*,* $\tau_{n-j}+p < \tau < \tau_{n-(j+1)}$ *(with* $\tau_0 := \infty$*),* $x \in \mathbb{R}$*, we have*

$$U(x,\tau) = \underbrace{\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}}_{j+1}\underbrace{\int_{0}^{L}\dots\int_{0}^{L}}_{j+1}\sum_{k_1=0}^{\infty}\cdots\sum_{k_{j+1}=0}^{\infty}$$

$$h_j(k_1,\dots,k_{j+1};x_1,\dots,x_{j+1};y_1,\dots,y_{j+1};x,\tau)\,dx_1\dots dx_{j+1}dy_1\dots dy_{j+1}. \quad (12)$$

*where* $h_j$ *and* $g_j$ *are defined recursively by*

## Theorem

$$
\begin{aligned}
&h_j(k_1,\dots,k_{j+1};x_1,\dots,x_{j+1};y_1,\dots,y_{j+1};x,\tau)\\
&\qquad := \frac{1}{\sqrt{2\pi}} e^{-y_{j+1}^2/2}\, \mathbb{I}_{\left[-\frac{x}{\sqrt{2(\tau-(\tau_{n-j}+p))}},\frac{L-x}{\sqrt{2(\tau-(\tau_{n-j}+p))}}\right]}(y_{j+1})\\
&\cdot g_j(k_1,\dots,k_{j+1};x_1,\dots,x_{j+1};y_1,\dots,y_j;x+y_{j+1}\sqrt{2(\tau-(\tau_{n-j}+p))},\tau_{n-j}+p)
\end{aligned}
$$

*and*

$$
\begin{aligned}
&g_j(k_1,\dots,k_{j+1};x_1,\dots,x_{j+1};y_1,\dots,y_j;x,\tau)\\
&\qquad := \frac{2}{L}\sin\frac{k_{j+1}\pi x_{j+1}}{L}\sin\frac{k_{j+1}\pi x}{L} e^{-(k_{j+1}\pi/L)^2(\tau-\tau_{n-j})}\\
&\qquad\qquad \cdot h_{j-1}(k_1,\dots,k_j;x_1,\dots,x_j;y_1,\dots,y_j;x_{j+1},\tau_{n-j}),
\end{aligned}
$$

*with the recursion starting at*

$$
g_0(k_1;x_1;;x,\tau) := \frac{2}{L} e^{-\alpha x_1}\sin\frac{k_1\pi x_1}{L}\sin\frac{k_1\pi x}{L} e^{-(k_1\pi/L)^2\tau}. \tag{13}
$$

**Idea of the Proof:**

- To iterate the argument of Proposition (see Figure 1)
- Using separation of variables in the barrier periods, and convolution with the heat kernel for the periods in between
- The required initial condition at the left boundary comes from the previous step of the iteration (for $j = 0$ also from the payoff).

**Remarks:**

- Proposition corresponds to (12) for $j = 0$
- If a different option (a call, say) with the same barrier conditions is to be priced instead of a digital payoff, the quantity $e^{-\alpha x_1}$ in (13) should be replaced by the appropriate payoff $U(x_1, 0)$.

