Dynamic Assessment Indices

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Motivation

Assessment Index $\alpha(x)$:

- diversification is good: quasiconcavity
- better for sure is less risky: monotonicity

Duality $\rightsquigarrow$ $\Updownarrow$ Drapeau, Kupper; Voglio et al; Penot, Volle

$$\alpha(x) = \inf_{x^*} R\left(x^*, \langle x^*, x \rangle\right)$$

Dual Representation $\rightsquigarrow$ generic interpretation of risk perception

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Context Dependent Interpretation: D., Kupper 2010

Setting's specification $\rightsquigarrow$ differentiated interpretation of risk perception

- Random variables X $\rightsquigarrow$ risk under model uncertainty

$$\alpha(X) = \underbrace{\inf_{Q}}_{\text{probability measures}} R\left(Q, E_Q[X]\right)$$

- Processes X $\rightsquigarrow$ risk under both discounting and model uncertainty

$$\alpha(X) = \underbrace{\inf_{Q \otimes D}}_{\text{probability measures on processes}} R\left(Q \otimes D, E_Q\left[\sum D_s \Delta X_s\right]\right)$$

Motivation

Some questions

- To what extent does the duality result hold when conditioning?
(Detlefsen, Scandolo; Fritelli, Maggis; Biagini, Bion-Nadal etc.)
- How to condition properly for stochastic processes without cash additivity?
(Cheridito, Delbaen, Kupper; Föllmer, Penner, etc.)
- Is the past component relevant in our assessment of risk?
- Dynamic (in)consistency(ies).

Outline

L^0-Theory, (Cheridito, Fillipovic, Kupper, Vogelpoth)

- $(\Omega, \mathcal{F}, P)$ probability space.
- $L^0, \bar{L}^0$: random variables mapping to $(-\infty, +\infty)$ or $[-\infty, +\infty]$.
- On L^0 we consider the topology induced by all balls

$$B_\varepsilon(\lambda) = \{\beta \in L^0 : |\lambda - \beta| \leq \varepsilon, P\text{-a.s.}\}, \ \varepsilon \in L^0_{++}.$$

  σ-stability of this basis:

$$B_\varepsilon(\lambda) = \sum 1_{A_n} B_{\varepsilon_n}(\lambda_n)$$

  where $\lambda = \sum 1_{A_n}\lambda_n$, $\varepsilon = \sum 1_{A_n}\varepsilon_n$.
- Agent will assess positions in an L^0-module $\mathcal{X}$.

Conditional Assessment Index

Definition

A Conditional Assessment Index is a function $\alpha : \mathcal{X} \to \bar{L}^0$ satisfying

1. Quasiconcavity: For $\lambda \in L^0$ with $0 \leq \lambda \leq 1$ and $X, Y \in \mathcal{X}$ holds
$$\alpha(\lambda(X) + (1-\lambda)Y) \geq \alpha(X) \wedge \alpha(Y).$$
2. Monotonicity: $\alpha(X) \geq \alpha(Y)$, whenever $X - Y \in \mathcal{K}$ (e.g. $\mathcal{K} = \bar{L}^0_+$).
3. Locality: For $A \in \mathcal{F}$ and $X, Y \in \mathcal{X}$ holds
$$\alpha(1_A X + 1_{A^c} Y) = 1_A \alpha(X) + 1_{A^c} \alpha(Y).$$

Further Properties

- An acceptability index is a scale invariant CAI, that is

$$\alpha(X) = \alpha(\lambda X),\ \lambda \in L^0_{++},\ X \in \mathcal{X}.$$

(Cherny, Madan; Biagini, Bion Nadal)

- A monetary utility function is a cash-additive CAI, that is

$$\alpha(X + m1) = \alpha(X) + m,\ m \in L^0,\ X \in \mathcal{X}.$$

For this $-\alpha$ is called monetary risk measure.

- Additional properties lead to convex risk measures, coherent risk measures etc.

