Optimal dual martingales and new algorithms for Bermudan products

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- Snell envelope
- Dual methods

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- Backward recursion
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Part I

Rehash of optimal stopping

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- asset prices $(S_i)_{i=0,\ldots,T}$ with values in $\mathbb{R}_+^D$
- cashflow $(Z_i)_{i=0,\ldots,T} \in \sigma(S)$ such that $\max_{i=0,\ldots,T} |Z_i| \in L^1$

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Optimal stopping problem

Find **spot price** by solving

$$Y_0^* := \sup_{\tau \in \{0,\ldots,T\}} \mathbb{E} Z_\tau, \quad \tau \text{ stopping time}$$

Snell envelope approach

- Dynamize the problem via
- $Y_i^* := \sup_{\tau \in \{i,\dots,T\}} \mathbb{E}_i Z_\tau := \operatorname{esssup}_{\tau \in \{i,\dots,T\}} \mathbb{E}\left[Z_\tau | \mathcal{F}_i\right]$

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Properties

- Bellman principle $\rightsquigarrow$ $Y_i^* = \max\big\{Z_i, \mathbb{E}_i Y_{i+1}^*\big\}$
- Y^* smallest supermartingale dominating Z
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Consequences

- $\tau \in \{0, \ldots, T\}$ any stopping time $\Longrightarrow$ $Y_0^{low}(\tau) := \mathbb{E} Z_\tau$
- $Y_0^{low}(\tau) \le Y_0^*$ and $Y_i^{low}(\tau_0^*) = Y_0^*$

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Question: Is the Doob martingale the **only martingale** which attains the infimum?

Part II

Surely optimal martingales

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Example

By taking $M = M^*$ being the Doob martingale of Y^*, i.e.

$$Y_j^* = Y_0^* + M_j^* - A_j^* \Longrightarrow$$

$$Z_j - Y_0^* - M_j^* = Z_j - Y_j^* - A_j^* \leq 0$$

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- $M_j^\circ = Y_0^* \left(\frac{Y_j^*}{Y_0^* B_j^*} - 1 \right) \geq Y_0^* \left(\frac{Y_j^*}{Y_0^*} - 1 \right) = Y_j^* - Y_0^* \geq Z_j - Y_0^*$

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- by previous Lemma

$$\mathbf{Y_0^*} = \max_{\mathbf{0 \leq j \leq T}} (\mathbf{Z_j} - \mathbf{M_j^\circ}) \quad a.s.$$

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where M^* is the Doob martingale and A_i^* the predictable part of the Snell envelope Y^*.

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Let $Z > 0$ and consider $Y_i^* = Y_0^* N_i^* B_i^* > 0$. For $\alpha \in [0,1]$ put

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Hence, for every $0 \leq \alpha \leq 1$

$$M_i = M_i^* - \alpha A_i^* + \alpha \sum_{l=1}^{i} \left(A_l^* - A_{l-1}^* \right) \frac{N_l^*}{N_{l-1}^*},$$

is an **surely optimal** for $i = 0, ..., T$.

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Let Y^* be the Snell envelope of the cashflow Z and let M be any martingale with $M_0 = 0$. Then, for every $i \in \{0, ..., T\}$ it holds that

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Checking sure optimality

- $i = 0$ and $\max_{0 \le j \le T} (Z_j - M_j)$ is **deterministic**
- Theorem yields that M is **surely optimal**

Define $\theta_{\mathbf{i}}^{(\mathbf{n})} := \max_{\mathbf{i} \leq \mathbf{j} \leq \mathbf{T}} \left(\mathbf{Z_j} - \mathbf{M_j^{(n)}} + \mathbf{M_i^{(n)}} \right)$.

Stability theorem

Let $M^{(n)}$ be a sequence of martingales with $M_0^n = 0$ that is **uniformly integrable**. If

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$$Var_i(\theta_i^{(n)}) := \mathbb{E}_i \left| \theta_i^{(n)} - \mathbb{E}_i \theta_i^{(n)} \right|^2 \longrightarrow 0 \text{ in prob. as } n \to \infty$$

then it holds that

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$\Longrightarrow$ **good martingales via**

$\mathbb{E} Var_i(\theta_i) = \mathbb{E} Var_i \max_{i \le j \le T} (Z_j - M_j + M_i)$ **low for all i = 0, ..., T**

Part III

Designing new dual algorithm

Useful recursion

$$\begin{aligned}\theta_i(M) &= \max\left(Z_i, \max_{i+1\le j\le T}(Z_j - M_j + M_i)\right)\\ &= \max\left(Z_i, \theta_{i+1}(M) + M_i - M_{i+1}\right)\\ &= Z_i + \left(\theta_{i+1}(M) + M_i - M_{i+1} - Z_i\right)^+\end{aligned}$$

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- $\xi_{i+1} = M_{i+1} - M_i$ with $\mathbb{E}_i\xi_{i+1} = 0$ solving

$$\xi_{i+1} = \operatorname*{arg\,min}_{\xi\in\Delta\mathcal{F}_{i,i+1},\ \mathbb{E}_i\xi=0} \mathbb{E}\Big[Var_i\left(\vartheta_{i+1}(M) - \xi - Z_i\right)^+\Big]$$

- set $\theta_i(M) = Z_i + \left(\theta_{i+1}(M) - \xi_{i+1} - Z_i\right)^+$

Structured increments

- let $\xi_i := \xi_i(\beta) := \sum_{k=1}^{K} \beta_k \mathfrak{m}_{i+1}^{(k)}$

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$$= \underset{\beta \in \mathbb{R}^K}{\arg\min}\ \mathbb{E}\Big[Var_i\Big(\vartheta_{i+1}(M) - \sum_{k=1}^{K} \beta_k \mathfrak{m}_{i+1}^{(k)}\Big)\Big]$$