# Numerical Implementation

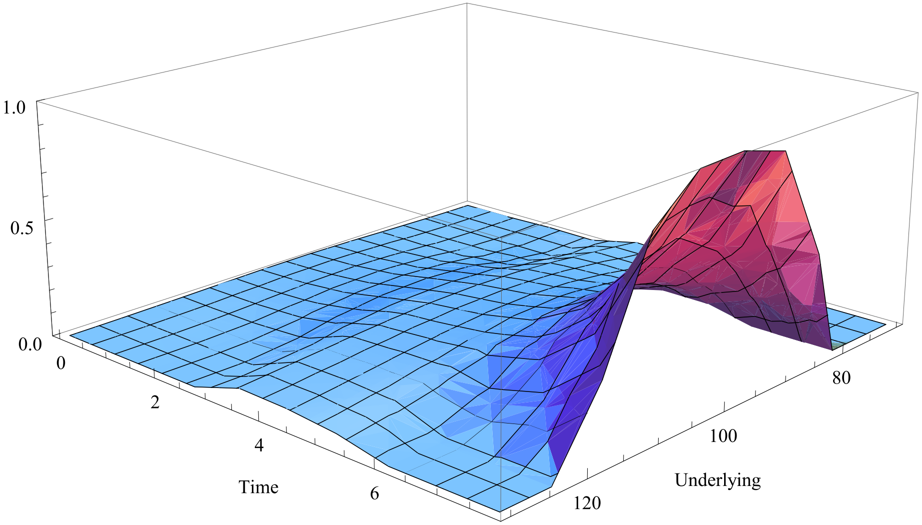


Figure: Value function of a double barrier digital with two barrier periods with parameters $r = 0.01$, $\sigma = 0.15$, $B_{\text{low}} = 80$, $B_{\text{up}} = 120$, $\{T_0, T_1\} = \{1, 6\}$, and $P = 2$.

# Structured Notes and Floors

- Tenor structure satisfies $T_{i-1} + P = T_i$ for $i \in \{1, \ldots, n-1\}$, and define $T_n := T_{n-1} + P$.
- Consider a structured note with $n$ coupons, where the $i$-th coupon consists of a payment of

$$C_i = \mathbb{I}_{\{B_{\text{low}} < S_t < B_{\text{up}},\ \forall t \in [T_{i-1}, T_i]\}}, \qquad i \in \{1, \ldots, n\}, \tag{14}$$

  at time $T_i$.
- In addition, the holder receives the terminal compensation

$$\left(F - \sum_{i=1}^{n} C_i\right)^+ \tag{15}$$

  at $T_n$, where $F > 0$.
- These coupons can be priced by the Proposition (replace $T_0$ by $T_{i-1}$).

(Question):HOWEVER, HOW TO GET A HANDLE ON THE LAW OF $A := \sum_{i=1}^{n} C_i$?

# Structured Floor

(Answer): THE LAW OF $A := \sum_{i=1}^{n} C_i$ IS LINKED TO BARRIER OPTIONS WITH SEVERAL BARRIER PERIODS.

**Observation:** The moments

$$\mathbf{E}[A^\nu] = \sum_{i=0}^{n} i^\nu \mathbf{P}[A = i], \qquad \nu \in \{1, \ldots, n-1\}, \tag{16}$$

of $A$ are linear combinations of multi-period double barrier option prices, with coefficients

$$c(\nu, J) := \sum_{\substack{0 \le i_1, \ldots, i_n \le \nu \\ \mathrm{supp}(\mathbf{i}) = J}} \binom{\nu}{i_1, \ldots, i_n}, \qquad J \subseteq \{1, \ldots, n\}. \tag{17}$$

(The notation $\mathrm{supp}(\mathbf{i}) = J$ means that $J$ is the set of indices such that the corresponding components of the vector $\mathbf{i} = (i_1, \ldots, i_n)$ are non-zero.)

# Structured Floor

## Theorem

*The price of the structured floor* (15) *at time* $t = 0$ *can be expressed as*

$$e^{-rT_n}\mathbf{E}[(F-A)^+] = e^{-rT_n}\sum_{i=0}^{n\wedge\lfloor F\rfloor}(F-i)\,\mathbf{P}[A=i], \tag{18}$$

*where*

$$\mathbf{P}[A=n] = BD(S_0, 0; \{T_0\}, T_n - T_0, B_{\mathrm{low}}, B_{\mathrm{up}}, 0). \tag{19}$$

*The other point masses* $\mathbf{P}[A=i]$ *in* (18) *can be recovered from the moments of* $A$ *by solving* (16) *(including* $\nu = 0$*, of course). The moments in turn can be computed from barrier digital prices by* ($\nu \in \{1, \ldots, n-1\}$)

$$\mathbf{E}[A^\nu] = \sum_{J\subseteq\{1,\ldots,n\}} c(\nu, J)\cdot BD(S_0, 0; \{T_j : j\in J\}, P, B_{\mathrm{low}}, B_{\mathrm{up}}, 0), \tag{20}$$

*where the coefficients* $c(\nu, J)$ *are defined in* (17).