Risk Acceptance Family

Definition

A conditional risk acceptance family is a collection of sets $\mathcal{A} = (\mathcal{A}^m)_{m \in \bar{L}^0}$ with $\mathcal{A}^m \subseteq \mathcal{X}$ and such that

1. convex: any $\mathcal{A}^m$ is L^0-convex.
2. decreasing: $\mathcal{A}^m \subseteq \mathcal{A}^n$, whenever $m \geq n$.
3. monotone: $\mathcal{A}^m + \mathcal{K} = \mathcal{A}^m$.
4. jointly σ-stable: for every partition $(A_i) \subseteq \mathcal{F}$, $X_i \in \mathcal{A}^{m_i}$ it holds
$$\sum 1_{A_i} X_i \in \mathcal{A}^{\sum 1_{A_i} m_i}.$$
5. left-continuous:
$$\mathcal{A}^m = 1_{A^c} \bigcap_{\varepsilon > 0} \mathcal{A}^{m-\varepsilon} + 1_A \mathcal{X}$$
where $A = \{m = -\infty\}$.

Assessment Index $\Leftrightarrow$ Risk Acceptance Family

Theorem: (BCDK)

For a given conditional assessment index α, the family $\mathcal{A} = (\mathcal{A}^m_\alpha)_{m \in \bar{L}^0}$ defined by

$$\mathcal{A}^m_\alpha := \{X \in \mathcal{X} : \alpha(X) \geq m\}\,,\ m \in \bar{L}^0,$$

is a conditional risk acceptance family.
For a given conditional risk acceptance family $\mathcal{A} = (\mathcal{A}^m)_{m \in \bar{L}^0}$ the function $\alpha_\mathcal{A}$ defined by

$$\alpha_\mathcal{A}(X) := \operatorname{ess\,sup}\left\{m \in \bar{L}^0 : X \in \mathcal{A}^m\right\},\ X \in \mathcal{X},$$

is a conditional assessment index. Further, $\alpha = \alpha_{\mathcal{A}_\alpha}$ and $\mathcal{A} = \mathcal{A}_{\alpha_\mathcal{A}}$.

See also D., Kupper; Frittelli, Maggis '10.

However, we have to pay attention to $-\infty$!

Robust Representation of Conditional Assessment Indices

Theorem: BCDK

Let $\alpha : \mathcal{X} \to \bar{L}^0$ be an upper semicontinuous conditional assessment index. Then

$$\alpha(X) = \operatorname*{ess\,inf}_{X^*} R(X^*, \langle X^*, X \rangle)$$

for a unique minimal risk function R.

This minimal risk function is uniquely determined within the set of functions $R : \mathcal{K}^\circ \times \bar{L}^0 \to \bar{L}^0$ which

1. are jointly local;
2. are increasing and right-continuous in the second argument;
3. are jointly quasiconvex;
4. have a left-continuous version which is lower semicontinuous;
5. have an uniform asymptotic minimum.

Main challenges of the proof: L^0 tricky parts, and non-trivial duality between conditionally increasing functions and their general inverse.

Certainty Equivalent

A κ-Conditional Certainty Equivalent for an assessment index α is a functional $C : \mathcal{X} \to L^0$ such that

$$\alpha(C(X)\kappa) = \alpha(X), \; X \in \mathcal{X}.$$

Proposition BCDK

If α is "regular", then $C(X) := \operatorname{ess\,inf}\{m \in L^0 : \alpha(m\kappa) \geq \alpha(X)\}$ is a certainty equivalent for α. Moreover

$$\alpha(X) \geq \alpha(Y) \quad \Longleftrightarrow \quad C(X) \geq C(Y)$$

and C itself is a regular conditional assessment index.

Regular means sensitive, essentially bounded and increasing along κ.

The concept of certainty equivalent is used to study dynamic consistency. (Almost all risk measures are regular)

See also Cheridito, Kupper '08.

Assessment Indices for Processes

Framework

- Processes of cumulative cash flows $X_0, X_1, \dots, X_T$ on $(\Omega, \mathcal{F}, (\mathcal{F}_s), P)$.
- Process $\cong$ random variable $X : \tilde{\Omega} = \Omega \times \{0, \dots, T\} \to R$.
- $\mathcal{O}_t$ the optional σ-algebra up to time t, $\mathcal{O} := \mathcal{O}_T$.
  - $X = (X_0, X_1, \dots, X_T) \in L^0(\mathcal{O}) \Leftrightarrow X_0, \dots, X_T$ is an adapted process.
  - $\lambda = (\lambda_0, \dots, \lambda_T) \in L^0(\mathcal{O}_t) \Leftrightarrow \lambda_s$ remains constant after t.

