Option Pricing and Calibration with Time-changed Lévy processes

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1. How to find a perfect model that captures essential features of financial returns
    - empirical findings: negative skewness, high kurtosis, stochastic volatility and jumps
    - available stochastic processes: Brownian motion and jump processes (Lévy processes)
2. How to keep the tractability
    - Carr-Madan formula with FFT method
3. Empirical analysis
    - estimation
    - daily calibration

- Lévy processes are widely used recently to model Financial returns. A Lévy process can generate a variety of distributions at a fixed time horizon. Brownian motion is a special case of Lévy processes.
- A stochastic process $(X_t)_{t \geq 0}$ is said to be a Lévy process if:
  1. $X_0 = 0$ a.s. ;
  2. $X_t - X_s \perp X_s$, for any $t > s$;
  3. $X_t - X_s$ is equal in distribution to X_{t-s} , for any $t > s$.
- Lévy processes are fully characterized by its characteristic function

$$\mathbb{E}[\exp(iuX_t)] = \exp\left(i\mu ut - \frac{1}{2}\sigma^2 u^2 t + t\int_{\mathbb{R}_0} (e^{iux} - 1 - iux 1_{|x|<1})\pi(dx)\right)$$

which is the Levy-Khintchine representation.

What is time-change?

- Let $t \to T_t$ be an increasing right-continuous process with left limits satisfying the usual conditions.
- The random time T_t can be modelled as a nondecreasing semimartingale

$$T_t = \alpha_t + \int_0^t \int_0^\infty y\mu(dt, dy)$$

- Simply, we can model the random time as

$$T_t = \int_0^t v_s ds$$

where v(t) is the activity rate.

- T_t can be viewed as the business time at time t. It is driven by a stochastic activity process. A more active business day can generate higher volatility.

- Time-changed Lévy process: $Y_t = X_{T_t}$
- Lévy processes are natural to be applied with time-change technique
  - infinitely divisible distribution
- Common choices of the activity rate of random time:
  - CIR process
  - Ornstein-Uhlenbeck (OU) process
  - Non-Gaussian OU process
- Introducing Leverage Effect
  - Pure jump innovation cannot have non-zero correlation with a pure-diffusion modelling the random time

- Economic intuition and explanation:
  - small movements and jumps
  - stochastic business time with stochastic intensity
- Flexible distribution for innovation:
  - Non-Gaussian, asymmetry and high kurtosis
- Tractability:
  - known explicit characteristic function and tractable Laplace transform of time-change
  - Fast calibration
- Fitness of modelling
  - infinite-activity jump models outperform existing models
  - potential development with the rapidly research on infinitely divisible distribution

- Empirical research suggests that diffusion models cannot be used for modelling financial returns in a quantitative sense while MJD models can only capture large movements
- Infinite activity jumps are essential and capable of modelling both large and small movements, in the absent of diffusion components:
  - VG model: Madan et al. (1998) EFR paper
  - CGMY model: Carr et al. (2002) JB paper
  - FMLS model: Carr and Wu (2003) JF paper
- Time-change technique produces stochastic volatility; however, it is very daunting task to introduce the leverage effect for time-changed Lévy models:
  - Carr and Wu (2004) JFE paper
  - Carr et al. (2003) MF paper

- Carr and Wu (2003) propose a leverage-neutral measure with which correlation can be introduced

$$\Phi(\theta) = \mathbb{E}[e^{i\theta Y_t}] = \mathbb{E}^{\theta}[e^{-T_t \Psi(\theta)}] = L^{\theta}_{T_t}(\Psi(\theta)) \tag{1}$$

- A sketch of proof:

$$\begin{aligned} \mathbb{E}[e^{i\theta Y_t}] &= \mathbb{E}[e^{i\theta Y_t + T_t \Psi(\theta) - T_t \Psi(\theta)}] \\ &= \mathbb{E}[M_t(\theta) e^{-T_t \Psi(\theta)}] \\ &= \mathbb{E}^{\theta}[e^{-T_t \Psi(\theta)}] \end{aligned}$$

where $M_t(\theta) = e^{i\theta Y_t + T_t \Psi(\theta)}$ can be easily proved to be a martingale under measure $\mathbb{Q}$.

