Long-range dependence in finance

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Models with long-memory

Modern stochastic finance tends to shift away from the standard schemes based on independent random variables and the increments of the Wiener process. Finance models are well-developed for diffusion processes, Lévy processes, semimartingales but is still developing for the processes with long-range dependence. The latter is an important component of the theory of stochastic processes, featuring a wide spectrum of applications in economics, physics, finance and other fields.

Long-range dependence can be modeled via fractional Brownian motion. So, we consider the models involving fractional Brownian motion (fBm) with Hurst parameter $H > \frac{1}{2}$ which is a well-known long-memory process. We study also a mixed model based on both standard and fractional Brownian motion which turns out to be more flexible. One of the reasons to consider such model is that it has become very popular in stochastic finance to assume that the underlying random noise consists of two parts: the fundamental part, describing the economical background for the stock price, and the trading part, related to the randomness inherent to the stock market. In our case the fundamental part of the noise has a long memory while the trading part is a white noise.

We shall consider two problems that involve fractional Brownian motion. The 1st one concerns the representation of the investors capital as the result of the investment into the asset with long range dependence and connected non-arbitrage questions. The 2nd one is devoted to the problem of efficient hedging on the market with long-range dependence.

Capital problem formulation

The results of this part are common with G. Shevchenko and E. Valkeila ([M., Shevchenko, Valkeila (2013)]). At the beginning, we state purely mathematical problem.

Question

Which random variables ξ can be represented as

$$\xi = \int_0^1 \phi_s \, dB_s^H,$$

where B^H is fractional Brownian motion, ϕ adapted?

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For Wiener process W, the answer is very well- known:

- $\xi = \int_0^1 \phi_s \, dW_s$ with adapted $\phi \in L^2([0,1] \times \Omega)$ iff ξ is W-measurable, centered, and square integrable;

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Idea: since $\int_0^1 (1-s)^{-2} ds = \infty$, it holds $\underline{\lim}_{t\to 1-} \int_0^t (1-s)^{-1} dW_s = -\infty$ and $\overline{\lim}_{t\to 1-} \int_0^t (1-s)^{-1} dW_s = +\infty$. So, for example, to represent $\xi = 1$ as stochastic integral, let $v_t = \int_0^t (1-s)^{-1} dW_s$, $\tau = \inf\{t : v_t = 1\}$ and put $\phi_t = (1-t)^{-1} 1_{t\le\tau}$.

Preliminaries

Definition 1

The fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$ is a centered Gaussian process $B^H = \{B_t^H, t \geq 0\}$ with stationary increments and the covariance function

$$\mathrm{E}\left[B_t^H B_s^H \right] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

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For $H > 1/2$ (the case we consider here) fBm has long memory.
B^H is almost surely Hölder continuous with any exponent $\gamma < H$.

Integration

For $\alpha \in (0,1)$ fractional derivatives

$$\left(D_{a+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x)-f(u)}{(x-u)^{1+\alpha}} du\right) 1_{(a,b)}(x),$$

$$\left(D_{b-}^{1-\alpha} g\right)(x) = \frac{e^{i\pi\alpha}}{\Gamma(\alpha)}\left(\frac{g(x)}{(b-x)^{1-\alpha}} + (1-\alpha) \int_x^b \frac{g(x)-g(u)}{(u-x)^{2-\alpha}} du\right) 1_{(a,b)}(x).$$

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Assume that $D_{a+}^{\alpha} f \in L_p[a,b]$, $D_{b-}^{1-\alpha} g_{b-} \in L_q[a,b]$ for some $p \in (1, 1/\alpha)$, $q = p/(p-1)$, where $g_{b-}(x) = g(x) - g(b)$.

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We can define

$$\int_a^b f(x) dg(x) = e^{-i\pi\alpha} \int_a^b \left(D^{\alpha}_{a+} f\right)(x) \left(D^{1-\alpha}_{b-} g_{b-}\right)(x) dx.$$

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$$\int_a^b f(x)dg(x) = e^{-i\pi\alpha} \int_a^b \left(D^{\alpha}_{a+}f\right)(x)\left(D^{1-\alpha}_{b-}g_{b-}\right)(x)dx.$$

If $f \in C^{\mu}[a,b]$, $g \in C^{\nu}[a,b]$ with $\mu + \nu > 1$, then $\int_a^b f(x)dg(x)$ is a limit of integral sums.

For any $\alpha \in (1-H, 1)$, $D_{b-}^{1-\alpha} B_{b-}^H \in L_\infty[a,b]$, so we can define for f with $D_{a+}^{\alpha} f \in L_1[a,b]$

$$\int_a^b f_s dB_s^H = e^{-i\pi\alpha} \int_a^b \left(D_{a+}^{\alpha} f\right)(x) \left(D_{b-}^{1-\alpha} B_{b-}^H\right)(x) dx.$$

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Consider the following norm for $\alpha \in (1-H, 1/2)$:

$$\|f\|_{1,\alpha,[a,b]} = \int_a^b \left(\frac{|f(s)|}{(s-a)^\alpha} + \int_a^s \frac{|f(s)-f(z)|}{(s-z)^{1+\alpha}} dz \right) ds.$$

For simplicity we will abbreviate $\|\cdot\|_{\alpha,t} = \|\cdot\|_{1,\alpha,[0,t]}$.

