Worst-Case Portfolio Optimization in a Market with Bubbles

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6th AMaMeF Conference, Warsaw, Poland
June 11, 2013
Overview

1. Introduction and Model Setup

2. The HJB System and Verification

3. Numerical Results
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We study

- an optimal investment problem, in which the investor aims to maximize expected utility from terminal wealth,
Introduction

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   expected utility from terminal wealth,

2. while assuming that bubbles may be present in the market which may lead 
   to crashes,
Worst-Case Portfolio Optimization in a Market with Bubbles

We study

1. an optimal investment problem, in which the investor aims to maximize expected utility from terminal wealth,
2. while assuming that bubbles may be present in the market which may lead to crashes,
3. and we assume that the investor takes a worst-case perspective towards the impact of these crashes.
Worst-Case Portfolio Optimization in a Market with Bubbles

We study

1. an optimal investment problem, in which the investor aims to maximize expected **utility from terminal wealth**,
2. while assuming that bubbles may be present in the market which may **lead to crashes**, and
3. we assume that the investor takes a **worst-case perspective** towards the impact of these crashes.

### The Worst-Case Problem

\[
\sup_{\pi} \inf_{\theta} \mathbb{E} \left[ U(X_T^{\pi,\theta}) \right].
\]
Literature related to **Bubbles**:

Loewenstein and Willard (2000), Cox and Hobson (2005), Jarrow, Protter and Shimbo (2007, 2010), Biagini, Föllmer and Nedelcu (2013), ...

⇒ Compare with Föllmer’s talk.

Literature related to the **Worst-Case Approach**:

Korn and Willmot (2002), Korn and Menkens (2005), Korn and Steffensen (2007), Seifried (2010), ...

⇒ Compare with Menkens’ talk.
We start with a **Black-Scholes market**:

\[
\begin{align*}
\frac{dB_t}{B_t} &= 0, \\
\frac{dS_t}{S_t} &= \alpha S_t dt + \sigma S_t dW_t.
\end{align*}
\]
We start with a **Black-Scholes market**:

\[ dB_t = 0, \quad dS_t = \alpha S_t \, dt + \sigma S_t \, dW_t. \]

Let \( Z_t \) be an **observable** continuous-time Markov chain with **finite state space** \( \{0, 1, \ldots, d\} \).
The Market

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- **State \( i \in \{1, \ldots, d\} \)** corresponds to a regime in which a **bubble is present**. The bubble may burst, **leading to a crash** of maximum relative size \( \kappa^i \), i.e.

\[ S_\tau = (1 - \kappa)S_{\tau^-}, \quad 0 \leq \kappa \leq \kappa^i. \]
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- After a crash, \( Z_t \) is assumed to **jump back to state 0**.
Bubbles and Crashes

\[ Z_t \text{ in state } 0 \]

\[ \rightarrow \text{ No crash possible} \]
Bubbles and Crashes

\[ Z_t \text{ in state } 0 \]
\[ \rightarrow \text{ No crash possible} \]

\[ \downarrow \]

\[ Z_t \text{ jumps to state } i \]
\[ \rightarrow \text{ Investor receives warning} \]
\[ \rightarrow \text{ Crash of maximum size } \kappa^i \text{ possible} \]
Bubbles and Crashes

**Z\_t in state 0**

→ No crash possible

↓

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→ Investor receives warning
→ Crash of maximum size $\kappa_i$ possible

↘

**Crash ($\tau, \kappa$) occurs**

→ Stock price crashes by a fraction of $\kappa$
→ $Z_t$ jumps back to state 0
**Bubbles and Crashes**

**Z_t in state 0**
- No crash possible

**Z_t jumps to state i**
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- Crash of maximum size \( \kappa^i \) possible

**Crash \((\tau, \kappa)\) occurs**
- Stock price crashes by a fraction of \( \kappa \)
- \( Z_t \) jumps back to state 0

**Z_t jumps to state j**
- Investor receives new information
- Crash of maximum size \( \kappa^j \) possible
The Investor

In each state $i = 0, \ldots, d$, the investor can choose which fraction $\pi^i_t$ of her total wealth $X_t$ to invest in the risky asset.
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$$X_{\tau_k} = (1 - \pi^i_{\tau_k} \kappa_k) X_{\tau_k-}, \quad \text{on} \ \{Z_{\tau_k-} = i\}.$$
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We say that a strategy $\pi = (\pi^0, \ldots, \pi^d)$ is admissible, if it leads to nonnegative wealth for all possible crash scenarios $\theta = (\tau_k, \kappa_k)_{k \in \mathbb{N}}$. This holds if

$$\pi^i \leq \frac{1}{\kappa^i},$$

for all $i$. 