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a complete probability space and $(F_t)_{t \geq 0}$ be a Filtration satisfying the usual conditions. For a time-changed Lévy process $Y_t = X_{T_t}$ under the $\mathbb{Q}$ measure, the characteristic function of Y_t is

$$\Phi_{Y_t}(\theta) = \mathbb{E}[\exp(iuY_t)] = \mathbb{E}^M[\exp(-T_t \Psi(u))] = L_T^M(\Psi_X(u))$$

where $\mathbb{E}[\cdot]$ and $\mathbb{E}^M[\cdot]$ denote expectations under measure $\mathbb{Q}$ and $\mathbb{M}$, respectively. The complex-valued measure $\mathbb{M}$ is absolutely continuous with respect to $\mathbb{Q}$ and the Radon-Nikodym derivative is defined by

$$M_t(u) = \frac{dM(u)}{dQ} = \exp(iuY_t + T_t \Psi_X(u))$$

- Last task: derive $L_T(\theta) = \mathbb{E}\left[\exp\left(-\theta \int_0^t v_s ds\right)\right]$
- Affine activity rate models

$$L_{T_t}(u) = \exp\left(-b(t)z_0 - c(t)\right) \tag{2}$$

where $b(t)$ and $c(t)$ are scalar functions.

- Filipovic (2001) shows that the infinite generator of a activity rate process $v(t)$ has the representation of

$$\begin{aligned} Af(x) = &\frac{1}{2} x f''(x) + (a' - \kappa x) f'(x) + \\ &+ \int_{R_0^+} \left(f(x+y) - f(x) - f(y)(1 \wedge y)\right)\left(m(dy) + \mu(dy)\right) \end{aligned} \tag{3}$$

- Is there any problem? not practical

- Since closed-form solutions of ODEs are not obtainable, numerical methods are needed.
- Traditional Pricing of Heston model and Lévy models rely on the Carr-Madan formula and Fast Fourier Transform (FFT).
  - Thousands of ODEs must be solved numerically and simultaneously: $N \geq 4096$
  - Adaptive Runge-Kutta methods do not perform well as solving $c(t)$ requires the whole information of $b(t)$

- Fast Fourier Transform (FFT) and Carr-Madan Formula (See Carr and Madan(1999))
  - stable, easy-implemented
  - a large number of sampling points required ($N \geq 4096$)
  - restrictive as sampling must be equally spaced
- Fractional FFT (FrFT) (See Chourdakis (2005))
  - faster than FFT as less sampling points are needed
  - equally spacing still required
- Direct Integration (See Attari (2004))
  - very fast
  - accuracy is unstable
- COS Expansion introduced in Fang and Oosterlee (2008)
  - very limited sampling points are required to have the desired accuracy

- The underlying process used is a special case of the α-stable process. The characteristic function of an α-stable process $L_t$is

$$\Phi(u) = E[e^{iuL_1}] = \exp\left(iu\theta - |u|^\alpha \sigma^\alpha \left(1 - i\beta(sgnu)\tan\frac{\pi\alpha}{2}\right)\right) \tag{4}$$

- Carr and Wu (2003) modify the original α-stable process and name the "new" process Finite Moment Log Stable (FMLS) process by setting $\beta = -1$, in order to ensure finite moments of returns. It can be further simplified by normalization of $\sigma = 1$ and abandon the drift θ.

- It admits only negative jumps and is of infinite activity and infinite variation.

- Fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions. Suppose the spot price follows:

$$
\begin{aligned}
S_t &= S_0 \exp\left((r-q)t + \sigma L_{T_t}^{\alpha_1,-1} - \xi T_t\right) \\
T_t &= \int_0^t (v_s^1 + v_s^2) ds \\
dv_t^1 &= \kappa^1(1 - v_t^1)dt + \beta^1 dL_t^{\alpha_1,1} \\
dv_t^2 &= \kappa^2(1 - v_t^2)dt + \beta^2 dL_t^{\alpha_2,1}
\end{aligned} \tag{5}
$$

  where r and q are the risk-free rate and dividend rate, and ξ is the convexity correction. $L_t^{\alpha_1,-1}$ is a standard FMLS process with parameter α_1. $L_t^{\alpha_1,1}$ is the mirror image of $L_t^{\alpha_1,-1}$.