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Theorem 2 (Azmoodeh, M., Valkeila (2011))

Let $f : \mathbb{R} \to \mathbb{R}$ be a function of locally bounded variation, $F(x) = \int_0^x f(y)dy$. Then for any $\alpha \in (1-H, 1/2)$ $\|f(B_\cdot^H)\|_{\alpha,1} < \infty$ a.s. and

$$F(B_t^H) = \int_0^t f(B_s^H) dB_s^H.$$

For $F(x) = |x|$:

$$\left|B_t^H\right| = \int_0^t \text{sign}\, B_s^H dB_s^H.$$

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Fact: for $0 < s \le t \le 1$ $\mathsf{P}(B_s^H B_t^H < 0) \le C(t-s)^H t^{-H}$.

$$\begin{aligned}
\mathsf{E}\|\text{sign}\, B^H\|_{\alpha,t} &= \mathsf{E}\int_0^t \left(\frac{\left|\text{sign}\, B_s^H\right|}{s^\alpha} + \int_0^s \frac{\left|\text{sign}\, B_s^H - \text{sign}\, B_z^H\right|}{(s-z)^{1+\alpha}} dz\, ds \right) \\
&\le C + \int_0^t \int_0^s \frac{\mathsf{E}\left|\text{sign}\, B_s^H - \text{sign}\, B_z^H\right|}{(s-z)^{1+\alpha}} dz\, ds \\
&= C + 2\int_0^t \int_0^s \frac{P(B_s^H B_z^H < 0)}{(s-z)^{1+\alpha}} dz\, ds \\
&\le C + C\int_0^t \int_0^s (s-z)^{H-1-\alpha} s^{-H} dz\, ds < \infty.
\end{aligned}$$

Results

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a complete probability space endowed with a P-complete *left-continuous* filtration $\mathbb{F} = \{\mathcal{F}_t, t \in [0,1]\}$, and B^H be $\mathbb{F}$-adapted fractional Brownian motion.

Results

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a complete probability space endowed with a P-complete *left-continuous* filtration $\mathbb{F} = \{\mathcal{F}_t, t \in [0,1]\}$, and B^H be $\mathbb{F}$-adapted fractional Brownian motion.

Lemma 3

There exists an $\mathbb{F}$-adapted process $\varphi = \{\varphi_t, t \in [0,1]\}$ such that

- *For any $t < 1$ and $\alpha \in (1-H, 1/2)$ $\|\varphi\|_{\alpha,t} < \infty$ a.s., so integral $v_t = \int_0^t \varphi_s dB_s^H$ exists as a generalized Lebesgue–Stieltjes integral.*
- $\lim_{t \to 1-} v_t = \infty$ *a.s.*

Key ingredient of the proof is small ball estimate for fBm:

$$P\left(\sup_{t\in[0,T]} |B_t^H| < \epsilon\right) \le e^{-cT\epsilon^{-1/H}} \text{ for } \epsilon \le T^H.$$

Theorem 4

For any distribution function G there exists an adapted process ζ such that $\|\zeta\|_{\alpha,1} < \infty$ and the distribution function of $\int_0^1 \zeta_s dB_s^H$ is G.

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Proof. Take a monotone function $g : \mathbb{R} \to \mathbb{R}$ such that $g(B^H_{1/2})$ has distribution G. Let φ be the process constructed in lemma, $v_t = \int_{1/2}^t \varphi_s dB_s^H$. Define $\tau = \min\left\{t \geq 1/2 : v_t = |g(B^H_{1/2})|\right\}$. Since $v_t \to \infty$ as $t \to 1-$ a.s., we have $\tau < 1$ a.s. Now put

$$\zeta_t = \varphi_t \operatorname{sign} g(B^H_{1/2}) \mathbf{1}_{[1/2,\tau]}(t).$$

Theorem 5

For any $\mathcal{F}_1$-measurable variable ξ there exists an $\mathbb{F}$-adapted process ψ such that

- *For any $t < 1$ and $\alpha \in (1 - H, 1/2)$ $\|\psi\|_{\alpha,t} < \infty$ a.s.*
- $\lim_{t\to 1} \int_0^t \psi_s dB_s^H = \xi$ *a.s.*

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Proof. $z_t = \tan \mathbb{E}[\arctan \xi | \mathcal{F}_t]$ is $\mathbb{F}$-adapted and $z_t \to \xi$, $t \to 1-$.
Let $\{t_n, n \geq 1\}$ be arbitrary increasing sequence of points from $[0,1]$ converging to 1.
By Lemma, there exists an $\mathbb{F}$-adapted process φ^n on $[t_n, t_{n+1}]$ such that $v_t^n = \int_{t_n}^t \varphi_s^n dB_s^H \to +\infty$, $t \to t_{n+1}-$.
Now denote $\xi_n = z_{t_n}$ and $\delta_n = \xi_n - \xi_{n-1}$, $n \geq 2$, $\delta_1 = \xi_1$. Take $\tau_n = \min\{t \geq t_n : v_t^n = |\delta_n|\}$ and define

$$\psi_t = \sum_{n\geq 1} \varphi_t^n \mathbb{1}_{[t_n,\tau_n]}(t) \operatorname{sign} \delta_n.$$

Main theorem

Theorem 6

Let for a random variable ξ there exist an $\mathbb{F}$-adapted almost surely a-Hölder continuous process $\{z_t, t \in [0,1]\}$ such that $z_1 = \xi$. Then for any $\alpha \in (1-H, (1-H+a) \wedge 1/2)$ there exists an $\mathbb{F}$-adapted process ψ such that $\|\psi\|_{\alpha,1} < \infty$ and $\int_0^1 \psi_s dB_s^H = \xi$.

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(Blue) assumption is equivalent to: there exist $a > 0$, sequence $\{t_n, n \geq 1\}$, $t_n \uparrow 1$, sequence of rv's $\{\xi_n, n \geq 1\}$ such that ξ_n is $\mathcal{F}_{t_n}$-measurable and $|\xi_n - \xi| = O(|t_n - 1|^a)$, $n \to \infty$.

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Theorem 7

Let ξ be an $\mathcal{F}_1$-measurable random variable and let there exist an $\mathbb{F}$-adapted continuous process ψ such that for some $\alpha > 1-H$ $\|\psi\|_{\alpha,1} < \infty$ a.s. and $\int_0^1 \psi_s dB_s^H = \xi$. Then the assumption of main theorem is satisfied.

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Let for a random variable ξ there exist an $\mathbb{F}$-adapted almost surely a-Hölder continuous process $\{z_t, t \in [0,1]\}$ such that $z_1 = \xi$. Then for any $\alpha \in (1-H, (1-H+a) \wedge 1/2)$ there exists an $\mathbb{F}$-adapted process ψ such that $\|\psi\|_{\alpha,1} < \infty$ and $\int_0^1 \psi_s dB_s^H = \xi$.

