BSDEs with singular terminal condition and control problems with terminal state constraint

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Optimal position closure

Bertsimas&Lo framework:

- $T < \infty$: time horizon
- $x \in \mathbb{R}$: initial position
- X_t: position size at time $t \in [0, T]$

  Constraint: $X_T = 0$

- $\dot{X}_t$: trading rate ($\dot{X} \geq 0$: buying, $\dot{X} \leq 0$: selling)
- S_t: uninfluenced price (a martingale)
- $\eta > 0$: price impact parameter
- $\tilde{S}_t = S_t + \eta \dot{X}_t$: realized price

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- Expected costs:

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A control problem with terminal state constraint

- Brownian set-up: $(\Omega, \mathcal{F}, P, (\mathcal{F}_t), (W_t))$
- (η_t): positive, progressively measurable
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- Control problem:

$$v(t, x) = \inf_{X \in \mathcal{A}_0(t,x)} E\left[\int_t^T \left(\eta_s|\dot{X}_s|^p + \gamma_s|X_s|^p\right) ds \middle| \mathcal{F}_t\right]$$

Our aim & related literature

- We aim at providing a purely probabilistic solution of the control problem
- Characterize the optimal control by means of a BSDE with singular terminal condition
- Schied 2013: Solves a variant of this problem in a Markovian framework using superprocesses

A maximum principle

$$v(t,x) = \inf_{X \in \mathcal{A}_0(t,x)} E\left[\int_t^T \left(\eta_s|\dot{X}_s|^p + \gamma_s|X_s|^p\right) ds \Big| \mathcal{F}_t\right] \qquad (1)$$

Proposition (Maximum Principle)

Let $X \in \mathcal{A}_0(t,x)$ such that

$$M_s = \eta_s|\dot{X}_s|^{p-1} + \int_t^s \gamma_r|X_r|^{p-1} dr$$

is a martingale. Then X is optimal in (1).

A maximum principle

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Proposition (Maximum Principle)

Let $X \in \mathcal{A}_0(t,x)$, i.e.

$$dX_s = \dot{X}_s ds, \quad X_t = x \quad \& \quad X_T = 0$$

such that

$$M_s = \eta_s|\dot{X}_s|^{p-1} + \int_t^s \gamma_r|X_r|^{p-1} dr$$

is a martingale. Then X is optimal in (2).

Derivation of the BSDE

$$v(t,x) = \inf_{X \in \mathcal{A}_0(t,x)} E\left[\int_t^T \left(\eta_s|\dot{X}_s|^p + \gamma_s|X_s|^p\right) ds \Big| \mathcal{F}_t\right]$$

- The value function is explicit in the x variable:

$$v(t,x) = Y_t|x|^p$$

  for some coefficient process Y.
- The maximum principle implies:

$$dY_t = \left((p-1)\frac{Y_t^q}{\eta_t^{q-1}} - \gamma_t\right) dt + Z_t dW_t$$

- Terminal constraint leads to singular terminal condition: $Y_T = \infty$
- Optimal control: $\dot{X}_t = -\left(\frac{Y_t}{\eta_t}\right)^{q-1} X_t$,
  i.e.

$$X_t = xe^{-\int_0^t \left(\frac{Y_s}{\eta_s}\right)^{q-1} ds}$$

BSDEs with singular terminal condition

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$$dY_t = \left((p-1)\frac{Y_t^q}{\eta_t^{q-1}} - \gamma_t \right) dt + Z_t dW_t \tag{3}$$

Definition

(Y, Z) is a solution of the BSDE (3) with singular terminal condition $Y_T = \infty$ if it satisfies

(i) *for all $0 \le s \le t < T$:*
$Y_s = Y_t - \int_s^t \left((p-1)\frac{Y_r^q}{\eta_r^{q-1}} - \gamma_r \right) dr - \int_s^t Z_r dW_r$;

(ii) $\liminf_{t \nearrow T} Y_t = \infty$, *a.s.*

(iii) *for all $0 \le t < T$: $E\left[\sup_{0 \le s \le t} |Y_s|^2 + \int_0^t |Z_r|^2 dr\right] < \infty$;*

