Option Pricing and Calibration with Time-changed Lévy processes

Yan Wang and Kevin Zhang

Warwick Business School

12th Feb. 2013
1. How to find a perfect model that captures essential features of financial returns
   - empirical findings: negative skewness, high kurtosis, stochastic volatility and jumps
   - available stochastic processes: Brownian motion and jump processes (Lévy processes)

2. How to keep the tractability
   - Carr-Madan formula with FFT method

3. Empirical analysis
   - estimation
   - daily calibration
Lévy processes are widely used recently to model Financial returns. A Lévy process can generate a variety of distributions at a fixed time horizon. Brownian motion is a special case of Lévy processes.

A stochastic process \((X_t)_{t \geq 0}\) is said to be a Lévy process if:
1. \(X_0 = 0\) a.s. ;
2. \(X_t - X_s \perp X_s\), for any \(t > s\);
3. \(X_t - X_s\) is equal in distribution to \(X_{t-s}\), for any \(t > s\).

Lévy processes are fully characterized by its characteristic function

\[
\mathbb{E}[\exp(iuX_t)] = \exp\left(i\mu ut - \frac{1}{2}\sigma^2 u^2 t + t \int_{\mathbb{R}_0} (e^{iux} - 1 - iux1_{|x|<1}) \pi(dx)\right)
\]

which is the Levy-Khintchine representation.
Theoretical Background: What is time-changed Lévy process?

What is time-change?

- Let $t \rightarrow T_t$ be an increasing right-continuous process with left limits satisfying the usual conditions.

- The random time $T_t$ can be modelled as a nondecreasing semimartingale

$$T_t = \alpha_t + \int_0^t \int_0^\infty y\mu(dt, dy)$$

- Simply, we can model the random time as

$$T_t = \int_0^t v_s ds$$

where $v(t)$ is the activity rate.

- $T_t$ can be viewed as the business time at time $t$. It is driven by a stochastic activity process. A more active business day can generate higher volatility.
Theoretical Background: What is time-changed Lévy process?

- Time-changed Lévy process: $Y_t = X_{T_t}$

- Lévy processes are natural to be applied with time-change technique
  - infinitely divisible distribution

- Common choices of the activity rate of random time:
  - CIR process
  - Ornstein-Uhlenbeck (OU) process
  - Non-Gaussian OU process

- Introducing Leverage Effect
  - Pure jump innovation cannot have non-zero correlation with a pure-diffusion modelling the random time
Theoretical Background: Advantages of time-changed Lévy processes

- Economic intuition and explanation:
  - small movements and jumps
  - stochastic business time with stochastic intensity

- Flexible distribution for innovation:
  - Non-Gaussian, asymmetry and high kurtosis

- Tractability:
  - known explicit characteristic function and tractable Laplace transform of time-change
  - Fast calibration

- Fitness of modelling
  - infinite-activity jump models outperform existing models
  - potential development with the rapidly research on infinitely divisible distribution
Motivations: How to introduce Leverage Effect

• Empirical research suggests that diffusion models cannot be used for modelling financial returns in a quantitative sense while MJD models can only capture large movements.

• Infinite activity jumps are essential and capable of modelling both large and small movements, in the absence of diffusion components:
  – VG model: Madan et al. (1998) EFR paper
  – CGMY model: Carr et al. (2002) JB paper

• Time-change technique produces stochastic volatility; however, it is very daunting task to introduce the leverage effect for time-changed Lévy models:
  – Carr and Wu (2004) JFE paper
  – Carr et al. (2003) MF paper
What is the solution?