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Example 8

$\xi = F(B^H_{s_1}, \dots, B^H_{s_n})$, where $F : \mathbb{R}^n \to \mathbb{R}$ is locally Hölder continuous with respect to each variable. (Set $z_t = F(B^H_{s_1 \wedge t}, \dots, B^H_{s_n \wedge t})$.)

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$\xi = G(\{B^H_s, s \in [0,1]\})$, where $G \colon C[0,1] \to \mathbb{R}$ is locally Hölder continuous with respect to the supremum norm on $C[0,1]$. In the case one can set $z_t = G(\{B^H_{s \wedge t}, s \in [0,1]\})$.

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Example 11

Assume that $\mathbb{F} = \{\mathcal{F}_t = \sigma(B^H_s, s \in [0,t]), t \in [0,1]\}$. It is well known that there exists a Wiener process W such that its natural filtration coincides with $\mathbb{F}$. Define $\xi = \int_{1/2}^1 g(t)dW_t$, where $g(t) = (1-t)^{-1/2} \left|\log(1-t)\right|^{-1}$. Then ξ does *not* satisfy the main theorem assumption.

Application to finance

Consider a fractional (B, S)-market:

$$B_t = \exp\left\{\int_0^t r_s ds\right\}$$
$$S_t = S_0 \exp\left\{\mu t + \sigma B_t^H\right\}.$$

Interest rate r can be random. Let $\mathbb{F}$ be the filtration generated by B and S.

Definition 12

Portfolio is $\mathbb{F}$-adapted process $\Pi = (\Pi_t)_{t\in[0,1]} = (\pi_t^0, \pi_t^1)_{t\in[0,1]}$.
Value of portfolio Π at time t is

$$V_t^\Pi = \pi_t^0 B_t + \pi_t^1 S_t.$$

Portfolio is *self-financing* (SF) if

$$dV_t^\Pi = \pi_t^0 dB_t + \pi_t^1 dS_t.$$

Discounted value of a self-financing portfolio

$$C_t^{\Pi} = V_t^{\Pi} B_t^{-1}.$$

It is easy to check that

$$dC_t^{\Pi} = \pi_t^1 dX_t,$$

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Definition 13

A SF portfolio Π is *arbitrage* if $V_0^{\Pi} = 0$, $V_1^{\Pi} \geq 0$ a.s., and $\mathrm{P}\left(V_1^{\Pi} > 0\right) > 0$. It is *strong arbitrage* if there is $c > 0$ s.t. $V_1^{\Pi} \geq c$ a.s.

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Definition 14

Contingent claim is $\mathcal{F}_1$-measurable r.v. $\xi \geq 0$.

Discounted value of a self-financing portfolio

$$C_t^{\Pi} = V_t^{\Pi} B_t^{-1}.$$

It is easy to check that

$$dC_t^{\Pi} = \pi_t^1 dX_t,$$

where $X_t = S_t B_t^{-1}$ is the discounted risky asset price process.

Definition 13

A SF portfolio Π is *arbitrage* if $V_0^{\Pi} = 0$, $V_1^{\Pi} \geq 0$ a.s., and $\mathrm{P}\left(V_1^{\Pi} > 0\right) > 0$.
It is *strong arbitrage* if there is $c > 0$ s.t. $V_1^{\Pi} \geq c$ a.s.

Definition 14

Contingent claim is $\mathcal{F}_1$-measurable r.v. $\xi \geq 0$.
ξ is *hedgeable*, if there is SF portfolio Π (a *hedge* or *replicating portfolio* for ξ) s.t. $V_1^{\Pi} = \xi$ a.s.

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ξ is *hedgeable*, if there is SF portfolio Π (a *hedge* or *replicating portfolio* for ξ) s.t. $V_1^\Pi = \xi$ a.s.
ξ is *weakly hedgeable* if there is SF portfolio Π (a *weak hedge*), s.t. $\lim_{t\to 1-} V_t^\Pi = \xi$ a.s.
Initial portfolio value V_0^Π is *hedging cost* (*weak hedging cost*).

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Any contingent claim ξ in the fractional (B, S)-market is weakly hedgeable. Moreover, its weak hedging cost can be any real number.

Theorem 18

Assume that for a contingent claim ξ there exists an $\mathbb{F}$-adapted almost surely Hölder continuous process $\{z_t, t \in [0, 1]\}$ with $z_1 = \xi$. Then ξ is hedgeable and its hedging cost can be any real number.

Mixed financial markets with $H > 3/4$

We consider financial market with risky asset governed by both the Wiener process and fractional Brownian motion with Hurst parameter $H > 3/4$. Using [Hitsuda] and [Cheridito] representations for the mixed Brownian–fractional Brownian process, we present the solution of the problem of efficient hedging. The results of this part are common with A. Melnikov [Melnikov, M. (2011)]

Let we have a financial market with two assets: non-risky asset

$$B_t = B_0 e^{rt}, \quad t \geq 0, \quad B_0 > 0, \tag{1}$$

$r > 0$ is a constant risk-free rate, and risky asset that is governed by the linear combination of W and B^H

$$S_t = S_0 \exp\{\mu t + \sigma_1 W_t + \sigma_2 B_t^H\}, \quad t \geq 0, \tag{2}$$

where $S_0 > 0$, $\mu \in \mathbb{R}$ is a drift coefficient, $\sigma_1 > 0$ is a volatility coefficient for standard Brownian motion W, $\sigma_2 > 0$ is a volatility coefficient for fBm B^H. Such model will be called the mixed Brownian-fractional-Brownian one.