  So is $L^0(\mathcal{O})$ an $L^0(\mathcal{O}_t)$ module (component-wise).
- The space

$$\mathcal{X} = L^p_t := \left\{ X \in L^0(\mathcal{O}) : E\left[|X|^p \,|\, \mathcal{O}_t\right]^{1/p} \in L^0(\mathcal{O}_t) \right\}$$

is a conditional $L^0(\mathcal{O}_t)$-topological module.

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- The space

$$\mathcal{X} = L_t^p := \left\{ X \in L^0(\mathcal{O}) : E\left[|X|^p \,\middle|\, \mathscr{O}_t \right]^{1/p} \in L^0(\mathcal{O}_t) \right\}$$

  is a conditional $L^0(\mathcal{O}_t)$-topological module.

Robust Representation

Theorem: BCDK

Let $\alpha : L^p_t(\mathscr{O}) \to \bar{L}^0(\mathscr{O}_t)$ be an upper semicontinuous conditional assessment index. Then α_t has a robust representation of the form

$$\alpha_t(X) := \operatorname*{ess\,inf}_{Q\otimes D} R\left(Q \otimes D, X_t + E_Q\left[\sum_{s=t+1}^{T} D_s \Delta X_s \middle| \mathcal{F}_t\right]\right)$$

for some unique minimal risk function R.

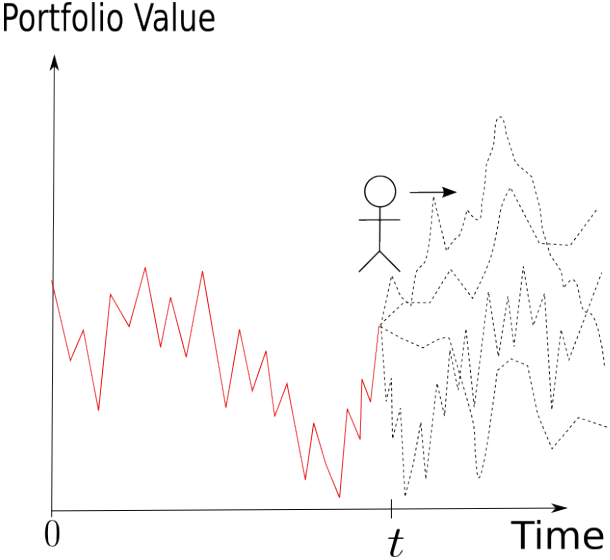
Here

- Q is a probability such that $Q << P$ with $Q = P$ on $\mathcal{F}_t$, $dQ/dP \in L^q$;
- $D := D(Q)$ is a discounting process, that is D is predictable, decreasing and starts in 1 at time t.

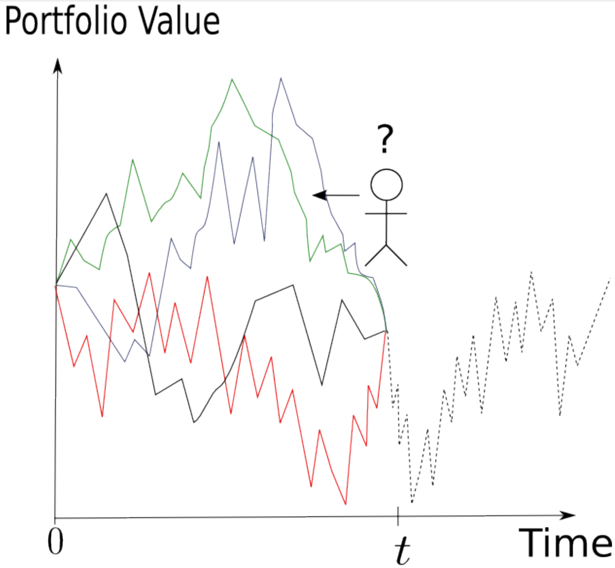
$\rightsquigarrow$ Model risk intertwined with discounting risk.

Cheridito, Kupper '09, Acciaio et al '11.