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Implies original problem

$$\underset{\beta\in\mathbb{R}^K}{\arg\min}\ \mathbb{E}\Big[Var_i\left(\theta_{i+1}(M) - \xi_{i+1}(\beta) - Z_i\right)^{+}\Big]$$

$$\leq \mathbb{E}\Big[Var_i\left(\theta_{i+1}(M) - \xi_{i+1}(\beta^\circ) - Z_i\right)\Big]$$

$$= \mathbb{E}\Big| \theta_{i+1}(M) - \xi_{i+1}(\beta^\circ) - \mathbb{E}_i\theta_i(M)\Big|^2$$

Linear regression problem

$$[\beta^\circ, \gamma^\circ] = \underset{\beta \in \mathbb{R}^K, \gamma \in \mathbb{R}^{K_1}}{\arg\min} \mathbb{E} \left| \theta_{i+1}(M) - \sum_{k=1}^{K} \beta_k \mathfrak{m}_{i+1}^{(k)} - \sum_{k=1}^{K_1} \gamma_k \psi_k(i, X_i) \right|^2$$

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- another set of basis function $\psi_k(t, x)$, $k = 1, ..., K_1$
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Primal dual algorithm

- $Y_0^* \leq Y_0^{up} = \mathbb{E} \max_{0 \leq j \leq T} \left(Z_j - \sum_{j=1}^{i} \sum_{k=1}^{K} \beta_k^\circ \mathfrak{m}_j\right)$
- $Y_0^* \geq Y_0^{low} = \mathbb{E} Z_\tau$

Remark

- Lévy-Itô setting typically has

$$\xi_{i+1}(\beta) = \sum_{k=1}^{N_1} \beta_k^c \int_{T_i}^{T_{i+1}} \varphi_k^c(s, X_s) dW_s + \sum_{k=1}^{N_2} \beta_k^d \int_{T_i}^{T_{i+1}} \varphi_k^d(s, X_s, u) d\widetilde{N}(ds, du)$$

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- thus the class of martingales is spanned by

$$\mathfrak{m}_{i+1}^{(k)} = \int_{T_i}^{T_{i+1}} \varphi_k^c(s, X_s) dW_s + \int_{T_i}^{T_{i+1}} \varphi_k^d(s, X_s, u) d\widetilde{N}(ds, du)$$

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- careful choice of φ^c, φ^d essential

Virtue of variance minimizing property

If the class of martingales spanned by $\mathfrak{m}_{i+1}^{(k)}$ is **"rich"** enough, the regression

$$[\beta^\circ, \gamma^\circ] = \underset{\beta \in \mathbb{R}^K, \gamma \in \mathbb{R}^{K_1}}{\arg\min} \mathbb{E} \left| \theta_{i+1}(M) - \sum_{k=1}^{K} \beta_k \mathfrak{m}_{i+1}^{(k)} - \sum_{k=1}^{K_1} \gamma_k \psi_k(i, X_i) \right|^2$$

can be realized with a **small sample size**

Numerical experiment

Market setup

- W_t^d independent Brownian motions
- risk-neutral dynamics of D assets

$$dX_t^d = (r-\delta)X_t^d dt + \sigma X_t^d dW_t^d, \quad d = 1, ..., D$$

Benchmark products

- **Bermudan basket-put:** $Z_t(X_t) = e^{-rt}(K - \frac{X_t^1 + ... + X_t^D}{D})^+$
- **Bermudan max-call:** $Z_t(X_t) = e^{-rt}(\max(X_t^1, \ldots, X_t^D) - K)^+$

Basket put

Table : Lower and upper bounds for Bermudan basket-put on 5 assets with parameters $r = 0.05$, $\delta = 0$, $\sigma = 0.2$, $K = 100$, $T = 3$ and different J = exercise rights and x_0 = spot price

J	x_0	Low (SD)	Up (SD)	BKS Price Interval
3	90	10.000 (0.000)	10.000 (0.000)	[10.000, 10.004]
	100	2.164 (0.007)	2.168 (0.005)	[2.154, 2.164]
	110	0.539 (0.004)	0.555 (0.003)	[0.535, 0.540]
6	90	10.000 (0.000)	10.000 (0.000)	[10.000, 10.000]
	100	2.407 (0.006)	2.432 (0.005)	[2.359, 2.412]
	110	0.573 (0.003)	0.608 (0.003)	[0.569, 0.580]
9	90	10.000 (0.0000)	10.000 (0.000)	[10.000, 10.005]
	100	2.475 (0.0063)	2.539 (0.006)	[2.385, 2.502]
	110	0.5915 (0.0034)	0.635 (0.003)	[0.577, 0.600]

Regression sample size: 1000
Upper bound simulation sample size: 1000

Bermudan max-call

Table : Lower and upper bounds for Bermudan max-call with parameters $r = 0.05$, $\delta = 0.1$, $\sigma = 0.2$, $K = 100$, $T = 3$ and different D and x_0.

D	x_0	Low (SD)	Up (SD)	A&B price interval
2	90	8.0556 (0.021)	8.15284 (0.014)	[8.053, 8.082]
	100	13.8850 (0.027)	14.0145 (0.019)	[13.892, 13.934]
	110	21.3671 (0.0319)	21.5187 (0.022)	[21.316, 21.359]
5	90	16.5973 (0.0296)	16.7718 (0.027)	[16.602, 16.655]
	100	26.1325 (0.0356)	26.3440 (0.031)	[26.109, 26.292]
	110	36.7348 (0.0403)	37.0431 (0.039)	[36.704, 36.832]

Regression sample size: 1000
Upper bound simulation sample size: 1000

Thanks for your attention!