Fix some finite horizon $T > 0$ and consider our market on the interval $[0, T]$. Denote the filtration $\mathbb{F}^S = \{\mathbb{F}^S_t, 0 \le t \le T\}$, where $\mathbb{F}^S_t = \sigma\{S_u, 0 \le u \le t\}$. Further, by $\bar{\mathbb{F}}^S = \{\bar{\mathbb{F}}^S_t, 0 \le t \le T\}$ we denote the smallest filtration that contains $\mathbb{F}^S$ and fulfils the usual assumptions. The following properties of the model (1) and (2) were established by [Hitsuda] and [Cheridito]:

1. The mixed process $M_t^{H,\sigma} = W_t + \sigma B_t^H$, $t \in [0, T]$ is equivalent (in measure) to Brownian motion if and only if $H \in (3/4, 1]$.
2. For $H \in (3/4, 1]$ there exists a unique real-valued Volterra kernel $\widetilde{r}_\sigma \in \mathbb{L}_2([0, T]^2)$ such that

$$B_t := M_t^{H,\sigma} - \int_0^t \int_0^s \widetilde{r}_\sigma(s, u) dM_u^{H,\sigma} ds, \quad t \in [0, T] \tag{3}$$

is a Brownian motion on $(\Omega, \mathbb{F}, P)$.

3. The "inverse" representation holds:

$$M_t^{H,\sigma} := B_t + \int_0^t \int_0^s r_\sigma(s,u) dB_u ds, \quad t \in [0, T],$$

where $r_\sigma \in \mathbb{L}_2([0, T]^2)$ is the negative resolvent kernel of $\widetilde{r}_\sigma$, r_σ is the unique solution of the equation

$$\sigma^2 H(2H-1)(t-s)^{2H-2} = r_\sigma(t,s) + \int_0^s r_\sigma(t,x) r_\sigma(s,x) dx, \quad 0 \le s < t \le T, \tag{4}$$

and this representation is unique in the following sense: if $\widetilde{B}_t$ is a Brownian motion on $(\Omega, \mathbb{F}, P)$ and $l \in \mathbb{L}_2([0, T]^2)$ a real-valued Volterra kernel such that

$$M_t^{H,\sigma} := \widetilde{B}_t + \int_0^t \int_0^s l(s,u) d\widetilde{B}_u ds, \quad t \in [0, T],$$

then $\widetilde{B} = B$ and $l = r_\sigma$.

4. As a consequence, the process $\sigma_1 W_t + \sigma_2 B_t^H$ is a semimartingale with respect to its natural filtration. Let the process $\Psi(s)$ be $\bar{\mathbb{F}}^S$-predictable and satisfy the condition $\int_0^T |\Psi_u|^2 du < \infty$ a.s. Then the stochastic integral $\int_0^T \Psi_u dS_u$ is correctly defined as the integral w.r.t. the semimartingale.

5. Let $\{B_t, t \in [0, T]\}$ be a Brownian motion on a probability space $(\Omega, \mathbb{F}, P)$ and $l \in \mathbb{L}_2([0, T]^2)$ a real-valued Volterra kernel. Then

$$E \exp\left(\int_0^t \int_0^s l(s,u)dB_u dB_s - \frac{1}{2}\int_0^t \left(\int_0^s l(s,u)dB_u\right)^2 ds\right) = 1$$

moreover, by Girsanov theorem,

$$B_t - \int_0^t \int_0^s l(s,u)dB_u ds, \quad t \in [0, T]$$

is a Brownian motion on $(\Omega, \mathbb{F}, \widetilde{P})$, where

$$\frac{d\widetilde{P}}{dP} = \exp\left(\int_0^T \int_0^s l(s,u)dB_u dB_s - \frac{1}{2}\int_0^T \left(\int_0^s l(s,u)dB_u\right)^2 ds\right).$$

6. If we consider the following class of strategies

$$\mathbb{S} = \{\Psi = (\Psi^1, \Psi^2)\} : \quad \Psi^1 \quad \text{and}$$

$$\Psi^2 \quad \text{are} \quad \bar{\mathbb{F}}^{\mathbb{S}} \quad \text{-predictable},$$

$$\int_0^T |\Psi_u^1| du < \infty, \quad \int_0^T |\Psi_u^2|^2 du < \infty \quad \text{a.s.},$$

$$V_t = V_0 + \int_0^t \Psi_u^2 dS_u, \quad t \in [0, T]$$

and there exists a constant $c \geq 0$ such that

$$\inf_{t\in[0,T]} \int_0^T \Psi_u^2 dS_u \geq -c \quad \text{a.s}\},$$

then the model (1)–(2) is arbitrage-free and complete.

Denote $\sigma = \frac{\sigma_2}{\sigma_1}$. It follows from Property 5 that for any $\sigma_1 > 0$, $\sigma_2 > 0$ and for a simplest mixed Brownian and fractional-Brownian model with non-random and constant coefficients, there exists a unique probability measure $P^* = P^*(\sigma_1, \sigma)$ such that the discounted process is a martingale with respect to P^* and the natural filtration $\bar{\mathbb{F}}^M$, and $\frac{dP^*}{dP}$ is defined by the relation

$$Z_t := \frac{dP^*}{dP}\Big|_{\bar{\mathbb{F}}^M_t} = \exp\Big\{ -\int_0^t \Big(\frac{\mu - r}{\sigma_1} + \frac{\sigma_1}{2} + \int_0^s r_\sigma(s,u)dB_u\Big)dB_s - \frac{1}{2}\int_0^t \Big(\frac{\mu - r}{\sigma_1} + \frac{\sigma_1}{2} + \int_0^s r_\sigma(s,u)dB_u\Big)^2 ds\Big\}. \tag{5}$$

Evidently, under measure P^* the process X_t has a form $X_t = S_0 \exp\left\{\sigma_1 W_t - \frac{\sigma_1^2}{2} t\right\}$, where W is some Wiener process.