- Carr and Wu (2003) propose a leverage-neutral measure with which correlation can be introduced

\[ \Phi(\theta) = \mathbb{E}[e^{i\theta Y_t}] = \mathbb{E}^\theta [e^{-T_t \Psi(\theta)}] = L^\theta_{T_t}(\Psi(\theta)) \] (1)

- A sketch of proof:

\[
\mathbb{E}[e^{i\theta Y_t}] = \mathbb{E}[e^{i\theta Y_t + T_t \Psi(\theta) - T_t \Psi(\theta)}] \\
= \mathbb{E}[M_t(\theta)e^{-T_t \Psi(\theta)}] \\
= \mathbb{E}^\theta [e^{-T_t \Psi(\theta)}]
\]

where \( M_t(\theta) = e^{i\theta Y_t + T_t \Psi(\theta)} \) can be easily proved to be a martingale under measure \( \mathbb{Q} \).
Proposition: Leverage-neutral Measure

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a complete probability space and $(F_t)_{t \geq 0}$ be a Filtration satisfying the usual conditions. For a time-changed Lévy process $Y_t = X_{T_t}$ under the $\mathbb{Q}$ measure, the characteristic function of $Y_t$ is

$$\Phi_{Y_t}(\theta) = \mathbb{E}[\exp(iuY_t)] = \mathbb{E}^M[\exp(-T_t \Psi(u))] = L_T^M(\Psi_X(u))$$

where $\mathbb{E}[\cdot]$ and $\mathbb{E}^M[\cdot]$ denote expectations under measure $\mathbb{Q}$ and $\mathcal{M}$, respectively. The complex-valued measure $\mathcal{M}$ is absolutely continuous with respect to $\mathbb{Q}$ and the Radon-Nikodym derivative is defined by

$$M_t(u) = \frac{dM(u)}{d\mathbb{Q}} = \exp(iuY_t + T_t \Psi_X(u))$$
• Last task: derive $L_T(\theta) = \mathbb{E} \left[ \exp \left( -\theta \int_0^t v_s ds \right) \right]$

• Affine activity rate models

$$L_{T_t}(u) = \exp \left( -b(t)z_0 - c(t) \right)$$ (2)

where $b(t)$ and $c(t)$ are scalar functions.

• Filipovic (2001) shows that the infinite generator of a activity rate process $v(t)$ has the representation of

$$Af(x) = \frac{1}{2} x f''(x) + (a' - \kappa x) f'(x) +$$

$$+ \int_{R_0^+} \left( f(x + y) - f(x) - f(y)(1 \wedge y) \right) (m(dy) + \mu(dy))$$ (3)

• Is there any problem? not practical
• Since closed-form solutions of ODEs are not obtainable, numerical methods are needed.

• Traditional Pricing of Heston model and Lévy models rely on the Carr-Madan formula and Fast Fourier Transform (FFT).
  – Thousands of ODEs must be solved numerically and simultaneously: \( N \geq 4096 \)
  – Adaptive Runge-Kutta methods do not perform well as solving \( c(t) \) requires the whole information of \( b(t) \)
Pricing Methods for European Options

- Fast Fourier Transform (FFT) and Carr-Madan Formula (See Carr and Madan (1999))
  - stable, easy-implemented
  - a large number of sampling points required \((N \geq 4096)\)
  - restrictive as sampling must be equally spaced

- Fractional FFT (FrFT) (See Chourdakis (2005))
  - faster than FFT as less sampling points are needed
  - equally spacing still required

- Direct Integration (See Attari (2004))
  - very fast
  - accuracy is unstable

- COS Expansion introduced in Fang and Oosterlee (2008)
  - very limited sampling points are required to have the desired accuracy
The underlying process used is a special case of the $\alpha$-stable process. The characteristic function of an $\alpha$-stable process $L_t$ is

$$\Phi(u) = E[e^{iuL_1}] = \exp \left( iu\theta - |u|\alpha \sigma^\alpha \left( 1 - i\beta(sgn u) \tan \frac{\pi \alpha}{2} \right) \right)$$

(4)

Carr and Wu (2003) modify the original $\alpha$-stable process and name the “new” process Finite Moment Log Stable (FMLS) process by setting $\beta = -1$, in order to ensure finite moments of returns. It can be further simplified by normalization of $\sigma = 1$ and abandon the drift $\theta$.