Christoph Belak, Sören Christensen, Olaf Menkens
Worst-Case Portfolio Optimization in a Market with Bubbles
The Worst-Case Problem

The aim of the investor is to find the strategy \( \pi^* = (\pi_0^*, \ldots, \pi_d^*) \) which performs best if the worst-possible crash scenario \( \theta = (\tau_k, \kappa_k)_{k \in \mathbb{N}} \) occurs.

\[
V(t, x, i) = \sup_{\pi} \inf_{\theta} \mathbb{E}_{(t,x,i)} \left[ U(X_T^{\pi, \theta}) \right].
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The Worst-Case Problem

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$$V(t, x, i) = \sup_{\pi} \inf_{\theta} \mathbb{E}_{(t, x, i)} \left[ U(X_{T}^{\pi, \theta}) \right].$$

Here, $U$ is assumed to be the power utility function:

$$U(x) = \frac{1}{\gamma} x^\gamma, \quad \gamma < 1, \gamma \neq 0.$$
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The value function $V(\cdot, \cdot, 0)$ and the corresponding optimal strategy in state 0 can be determined by solving the following \textbf{HJB equation}:

$$0 = \sup_{\pi} \left[ \mathcal{L}^{\pi} V(t, x, 0) + \sum_{j=0}^{d} q_{0,j} V(t, x, j) \right].$$
The HJB Equation for $i = 0$

Denote by $(q_{i,j})_{0 \leq i,j \leq d}$ the generator matrix of $Z_t$ and let

$$L^\pi V = V_t + \alpha \pi x V_x + \frac{1}{2} \sigma^2 \pi^2 x^2 V_{xx}.$$
Verification for $i = 0$

Using the HJB equation

$$0 = \sup_{\pi} \left[ \mathcal{L}^\pi V(t, x, 0) + \sum_{j=0}^{d} q_{0,j} V(t, x, j) \right],$$

it is easy to verify that the optimal strategy for state 0 is simply the Merton fraction

$$\pi_{t}^{0,*} = \frac{\alpha}{(1 - \gamma) \sigma^2}.$$
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The value function is given by

$$V(t, x, 0) = \frac{1}{\gamma} x^{\gamma} f_{0}^{0}(t),$$

where $f_{0}^{0}(t)$ solves

$$f_{t}^{0}(t) = -\frac{\gamma\alpha^{2}}{2(1 - \gamma)\sigma^{2}} f_{0}^{0}(t) - \sum_{j=0}^{d} q_{0,j} f^{j}(t), \quad f_{0}^{0}(T) = 1.$$
The HJB Equation for $i > 0$

Define the following sets:

$$A_1 := \left\{ \pi : V(t,x,i) \leq V(t,(1 - \kappa^i \pi_t)x,0) \right\},$$

The value function $V(\cdot, \cdot, i)$ and the corresponding optimal strategy in state $i$ can be determined by solving the **HJB system**

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$$0 \leq \sup_{\pi \in A_2} \left[ V(t, (1 - \kappa^i \pi_t)x, 0) - V(t, x, i) \right].$$
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$$\cdot \sup_{\pi \in A_2} \left[ V(t, (1 - \kappa^i \pi_t) x, 0) - V(t, x, i) \right].$$
Suppose both equations are equal to 0 at the same time. As before, we make the ansatz

\[ V(t, x, i) = \frac{1}{\gamma} x^{\gamma} f^i(t). \]

With this, the equations reduce to

\[ f^i(t) = (1 - \pi_t^{i,*} \kappa^i)^{\gamma} f^0(t), \]

\[ \frac{\partial}{\partial t} f^i(t) = -\gamma f^i(t) \left( \alpha \pi_t^{i,*} - \frac{1}{2} (1 - \gamma) \sigma^2 (\pi_t^{i,*})^2 \right) - \sum_{j=1}^{d} q_{i,j} f^j(t). \]
Verification for $i > 0$

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Taking the logarithm in the first equation and then the derivative with respect to $t$, this yields

$$\frac{\partial}{\partial t} \pi_t^{i, *} = \frac{1}{\gamma \kappa^i} (1 - \pi_t^{i, *} \kappa^i) \left[ \frac{1}{f^0(t)} \frac{\partial}{\partial t} f^0(t) - \frac{1}{f^i(t)} \frac{\partial}{\partial t} f^i(t) \right].$$
Plugging the ODEs for $f^0$ and $f^i$ into the last equation, we arrive at

$$
\frac{\partial}{\partial t} \pi_t^{i,*} = -\frac{1}{\kappa^i} (1 - \pi_t^{i,*} \kappa^i) \left[ \frac{1}{2} (1 - \gamma) \sigma^2 (\pi_t^{i,*} - \pi_t^{0,*})^2 \right. \\
- \frac{1}{\gamma} \sum_{j=1}^{d} q_{0,j} \left( (1 - \pi_t^{j,*} \kappa^j)^\gamma - 1 \right) \\
+ \frac{1}{\gamma} \sum_{j=1}^{d} q_{i,j} \left( \frac{(1 - \pi_t^{j,*} \kappa^j)^\gamma}{(1 - \pi_t^{i,*} \kappa^i)^\gamma} \right],
$$

with terminal condition $\pi_T^{i,*} = 0$. 
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Numerical Example
Numerical Example

The graph shows the behavior of risky asset fraction over time in a market with different crash sizes. The x-axis represents time, and the y-axis represents the risky fraction. Different lines correspond to different crash sizes, with the Merton fraction being represented by a blue line. The graph illustrates how the risky fraction decreases with time under varying crash scenarios.
We propose a regime switching model for an optimal investment problem in a financial market with bubbles.

We derive a system of HJB equations which lead to a coupled system of ordinary differential equations for the optimal strategies.

Depending on the choice of parameters, the optimal strategies may or may not make the investor indifferent towards the impact of the crashes.

It is straightforward to extend the results to state-dependent market coefficients.
Thank you for your attention!!!