The problem of efficient hedging on the mixed market

Let H be some contingent claim on our financial market. Consider the natural filtration $\bar{\mathbb{F}}^S$ generated by the process $W_t + \sigma B_t^H$, $0 \leq t \leq T$ and the class $\mathbb{S}$ of self-financing strategies described above. Consider the problem of efficient hedging, which purpose is to minimize the potential losses, weighted by the hedger's risk preference from imperfect hedging. Efficient hedging aims at finding an admissible self-financing strategy $\Psi^* = (\Psi^{*,1}, \Psi^{*,2}) \in \mathbb{S}$ that minimizes the shortfall risk

$$E\left(l\left(\left(H - V_T^{\Psi^*}\right)^+\right)\right) = \min_{\Psi} E\left(l\left(\left(H - V_T^{\Psi}\right)^+\right)\right)$$

with initial capital $V_0 \leq \nu_0 < H_0 := E_{P^*}(He^{-rT})$. Here l denotes the loss function that reflects the investor's risk preference.

According to [Foellmer, Leukert, 2000], the important particular case is the loss function $l(x) = \frac{x^p}{p}$, $p > 0$, where $p > 1$ corresponds to risk-averse investor, $p = 1$ corresponds to risk indifference, and $0 < p < 1$ means that the investor is risk-taker. In the general case, the minimization problem for some set of measures $P^* \in \mathcal{P}$ can be reformulated as follows [Foellmer, Leukert, 2000]:
(A) to find the randomized test, or minimizer, $0 \leq \psi^* \leq 1$, which is $\bar{\mathbb{F}}_T$-measurable, and which minimizes the shortfall risk $E[l(H(1-\psi))]$ among all $\bar{\mathbb{F}}_T$-measurable $0 \leq \psi \leq 1$ subject to constraints $E^*\psi H \leq \nu_0$ for all $P^* \in \mathcal{P}$.

Denote $L = (I')^{-1}$ the inverse function to I'. It was proved by [Foellmer, Leukert, 2000] that the solution of the problem of imperfect efficient hedging is the perfect hedge Ψ^* for the modified contingent claim $H^* = \varphi^* H$, where φ^* is determined as

$$\begin{aligned} \varphi^* &= 1 - \left(\frac{L\left(a^* e^{-rT} Z_T\right)}{H} \wedge 1 \right) \quad \text{for} \quad p > 1, \\ \varphi^* &= 1_{\{a^* e^{-rT} H^{1-p} Z_T < 1\}} \quad \text{for} \quad 0 < p < 1, \\ \varphi^* &= 1_{\{a^* e^{-rT} Z_T < 1\}} \quad \text{for} \quad p = 1, \end{aligned} \tag{6}$$

and Z_T refers to the density of equivalent martingale measure P^*:

$$Z_t = \frac{dP^*}{dP}\Big|_{\bar{\mathbb{F}}_t^S}, \quad t \in [0, T],$$

a^* is such constant that $E_{P^*}[H\varphi^*] = \nu_0$.

The formula (6) gives the general solution of our problem of imperfect efficient hedging. However, it is hard to proceed with some computations because the distribution of Z_T, according to (5), depends on the whole trajectory of Wiener process $\{B_t, 0 \le t \le T\}$. However, in turn, in the case when the objective measure P coincides with the measure P^* and the contingent claim H depends only on the final value of discounted risk asset: $H = H(X_T)$, the situation can be simplified.

Indeed, in this case

$$E\left(l\left(\left(H-V_T^{\Psi}\right)^+\right)\right)=E\left(\left(H(X_T)-\psi_T e^{-rT}-\psi_T^2 X_T\right)^+\right)^p,$$

where $E(\cdot)$ means mathematical expectation with respect to the measure for which X_T has the known log-normal distribution, $X_T = S_0 \exp\left\{\sigma_1 B_T - \frac{\sigma_1^2}{2} T\right\}$, B is Wiener process. The condition $E_{P^*} H \leq \nu_0$ is reduced to $EH(X_T) \leq \nu_0$, and we come to the standard problem of efficient hedging.

Imperfect hedging on an incomplete market. Approximations of fractional Brownian motion. Minimal martingale measure for an approximate market

Incomplete semimartingale market,
constructed as an approximation of initial market

Try to apply another approach to the solution of the problem of efficient hedging. One of reasons is: let the objective measure P does not coincide with the martingale measure P^* but we still want to obtain comparatively simple and computable distribution of the solution of the problem of efficient hedging.

Another reason: even if $H \in (\frac{1}{2}, \frac{3}{4})$, we can try to solve the problem of efficient hedging, using the properties of the involved processes W and B^H. In this case we can consider an incomplete market adapted to the filtration generated by the couple of two independent processes W and fBm B^H. Since B^H is not a semi-martingale we have no martingale measure P^*. However, we know the possibilities to approximate B^H with the help of the bounded processes of bounded variation.

On approximation of the fractional Brownian motion

As we know, the fractional Brownian motion is not a semimartingale unless $H = \frac{1}{2}$. However, there is a possibility (not unique, of course) to approximate B^H with the help of the bounded processes of bounded variation. To this end, we can apply the representation of fBm B^H via some Wiener process $\widetilde{W}$ on the finite interval $[0, T]$ that was obtained in the paper ([Norros, Valkeila, Virtamo]). This representation has the form

$$B_t^H = C_H^1 \int_0^t u^{-\alpha} \left(\int_u^t s^\alpha (s-u)^{\alpha-1} ds \right) d\widetilde{W}_u, \tag{7}$$

where $C_H^1 = \alpha \left(\frac{2H\Gamma(1-\alpha)}{\Gamma(1-2\alpha)\Gamma(\alpha+1)} \right)^{1/2}$, $\alpha = H - \frac{1}{2}$.