- The parameter set is $\{\alpha_1, \alpha_2, \beta_1, \beta_2, \sigma, \kappa^1, \kappa^2\}$. It has both long-run and short-run volatility effect with only 7 parameters, compared to 5 parameters of the Heston model.

- The proposed model admits the leverage effect, because there is dependence between S_t and T_t.
- Solving (5) is extremely hard as the iteration rule cannot be applied, due to the dependence.
- Applying the leverage-neutral measure, we can derive ODEs for (5); however, they cannot be solved analytically.
- Numerically solving is too time-consuming, especially because of the requirement of Carr-Madan method.
- COS expansion
  - a quick method
  - can be accelerated

- Generator of activity rate

$$\begin{aligned} Af(x) = (\kappa\eta - (\kappa+\delta)x)f'(x) + \\ \beta^{\alpha}\int_{R_0^+}(f(x+y) - f(x) - f'(x)(1\wedge y))\mu(dy) \end{aligned} \tag{6}$$

where $\mu(dy) = cy^{-\alpha-1}dy$ is the Lévy measure of the FMLS process, $c = -sec\frac{\pi\alpha}{2}\frac{1}{\Gamma(-\alpha)}$, and $\delta = \frac{c}{\alpha-1}$.

- The charactersitc function is

$$\Phi(u) = L_T^M(\Psi(u)) = \exp(-b(t)v_0 - c(t)) \tag{7}$$

where

$$b'(t) = \Psi(u) - \kappa b(t) + \sec\frac{\pi\alpha}{2}\beta[(b(t)+iu)^{\alpha} - (iu)^{\alpha}] \tag{8}$$

$$c'(t) = \kappa\eta b(t) \tag{9}$$

with initial conditions: $b(t) = 0$ and $c(t) = 0$. Unfortunately, $b(t)$ and $c(t)$ are not explicitly solvable.

- Solving ODEs with order $4, 5$ Runge-Kutta method to solve $b(t)$ and $c(t)$ simultaneously
- Vector calculation and cache technique must be used to accelerate the speed
- Use COS expansion pricing method to generate accurate prices based on very limited sampling points
- Apply global search combined with local search to achieve stable calibration results

A descriptive comparison with respect to the standard Carr-Madan method

COS			FFT		
N	Error	Time(msec.)	N	Error	Time(msec.)
32	5.18E-01	47	128	3.50E+08	2.9
64	1.73E-02	48	256	-3.71E+06	3.25
96	2.77E-03	66	512	2.17E+01	4.98
128	3.75E-04	68	1024	-1.92E+00	6.37
160	1.99E-05	78	2048	2.12E-03	11.24
192	3.17E-07	79	4096	-2.31E-07	19.25

Table 1: A Comparison of Error Convergence and computation time for COS Pricing and FFT Pricing.

	Heston	VG	CGMY	LS	VGSV	CGMYSV	LSSV
1998	0.2792	0.9222	0.9311	0.4741	0.3812	0.3922	0.2812
1999	0.3061	0.8568	0.8636	0.4639	0.4149	0.4039	0.3256
2000	0.3709	0.9005	1.0828	0.4679	0.4171	0.4289	0.2838
2001	0.1663	0.9393	0.9674	0.5486	0.3208	0.3770	0.2276
2002	0.3223	1.1011	0.9279	0.5446	0.4627	0.4069	0.2407
2003	0.2951	1.0608	0.9668	0.5450	0.4058	0.3828	0.3340

Table 2: Daily Calibration Results of Different Models

- Calibration results are obtained by minimizing the sum of squared pricing error between market prices and model prices. Market option data are S&P 500 index options which are collected from April 4, 1998 to May 31, 2003. The output is given in MSE(E+05).
- The model with the best performance is the LSSV model; it also exhibits excellent stability of parameters.