## Proof.

The expression (18) is clear. The event in (19) means that all of the $n$ coupons (14) are paid. By our assumption that $T_i = T_{i-1} + P$, its risk-neutral probability is the (undiscounted) price of a double barrier digital with one barrier period $[T_0, T_n]$, which yields (19). To prove (20), we calculate

$$\begin{aligned}
\mathbf{E}[A^\nu] = \mathbf{E}\left[\left(\sum_{i=1}^{n} C_i\right)^{\nu}\right] &= \sum_{i_1,\dots,i_n} \binom{\nu}{i_1,\dots,i_n} \mathbf{E}[C_1^{i_1}\dots C_n^{i_n}] \\
&= \sum_{i_1,\dots,i_n} \binom{\nu}{i_1,\dots,i_n} \mathbf{E}\Bigg[\prod_{\substack{j=1 \\ i_j>0}}^{n} C_j\Bigg] \\
&= \sum_{J\subseteq\{1,\dots,n\}} \Bigg(\sum_{\substack{i_1,\dots,i_n \\ \mathrm{supp}(\mathbf{i})=J}} \binom{\nu}{i_1,\dots,i_n}\Bigg) \mathbf{E}\Bigg[\prod_{j\in J} C_j\Bigg].
\end{aligned}$$

Now observe that $\prod_{j\in J} C_j$ is the payoff of a double barrier digital with barrier periods $[T_j, T_j + P]$ for $j \in J$. □

# Approximation of Structured Floor

- Numerical quadrature may be too involved for a large number of coupons.
- Let us fix a maturity $T = T_n$ and assume that the $n$ coupon periods

$$\mathcal{T}_i^n := \left]\tfrac{i-1}{n}T, \tfrac{i}{n}T\right], \qquad i \in \{1, \dots, n\},$$

  have length $T/n$.
- For large $n$, the proportion of intervals during which the underlying stays inside the barrier interval

$$\mathcal{B} := [B_{\mathrm{low}}, B_{\mathrm{up}}]$$

  is similar to the proportion of time that the underlying spends inside $\mathcal{B}$, i.e., the occupation time.

## Theorem

*Let $(S_t)_{t\geq 0}$ be a continuous stochastic process such that for each real c the level set $\{t \geq 0 : S_t = c\}$ has a.s. Lebesgue measure zero. Then we have a.s.*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}_{\{S_t \in \mathcal{B}\ \forall t \in \mathcal{T}_i^n\}} = \frac{1}{T} \int_0^T \mathbf{1}_{\mathcal{B}}(S_t) dt.$$

Theorem suggests the approximation

$$e^{-rT}\mathbf{E}[(F-A)^+] \approx e^{-rT} \frac{n}{T} \mathbf{E}\left[\left(\frac{FT}{n} - \int_0^T \mathbf{1}_{\mathcal{B}}(S_t)dt\right)^+\right] \tag{21}$$

for the price of the structure floor (15). It is obtained from replacing $F$ by $F/n$ in the relation

$$\mathbf{E}[(nF-A)^+] \sim n\mathbf{E}\left[\left(F - \frac{1}{T}\int_0^T \mathbf{1}_{\mathcal{B}}(S_t)dt\right)^+\right], \ n \to \infty.$$

which follows from above.

# Corridor Option

$$e^{-rT}\mathbf{E}[(F-A)^+] \approx e^{-rT}\frac{n}{T}\mathbf{E}\left[\left(\frac{FT}{n} - \int_0^T \mathbf{1}_{\mathcal{B}}(S_t)dt\right)^+\right]$$

- On the right-hand side, we recognize the price of a put on the occupation time of $(S_t)$, also called a corridor option.
- Fusai (2000) studied such options in the Black–Scholes model. In particular, his Theorem 1 gives an expression for the Laplace transform of the characteristic function of $\int_0^T \mathbf{1}_{\mathcal{B}}(S_t)dt$.
- By using numerical inversion techniques, Fusai (2000) shows how to price corridor options.