If we formally apply the stochastic Fubini theorem (see [Protter]) to the right-hand side of (7), we get

$$B_t^H = C_H^1 \int_0^t s^\alpha \left(\int_0^s u^{-\alpha}(s-u)^{\alpha-1} d\widetilde{W}_u \right) ds,$$

where the interior integral is divergent, due to singularity in the upper limit of integration. However, if we retreat from the singularity point, we can obtain the family of bounded processes of bounded variation $B_t^{H,\varepsilon}$ of the form

$$B_t^{H,\varepsilon} = \int_0^t \varphi_\varepsilon(s) ds, \tag{8}$$

where

$$\varphi_\varepsilon(s) = \left(C_H^1 s^\alpha \int_0^{(1-\varepsilon)s} u^{-\alpha}(s-u)^{\alpha-1} d\widetilde{W}_u \right) \wedge (\varepsilon)^{-1}, \quad 0 < \varepsilon < 1. \quad (9)$$

It was proved by Androshchuk, M. that for any $t > 0$ there is convergence in probability

$$B_t^{H,\varepsilon} \xrightarrow{P} B_t^H \quad (10)$$

as $\varepsilon \to 0$. This result was generalized in [Ralchenko, Shevchenko], where the convergence in probability of $B_t^{H,\varepsilon}$ to B_t^H in some Besov spaces was established for any $H \in (0,1)$. In particular, the uniform convergence was established: $\sup_{t\in[0,T]} |B_t^{H,\varepsilon} - B_t^H| \xrightarrow{P} 0$, $\varepsilon \to 0$. Uniform convergence together with the fact that the process B^H has a.s. the trajectories of unbounded variation on any interval implies that

$$P - \lim_{\varepsilon\to 0} \int_0^T |\varphi_\varepsilon(s)| ds = \infty. \quad (11)$$

since the limit equals to the variation of B^H on the interval $[0, T]$.

By $\bar{\mathbb{F}}^{W,\tilde{W}} = \{\bar{\mathbb{F}}_t^{W,\tilde{W}}, 0 \le t \le T\}$ we denote the smallest filtration that contains $\mathbb{F}^{W,\tilde{W}}$ and fulfils the usual assumptions. Introduce also the notation $\mathbb{F}_t^{\tilde{W}} := \sigma\{\tilde{W}_s, 0 \le s \le t\},\ t \in [0, T]$ and corresponding filtration $\bar{\mathbb{F}}^{\tilde{W}}$. The next result is an evident consequence of the structure of approximate market.

The approximate market has a form

$$B_t = B_0 e^{rt}, \quad t \geq 0, \quad B_0 > 0, \tag{12}$$

$r > 0$ is a constant risk-free rate, and risky asset that is governed by the linear combination of W and B^H

$$S_t = S_0 \exp\{\mu t + \sigma_1 W_t + \sigma_2 B_t^{H,\varepsilon}\}, \quad t \geq 0, \tag{13}$$

Lemma 19

All equivalent martingale measures P^* for the market described by the equations (12) and (13), equal a product of two likelihood ratios

$$\frac{dP^*}{dP}\Big|_{\bar{\mathbb{F}}_t^{W,\tilde{W}}} = Z_t^{\varepsilon,1} Z_t^2, \tag{14}$$

where

$$Z_t^{\varepsilon,1} = \exp\left\{\int_0^t \theta_\varepsilon(s)dW_s - \frac{1}{2}\int_0^t \theta_\varepsilon^2(s)ds\right\}, \tag{15}$$

$$\theta_\varepsilon(s) = \frac{r-\mu}{\sigma_1} - \frac{\sigma_1}{2} - \frac{\sigma_2}{\sigma_1}\varphi_\varepsilon(s), \tag{16}$$

$$Z_t^2 = \exp\left\{\int_0^t b(s)d\tilde{W}_s - \frac{1}{2}\int_0^t b^2(s)ds\right\}, \tag{17}$$

where $b = b(s)$ be any $\bar{\mathbb{F}}_s^{W,\tilde{W}}$-adapted function such that $EZ_T^{\varepsilon,1}Z_T^2 = 1$.

Minimal martingale measure (MMM) on the approximate market

Evidently, among all the measures, described by the relations (14) – (17), the simplest measure corresponds to the case $b(s) \equiv 0$, or $Z_t^2 \equiv 1$. Consider this measure in more detail.

Definition 20

(Foellmer, Schweizer) Probability measure $P^* \sim P$ is called minimal martingale measure if $E\left(\frac{dP^*}{dP}\right)^2 < \infty$ and any square-integrable P-martingale M, strongly orthogonal to the discounted price process $X_t = e^{-rt} S_t$, is also P^*-martingale.

According to the paper of Schweizer, minimal martingale measure can be found in the following way. Let the semi-martingale Y has a canonical decomposition of the form

$$Y = Y_0 + M + A = X_0 + M + \int \alpha d\langle M\rangle, \tag{18}$$

where M is a martingale, α is the predictable process, then the minimal martingale measure has a density

$$\widehat{Z}_T = \exp\left\{-\int_0^T \alpha_s dM_s - \frac{1}{2}\int_0^T \alpha_s^2 d\langle M\rangle_s\right\},$$

under the condition $E\widehat{Z}_T^2 < \infty$.

In our case, Y is discounted price process

$$X_t^\varepsilon = S_0 \exp\left\{(\mu - r)t + \sigma_1 W_t + \sigma_2 \int_0^t \varphi_\varepsilon(s)ds\right\},$$

and, according to Itô formula,

$$X_t^\varepsilon = S_0 + \sigma_1 \int_0^t X_s^\varepsilon dW_s + \frac{\sigma^2}{2}\int_0^t X_s^\varepsilon ds + \int_0^t X_s^\varepsilon[(\mu - r) + \sigma_2\varphi_\varepsilon(s)]ds.$$

Therefore, we can put in (18)

$$M_t = \sigma_1 \int_0^t X_s dW_s, \quad \langle M \rangle_t = \sigma_1^2 \int_0^t X_s^2 ds$$

and

$$\alpha_t = X_t^{-1} \left(\frac{1}{2} + \frac{\mu - r}{\sigma_1^2} + \frac{\sigma_2 \varphi_\varepsilon(t)}{\sigma_1} \right).$$

At last, it means that the density of MMM for approximate market has a form

$$\widehat{Z}_T = \widehat{Z}_T^\varepsilon = \exp \left\{ - \int_0^T \left(\frac{\sigma_1}{2} + \frac{\mu - r}{\sigma_1} + \sigma_2 \varphi_\varepsilon(s) \right) dW_s \right.$$
$$\left. - \frac{1}{2} \int_0^T \left(\frac{\sigma_1}{2} + \frac{\mu - r}{\sigma_1} + \sigma_2 \varphi_\varepsilon(s) \right)^2 ds \right\} = Z_T^{\varepsilon,1},$$

and minimal martingale measure (if it exists) corresponds to the case $b \equiv 0$ in (17), or $Z_t^2 \equiv 1$.