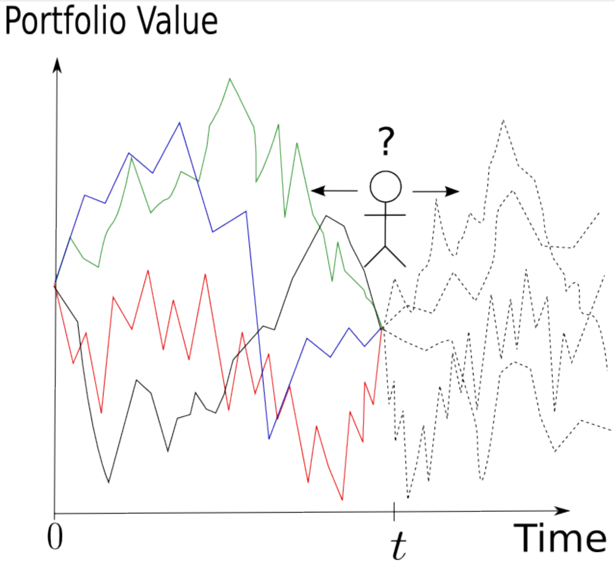
Prediction of the Future



Observation of the Past



Overall Inspection



Path Dependent Assessment Indices

A path dependent index $\alpha_t : L^p_t \to \bar{L}^0(\mathcal{F}_t)$ is such that

$$(X_t, \ldots, X_T) \mapsto \alpha_t(X_{[0,t-1]}, X_t, \ldots, X_T)$$

is an assessment index for every $X_{[0,t-1]} = (X_0, \ldots, X_{t-1})$.

Theorem: (BCDK)

If α_t is an upper semicontinuous path dependent assessment index, then

$$\alpha(X) = \operatorname*{ess\,inf}_{Q\otimes D} R\left(X_{[0,t-1]}, Q \otimes D, E_Q\left[X_t + \sum_{s=t+1}^{T} D_s \Delta X_s \middle| \mathcal{F}_t\right]\right)$$

for a unique minimal risk function R.

Example

$$\alpha(X) = \sum_{s=0}^{t-1} e^{-r_s(s-t)} \Delta X_s + \operatorname*{ess\,inf}_{Q \otimes D} \tilde{R} \left(Q \otimes D, X_t + E_Q \left[\sum_{s=t+1}^{T} D_s \Delta X_s \middle| \mathcal{F}_t \right] \right).$$

From a regulatory point of view, reasonable return 8%. Then, choose $r_s = \Delta X_s / X_s - 8\%$.

- recent returns well above 8% $\rightsquigarrow$ higher discounting of the actual level $\rightsquigarrow$ leverage dampening, and inversely.

Strong Time Consistency

Definition

A sequence $\alpha = (\alpha_t)$ of path dependent assessment indices is strongly time consistent, if

$$X_{[0,t]} = Y_{[0,t]} \text{ and } \alpha_{t+1}(X) \geq \alpha_{t+1}(Y) \quad \text{implies} \quad \alpha_t(X) \geq \alpha_t(Y).$$

Theorem: BCDK

If C is a certainty equivalent of a strongly time consistent α, then

$$C_t(X) = C_t\left(X_{[0,t]} + C_{t+1}(X)1_{[t+1,T]}\right).$$

If α is strongly time consistent and admits a certainty equivalent, then

$$\alpha_t(X) = H_t(X_{[0,t]}, \alpha_{t+1}(X)),$$

for some aggregator H_t.

$\rightsquigarrow$ decoupled FBSDE.

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Strong Time Consistency

Analogously to BSDE theory, the dual side fulfills a Bellmann principle

Proposition

Let C be a strongly time consistent upper semicontinuous path dependent certainty equivalent of an assessment index. Then it holds

$$C_t(X) = \operatorname*{ess\,inf}_{Q \otimes D} F_t\left(Q \otimes D, X\right) \tag{4.1}$$

where $F_T(Q \otimes D, X) = X_T$ and

$$\begin{aligned} &F_s(Q \otimes D, X) \\ &= \operatorname*{ess\,inf}_{\tilde{Q} \otimes \tilde{D}} R_{s,s+1}\left(X_{[0,t-1]}, Q \otimes D, E_Q\left[D\left(F_{s+1}\left(\tilde{Q} \otimes \tilde{D}, X\right) - X_s\right) + X_s \middle| \mathcal{F}_s\right]\right). \end{aligned}$$

Time Inconsistency

- Time consistency is good as long as you have certainty equivalent (monetary risk measures for instance).
- This is not the case if you are for instance scale invariant: Sharpe ratio, GLR or examples of Cherny, Madan.
- Ideas, but still work to be done there...

Thank You!