## Proof.

For $1 \leq i \leq n$, define processes $(X_{ni}(t))_{0 \leq t \leq T}$ by

$$X_{ni}(t) := \begin{cases} 1 & \text{if } t \in \mathcal{T}_i^n \text{ and } S_u \in \mathcal{B}\ \forall u \in \mathcal{T}_i^n \\ 0 & \text{otherwise.} \end{cases}$$

Put $X_n := \sum_{i=1}^n X_{ni}$. We claim that, a.s., the function $X_n(\cdot)$ converges pointwise on the set $[0, T] \setminus \{t : S_t = B_{\text{low}} \text{ or } S_t = B_{\text{up}}\}$, with limit $\mathbf{1}_{\mathcal{B}}(S_\cdot)$. Indeed, if $t \in [0, T]$ is such that $S_t \notin \mathcal{B}$, then $X_n(t) = 0$ for all $n$. If, on the other hand, $S_t \in \operatorname{int}(\mathcal{B})$, then $t$ has a neighborhood $V$ such that $S_u \in \mathcal{B}$ for all $u \in V$, by continuity. Hence $X_n(t) = 1$ for large $n$. Since we have pointwise convergence on a set of (a.s.) full measure, we can apply the dominated convergence theorem to conclude

$$\lim_{n\to\infty} \int_0^T X_n(t)dt = \int_0^T \mathbf{1}_{\mathcal{B}}(S_t)dt, \qquad \text{a.s.}$$

But this is the desired result, since

$$\begin{aligned}\int_0^T X_n(t)dt &= \sum_{i=1}^n \int_0^T X_{ni}(t)dt \\ &= \sum_{i=1}^n \int_{\mathcal{T}_i^n} X_{ni}(t)dt \quad = \sum_{i=1}^n |\mathcal{T}_i^n|\, \mathbf{1}_{\{S_t \in \mathcal{B}\ \forall t \in \mathcal{T}_i^n\}} = \frac{T}{n} \sum_{i=1}^n \mathbf{1}_{\{S_t \in \mathcal{B}\ \forall t \in \mathcal{T}_i^n\}}.\end{aligned}$$

# Numerical evaluation of the approximation

| coupons | structure floor | corridor option | relative error |
|---|---|---|---|
| $n = 1$ | 7.63696 | 9.91563 | 0.2298060 |
| $n = 2$ | 7.52979 | 9.24883 | 0.1858660 |
| $n = 3$ | 7.42262 | 8.66698 | 0.1435750 |
| $n = 4$ | 7.31545 | 8.06291 | 0.0927030 |
| $n = 5$ | 7.20827 | 7.44886 | 0.0322987 |
| $n = 6$ | 7.10110 | 7.31880 | 0.0297450 |
| $n = 7$ | 6.99393 | 7.18558 | 0.0266714 |
| $n = 8$ | 6.88677 | 7.03704 | 0.0213544 |
| $n = 9$ | 6.77962 | 6.92288 | 0.0206936 |
| $n = 10$ | 6.67232 | 6.80399 | 0.0193516 |

Table: Numerical evaluation of the approximation (21) with maturity $T = 4$, structure floor level $F = 10$, and $n$ coupons. The other parameters are $r = 0.01$, $\sigma = 0.15$, $B_{\mathrm{low}} = 80$, and $B_{\mathrm{up}} = 120$.

- Approximation by a corridor option works only for period lengths tending to zero.
- One could also let the number of coupons tend to infinity for a fixed period length $P$, so that maturity increases linearly with $n$.