Only, we must check that $E(\widehat{Z}_T^{\varepsilon})^2 < \infty$. Prove the following auxiliary result.

Lemma 21

The density $\widehat{Z}_T^{\varepsilon}$ has finite moments of any order.

Proof.

Recall that $\theta_\varepsilon(s)$ is bounded and take into account independence of $\theta_\varepsilon(s)$ and W. Then we obtain that for any $q > 0$

$$E(\widehat{Z}_T^{\varepsilon})^q = E[E((\widehat{Z}_T^{\varepsilon})^q / \bar{\mathbb{F}}_T^{\tilde{W}})] = E\left(\frac{q^2 - q}{2} \int_0^T \theta_\varepsilon^2(s) ds\right) < \infty. \tag{19}$$

□

So, in our situation we choose minimal martingale measure as "the best" equivalent martingale measure.

The solution of the "approximate" problem of efficient hedging with respect to MMM

In the case of arbitrage-free incomplete market we do not know the explicit solution for the efficient hedging problem, we only know that this solution exists and for the loss function l_p it is unique, see Proposition 3.1 and Theorem 3.2 from [Foellmer, Leukert, 2000]. However, even in the incomplete case we can state and solve explicitly the following "restricted" problem of efficient hedging for some selected measure $\widetilde{P}^* \in \mathcal{P}$:
(A') find a randomized test, or a minimizer, $0 \le \psi^* \le 1$, which is an $\widetilde{\mathbb{F}}_T$ - measurable random variable, and which minimizes the shortfall risk $E[l(\mathbb{H}(1-\psi))]$ among all $\widetilde{\mathbb{F}}_T$ - measurable random variables $0 \le \psi \le 1$ subject to constraints $E_{\widetilde{P}^*}(\psi\mathbb{H}) \le \nu_0$ for $\widetilde{P}^* \in \mathcal{P}$.
The solution for this problem for the loss function l_p will have the form (6), where Z_T refers to the final value of the density of the equivalent martingale measure $\widetilde{P}^*$.

Let φ_ε^* be the function from (6), where we substitute $\widehat{Z}_T^\varepsilon$ instead of Z_T. In what follows we suppose that the discounted contingent claim H is positive and depends only on the final value of discounted price process, $H = H(X_T^\varepsilon)$, where $H(\cdot) : \mathbb{R} \to \mathbb{R}^+$ is measurable function, and in this case present some results concerning simplifying of possible computations.

Lemma 22

The equality holds,

$$E\widehat{Z}_T^\varepsilon H(X_t^\varepsilon) = (2\pi)^{\frac{1}{2}} \int_{\mathbb{R}} H(e^{\sigma_1 T^{\frac{1}{2}} x - \frac{\sigma_1^2}{2} T}) e^{\frac{-x^2}{2}} dx, \tag{20}$$

under the condition that the last integral is finite.

In what follows we consider only the case of risk-averse investor, when $p > 1$ in (6) (other cases can be considered along the same lines). So, throughout this section $l(x) = x^p$, $p > 1$. We suppose that the following condition holds.

$$E\mathbb{H}^p < \infty. \tag{21}$$

At first note that it follows from (19) that in this case both parts of equation (20) are finite.

Introduce the following notations. Let $\varsigma(\cdot):\mathbb{R}\to\mathbb{R}$ be some measurable bounded function, $I_1(\varsigma)=\int_0^T\varsigma(s)ds$, $I_2(\varsigma)=\int_0^T\varsigma^2(s)ds$,
$H_1(x)=H(S_0\exp\{\sigma_1 T^{\frac{1}{2}}x-\frac{\sigma_1^2}{2}T\})$, $G(\varsigma(\cdot),y)=\exp\{y+\frac{1}{2}\int_0^T\varsigma^2(s)ds\}$,
and the matrix

$$P(\varsigma(\cdot))=\begin{pmatrix} T & \int_0^T\varsigma(s)ds \\ \int_0^T\varsigma(s)ds, & \int_0^T\varsigma^2(s)ds\end{pmatrix}$$

is non-degenerate. Also, let the set

$$A(\varsigma(\cdot),a)=\{(x,y)\in\mathbb{R}^2: a<H_1(x)^{p-1}G(\varsigma(\cdot),y)^{-1}\},$$

the function

$$\tilde{\Psi}(\varsigma(\cdot),a)=\int_{A(\varsigma(\cdot),a)}(H_1(x)-a^{\frac{1}{p-1}}G(\varsigma(\cdot),y)^{\frac{1}{p-1}})p(x,y,\varsigma(\cdot))dxdy,$$

$p(x,y,\varsigma(\cdot))$ is the density of bivariate Gaussian distribution with zero mean and covariance matrix $P(\varsigma(\cdot))$.

Theorem 23

Suppose that condition (21) holds.
1) Then the equation

$$E\widetilde{\Psi}(\theta_\varepsilon(\cdot), a) = \nu_0 \tag{22}$$

has a unique root. Denote it by a_ε^*.
2) The number a_ε^* and the function

$$\varphi_\varepsilon^* = 1 - \frac{(a_\varepsilon^*)^{\frac{1}{p-1}} (\widehat{Z}_T^\varepsilon)^{\frac{1}{p-1}}}{\mathbb{H}(X_T^\varepsilon)} \wedge 1 \tag{23}$$

solve the restricted problem (A') of efficient hedging for the approximate Brownian-fractional-Brownian market.

Asymptotic behavior of the solution of the problem of efficient hedging with respect to MMM

Now, let parameter value $\varepsilon \to 0$.

Lemma 24

Under conditions of Lemma 6 we have that $E[l(H(1-\varphi_\varepsilon^*))] \to 0$ *as* $\varepsilon \to 0$.