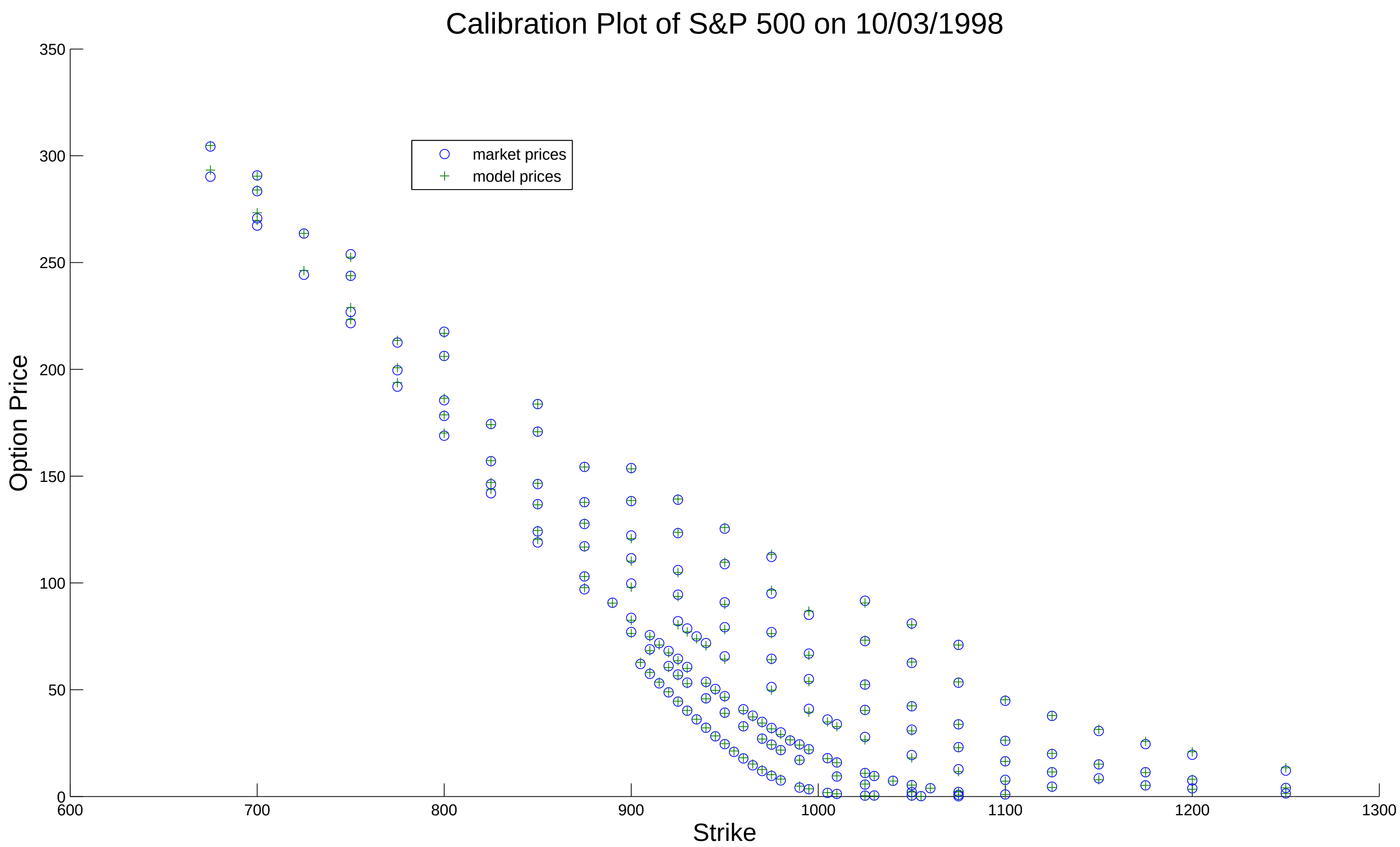

Figure 1: A Sample of Daily Calibration Result

- Jumps should play an important role in modelling the volatility/variance
- However, existing literature indicates that jump structure in the volatility process cannot improve the performance significantly
- Long-run and short-run volatility processes can provide better fitness
- The LSSV model outperforms the celebrated Heston model, and it also provides stable calibration results with parsimonious parameter space.
- It will lead a way to develop pure-jump stochastic volatility models incorporating the leverage effect, especially for Lévy processes.

Core contributions:

- The first attempt to investigate the fitness of time-changed Lévy models which admit the leverage effect
- Quantify the vital impact of leverage effect given time-changed Lévy models
- Construct a numerical framework that realize the leverage measure introduced by Carr and Wu (2003)
- It is a robust numerical framework that can be adopted for any kind of time-changed Lévy model
- A very decent model is proposed and evaluated, which admits leverage effect and multi-scale stochastic volatility.

Thank You