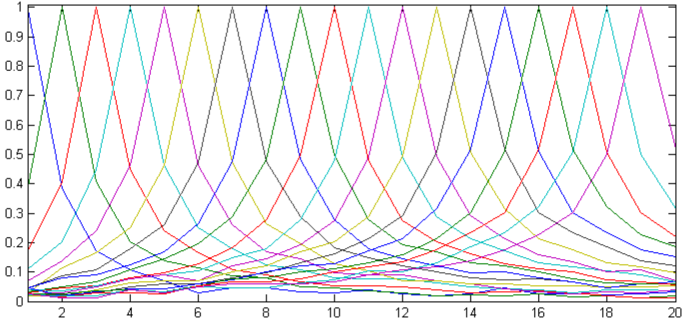


Figure: Correlation between individual coupons, n=20

# Approximation of Sum of Coupons

- The correlation of the random variables $C_i$ and $C_j$ decreases for large $|i-j|$.
- Therefore, it is a natural question whether a "central limit theorem" holds, i.e., whether the sum of coupons $A := \sum_{i=1}^{n} C_i$

$$\frac{A - \mathbf{E}[A]}{\sqrt{\mathbf{Var}[A]}}$$

  converges in law to a standard normal random variable as $n \to \infty$.
- To have CLT (or certain normal approximation methods) we have to verify any of the "mixing" conditions (see Bradley (2005)).
- "Mixing" means, roughly, that random variables temporally far apart from one another are nearly independent.
- However, we could not verify for example, the $\phi$ mixing condition.
- Numerical experiments also cast doubt on Gaussian limit law.

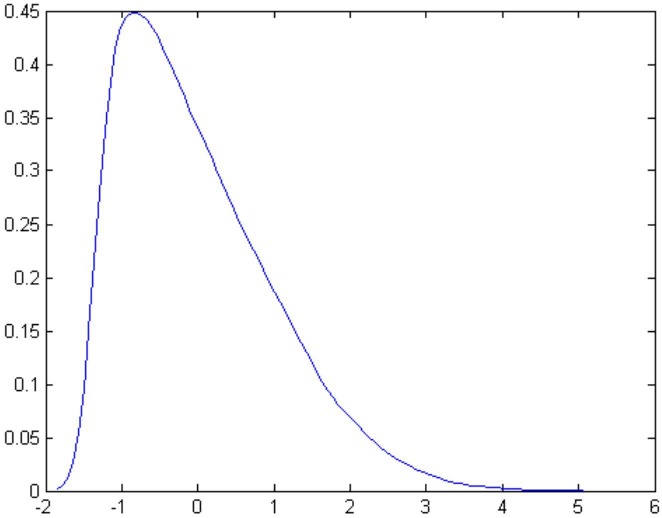


Figure: Density Approximation

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Thanks for listening...
Comments! and Questions?

# Approximation of Sum of Coupons and $\phi$-mixing

- On $(\Omega, \mathcal{F}, \mathbf{P})$, for any two $\sigma$-fields $\mathcal{A}$ and $\mathcal{B} \in \mathcal{F}$,
- $\phi(\mathcal{A}, \mathcal{B}) := \sup |\mathbf{P}(B|A) - \mathbf{P}(B)|$ given $A \in \mathcal{A}$ and $B \in \mathcal{B}$, where the supremum is taken over all pairs of (finite) partitions $\{A_1, \dots, A_i\}$ and $\{B_1, \dots, B_j\}$ of $\Omega$ such that $A_i \in \mathcal{A}$ for each $i$ and $B_i \in \mathcal{B}$ for each $j$.
- Now suppose $X := (X_k,\ k \in \mathbb{N})$ is a sequence of random variables. For $0 \leq J \leq L \leq \infty$, define the $\sigma$-field

$$\mathcal{F}_J^L := \sigma(X_k,\ J \leq k \leq L, (k \in \mathbb{N}))$$

  and for each $n \geq 1$ define

$$\phi(n) := \sup_{j \in \mathbb{N}} (\mathcal{F}_0^j, \mathcal{F}_{j+n}^\infty)$$

- The random sequence $X$ is called $\phi$-mixing if $\phi(n) \to 0$ as $n \to \infty$.
- In the context of our question, it can be shown that adapting the above notations in the obvious way, $\phi(\sigma(W_s, s \leq t), \sigma(W_s, s \geq t+u)) = 1$.