It admits only negative jumps and is of infinite activity and infinite variation.
• Fix a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with a filtration \(\{\mathcal{F}_t\}\) satisfying the usual conditions. Suppose the spot price follows:

\[
S_t = S_0 \exp \left( (r - q)t + \sigma L_{T_t}^{\alpha_1,-1} - \xi T_t \right)
\]

\[
T_t = \int_0^t (v_s^1 + v_s^2) ds
\]

\[
dv_t^1 = \kappa^1(1 - v_t^1)dt + \beta^1 dL_{t}^{\alpha_1,1}
\]

\[
dv_t^2 = \kappa^2(1 - v_t^2)dt + \beta^2 dL_{t}^{\alpha_2,1}
\]

(5)

where \(r\) and \(q\) are the risk-free rate and dividend rate, and \(\xi\) is the convexity correction. \(L_{t}^{\alpha_1,-1}\) is a standard FMLS process with parameter \(\alpha_1\). \(L_{t}^{\alpha_1,1}\) is the mirror image of \(L_{t}^{\alpha_1,-1}\).

• The parameter set is \(\{\alpha_1, \alpha_2, \beta_1, \beta_2, \sigma, \kappa^1, \kappa^2\}\). It has both long-run and short-run volatility effect with only 7 parameters, compared to 5 parameters of the Heston model.
The proposed model admits the leverage effect, because there is dependence between $S_t$ and $T_t$.

Solving (5) is extremely hard as the iteration rule cannot be applied, due to the dependence.

Applying the leverage-neutral measure, we can derive ODEs for (5); however, they cannot be solved analytically.

Numerically solving is too time-consuming, especially because of the requirement of Carr-Madan method.

COS expansion
  – a quick method
  – can be accelerated
Generator of activity rate

\[ Af(x) = (\kappa \eta - (\kappa + \delta)x)f'(x) + \]
\[ \beta^\alpha \int_{R_0^+} (f(x + y) - f(x) - f'(x)(1 \wedge y))\mu(dy) \]  

where \( \mu(dy) = cy^{-\alpha-1}dy \) is the Lévy measure of the FMLS process, \( c = -\sec\frac{\pi\alpha}{2}\frac{1}{\Gamma(-\alpha)} \), and \( \delta = \frac{c}{\alpha-1} \).

The characteristic function is

\[ \Phi(u) = L_T^M(\Psi(u)) = \exp(-b(t)v_0 - c(t)) \]  

where

\[ b'(t) = \Psi(u) - \kappa b(t) + \sec\frac{\pi\alpha}{2}\beta[(b(t) + iu)^\alpha - (iu)^\alpha] \]

\[ c'(t) = \kappa \eta b(t) \]  

with initial conditions: \( b(t) = 0 \) and \( c(t) = 0 \). Unfortunately, \( b(t) \) and \( c(t) \) are not explicitly solvable.
Numerical Pricing Framework

- Solving ODEs with order 4, 5 Runge-Kutta method to solve $b(t)$ and $c(t)$ simultaneously
- Vector calculation and cache technique must be used to accelerate the speed
- Use COS expansion pricing method to generate accurate prices based on very limited sampling points
- Apply global search combined with local search to achieve stable calibration results
A descriptive comparison with respect to the standard Carr-Madan method