Integrability Assumptions and Approximation

- For the remainder of the talk we assume that η satisfies

$$E\int_0^T \frac{1}{\eta_t^{q-1}}dt < \infty, \quad E\int_0^T \eta_t^2 dt < \infty$$

and that γ satisfies

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- Approximation

$$\begin{aligned} dY_t^L &= \left((p-1)\frac{(Y_t^L)^q}{\eta_t^{q-1}} - (\gamma_t \wedge L)\right)dt + Z_t^L dW_t \\ Y_T^L &= L \end{aligned}$$

Existence and Minimality

Proposition

There exists a solution (Y^L, Z^L). Y^L *is bounded from below*

$$Y_t^L \geq \frac{1}{\left(\frac{1}{L^{q-1}} + E\left[\int_t^T \frac{1}{\eta_s^{q-1}} ds \middle| \mathcal{F}_t\right]\right)^{p-1}}.$$

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Theorem

There exists a process (Y, Z) such that for every $t < T$ and as $L \nearrow \infty$

- $Y_t^L \nearrow Y_t$ *a.s.*
- $Z^L \to Z$ *in* $L^2(\Omega \times [0, t])$.

The pair (Y, Z) is the minimal solution to (3) with singular terminal condition $Y_T = \infty$.

Optimal Controls - Penalization

Consider the unconstrained minimization problem

$$v^L(0,x) = \inf_{X \in \mathcal{A}(0,x)} E\left[\int_0^T \left(\eta_s|\dot{X}_s|^p + (\gamma_s \wedge L)|X_s|^p\right) ds + L|X_T|^p\right] \quad (4)$$

Proposition

The control

$$X_t^L = xe^{-\int_0^t \left(\frac{Y_s^L}{\eta_s}\right)^{q-1} ds}$$

is optimal in (4) *and* $v^L(0,x) = Y_0^L|x|^p$.

Optimal Controls

Theorem

The control

$$X_t = xe^{-\int_0^t \left(\frac{Y_s}{\eta_s}\right)^{q-1} ds}$$

belongs to $\mathcal{A}_0(0, x)$ *and is optimal in* (1). *Moreover,* $v(t, x) = Y_t |x|^p$.

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Proof

Define $M_t = p\eta_t |\dot{X}_t|^{p-1} + \int_0^t p\gamma_s |X_s|^{p-1} ds$. *Then* $dM_t = X_t^{p-1} Z_t dW_t$. *Hence* M *is a nonnegative, local martingale on* $[0, T)$. *In particular* M *converges almost surely for* $t \nearrow T$. *This implies*

$$0 \leq X_t = \left(\frac{M_t - p\int_0^t \gamma_s X_s^{p-1} ds}{pY_t} \right)^{q-1} \leq \left(\frac{M_t}{pY_t} \right)^{q-1} \to 0$$

a.s. for $t \nearrow T$

Processes with uncorrelated multiplicative increments

Definition

η has uncorrelated multiplicative increments (umi) if

$$E\left[\frac{\eta_t}{\eta_s}\Big|\mathcal{F}_s\right] = E\left[\frac{\eta_t}{\eta_s}\right]$$

for all $s \leq t < T$.

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Examples

- η is deterministic
- η is a martingale
- $d\eta_t = \mu(t)\eta_t dt + \sigma(t, \eta_t)dW_t$
- $\eta_t = e^{Z_t}$ where Z is a Lévy process

umi processes $\leftrightarrow$ deterministic controls

Assume $\gamma = 0$.

Proposition

Suppose that η has umi, then

$$Y_t = \frac{1}{\left(\int_t^T \frac{1}{E[\eta_s|\mathcal{F}_t]^{q-1}} ds\right)^{p-1}}$$

is the minimal solution to (3) *with singular terminal condition. The deterministic control*

$$X_t = x \frac{1}{\int_0^T \frac{1}{E[\eta_s]^{q-1}} ds} \int_t^T \frac{1}{E[\eta_s]^{q-1}} ds$$

is optimal in (1)*.*

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is optimal in (1).

Vice versa, assume that the optimal control $X_t = x e^{-\int_0^t \left(\frac{Y_s}{\eta_s}\right)^{q-1} ds}$ is deterministic. Then η has umi.

Thank you!

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