<table>
<thead>
<tr>
<th>N</th>
<th>Error</th>
<th>Time (msec.)</th>
<th>N</th>
<th>Error</th>
<th>Time (msec.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>5.18E-01</td>
<td>47</td>
<td>128</td>
<td>3.50E+08</td>
<td>2.9</td>
</tr>
<tr>
<td>64</td>
<td>1.73E-02</td>
<td>48</td>
<td>256</td>
<td>-3.71E+06</td>
<td>3.25</td>
</tr>
<tr>
<td>96</td>
<td>2.77E-03</td>
<td>66</td>
<td>512</td>
<td>2.17E+01</td>
<td>4.98</td>
</tr>
<tr>
<td>128</td>
<td>3.75E-04</td>
<td>68</td>
<td>1024</td>
<td>-1.92E+00</td>
<td>6.37</td>
</tr>
<tr>
<td>160</td>
<td>1.99E-05</td>
<td>78</td>
<td>2048</td>
<td>2.12E-03</td>
<td>11.24</td>
</tr>
<tr>
<td>192</td>
<td>3.17E-07</td>
<td>79</td>
<td>4096</td>
<td>-2.31E-07</td>
<td>19.25</td>
</tr>
</tbody>
</table>

Table 1: A Comparison of Error Convergence and computation time for COS Pricing and FFT Pricing.
Empirical Results and Analysis: Daily Calibration

Table 2: Daily Calibration Results of Different Models

<table>
<thead>
<tr>
<th></th>
<th>Heston</th>
<th>VG</th>
<th>CGMY</th>
<th>LS</th>
<th>VGSV</th>
<th>CGMYSV</th>
<th>LSSV</th>
</tr>
</thead>
<tbody>
<tr>
<td>1998</td>
<td>0.2792</td>
<td>0.9222</td>
<td>0.9311</td>
<td>0.4741</td>
<td>0.3812</td>
<td>0.3922</td>
<td>0.2812</td>
</tr>
<tr>
<td>1999</td>
<td>0.3061</td>
<td>0.8568</td>
<td>0.8636</td>
<td>0.4639</td>
<td>0.4149</td>
<td>0.4039</td>
<td>0.3256</td>
</tr>
<tr>
<td>2000</td>
<td>0.3709</td>
<td>0.9005</td>
<td>1.0828</td>
<td>0.4679</td>
<td>0.4171</td>
<td>0.4289</td>
<td>0.2838</td>
</tr>
<tr>
<td>2001</td>
<td>0.1663</td>
<td>0.9393</td>
<td>0.9674</td>
<td>0.5486</td>
<td>0.3208</td>
<td>0.3770</td>
<td>0.2276</td>
</tr>
<tr>
<td>2002</td>
<td>0.3223</td>
<td>1.1011</td>
<td>0.9279</td>
<td>0.5446</td>
<td>0.4627</td>
<td>0.4069</td>
<td>0.2407</td>
</tr>
<tr>
<td>2003</td>
<td>0.2951</td>
<td>1.0608</td>
<td>0.9668</td>
<td>0.5450</td>
<td>0.4058</td>
<td>0.3828</td>
<td>0.3340</td>
</tr>
</tbody>
</table>

- Calibration results are obtained by minimizing the sum of squared pricing error between market prices and model prices. Market option data are S&P 500 index options which are collected from April 4, 1998 to May 31, 2003. The output is given in MSE(E+05).

- The model with the best performance is the LSSV model; it also exhibits excellent stability of parameters.
Empirical Results and Analysis: Daily Calibration

Figure 1: A Sample of Daily Calibration Result
Empirical Results and Analysis: Fitness and stability

- Jumps should play an important role in modelling the volatility/variance
- However, existing literature indicates that jump structure in the volatility process cannot improve the performance significantly
- Long-run and short-run volatility processes can provide better fitness
- The LSSV model outperforms the celebrated Heston model, and it also provides stable calibration results with parsimonious parameter space.
- It will lead a way to develop pure-jump stochastic volatility models incorporating the leverage effect, especially for Lévy processes.
Core contributions:

- The first attempt to investigate the fitness of time-changed Lévy models which admit the leverage effect
- Quantify the vital impact of leverage effect given time-changed Lévy models
- Construct a numerical framework that realize the leverage measure introduced by Carr and Wu (2003)
- It is a robust numerical framework that can be adopted for any kind of time-changed Lévy model
- A very decent model is proposed and evaluated, which admits leverage effect and multi-scale stochastic volatility.
Thank You