Remark 1

Relation $E[l(H(1-\varphi_\varepsilon^*))] \to 0$ means that under the filtration $\bar{\mathbb{F}}^{W,\tilde{W}}$ hedging of contingent claim H is asymptotically perfect for any $\nu_0 > 0$.

Estimate of success probability in the problem of quantile hedging
Now consider the simpler problem of quantile hedging on the complete non-arbitrage market. The problem of hedging contingent claims is well studied for complete arbitrage-free financial markets. Let an investor want to ensure that a contingent claim H will be hedged with probability 1 at a fixed moment $T > 0$ and assume that the investor capital is modeled by a semimartingale $X = \{X_t, t \geq 0\}$. As is known, a necessary and sufficient condition for such a perfect hedging is that the initial capital $H_0 = E_{P*}(H)$ where E_{P*} denotes the expectation with respect to the unique martingale measure P. (In other words, with respect to the measure such that X is a martingale with respect to P).

If the investor is unwilling or is unable to put up the initial amount of capital required by this condition, but he is ready to put up another amount $\nu < H_0$, then the arbitrage-free property of the market implies that the investor cannot replicate the claim H in all possible scenarios, i.e. he cannot hedge the claim H with probability 1. In such a case, general results concerning the hedging, called quantile hedging, are obtained in the paper [Foellmer, Leukert, 99], if the price process is a semimartingale. The main aim of the quantile hedging is to maximize the probability of success, that is the probability of a successful hedge for a contingent claim H.

It is easy to check that the shifted process $\frac{m}{\sigma}t + B_t + \sigma B_t^H$ also is equivalent to a Wiener process. Indeed, one can consider two independent processes $\frac{m}{\sigma}t + B_t$ and σB_t^H and remove the shift for the process $\frac{m}{\sigma}t + B_t$. Thus $M_t^{H,\sigma}$ is a Wiener process with respect to the measure P^* defined as follows

$$\begin{aligned} \frac{dP^*}{dP} = \exp\Bigg(& -\left(\frac{m}{\delta} + \frac{1}{2}\delta\right) W_t + \int_0^t \int_0^s r_\sigma(s,u)\, dW_u dW_s - \\ & - \frac{1}{2}\int_0^t \left(\frac{m}{\delta} + \frac{1}{2}\delta - \int_0^s r_\sigma(s,u)\, dW_u\right)^2 ds \Bigg). \end{aligned} \tag{24}$$

The general semimartingale approach to the solution of this problem using the Neyman– Pearson lemma is described in the paper [Foellmer, Schweizer]. We outline the solution of this problem for the general case and briefly discuss its features when applied to the mixed model.

Recall that the problem of quantile hedging for the case of $\nu < H_0$ is to maximize the probability of the event $P\{V_T \geq H\}$ in the class of all self-financing strategies ξ with the initial capital $V_0 \leq \nu$ that are admissible in the sense that the capital process

$$V_t = V_0 + \int_0^t \xi_s dX_s, \tag{25}$$

corresponding to the strategy ξ, is nonnegative almost surely for all $0 \leq t \leq T$. The general semimartingale approach to the solution of this problem using the Neumann– Pearson lemma is described in the paper [Foellmer, Leukert, 99]. We outline the solution of this problem for the general case and briefly discuss its features when applied to the mixed model.

A set $A = \{V_T \geq H\}$ is called the success set corresponding to the strategy ξ (see (5)). Consider the measure Q^* defined by

$$\frac{dQ^*}{dP^*} = \frac{H}{H_0}. \tag{26}$$

The optimization problem is to maximize the probability $P(A)$ with respect to all $\mathcal{F}_T$- measurable sets A that satisfy the condition

$$\frac{V_0}{H_0} = \frac{E_{P^*}(HI_A)}{H_0} = E_{Q^*}(I_A) \leq \frac{\nu}{H_0}. \tag{27}$$

Let $\bar{a} = \inf \left\{ a : Q^* \left(\frac{dP}{dP^*} > a \cdot H \right) \leq \frac{\nu}{H_0} \right\}$. Assume that

$$Q^* \left(\frac{dP}{dP^*} = \bar{a} \cdot H \right) = 0. \tag{28}$$

According to the Neumann–Pearson lemma, the set

$$A = A_{\bar{a}} := \left\{ \frac{dP}{dP*} > \bar{a} \cdot H \right\}. \tag{29}$$

is optimal. Due to the choice of $\bar{a}$ given the initial capital ν there exists a strategy that hedges the contingent claim $\bar{H} = HI_{A_{\bar{a}}}$ with probability one. Thus the same strategy allows one to hedge the claim H with probability $P(A_{\bar{a}})$. This is a strategy that maximizes the probability $P(V_T \geq H)$.

Consider one of the simplest cases where the claim H depends on the price of an asset at the terminal moment only, that is $H = H_T = f(X_T)$. Using the Neumann–Pearson lemma, the problem is to find a constant $\bar{a}$ such that the probability P of the set

$$A = A_a := \left\{ \frac{dP}{dP^*} > a \cdot H \right\}$$

is maximal provided $Q^*(A) \leq \frac{\nu}{H_0}$. Taking into account expression (35) for the density, the set A can be rewritten as follows

$$A = \left\{ \exp\left(\left(\frac{m}{\delta} + \frac{1}{2}\delta \right) W_T - \int_0^T \int_0^s r_\sigma(s,u)\, dW_u dW_s + \right.\right.$$
$$\left.\left. + \frac{1}{2} \int_0^T \left(\frac{m}{\delta} + \frac{1}{2}\delta - \int_0^s r_\sigma(s,u)\, dW_u \right)^2 ds \right) > a \cdot f(X_T) \right\}. \tag{30}$$

This set depends on the process $W = \{W_t, t \geq 0\}$ in the whole interval $[0, T]$ and thus the probabilities $Q^*(A)$ and $P(A)$ are hard to evaluate. Therefore the approach of [Foellmer, Leukert, 99] is not easy to apply even for simplest contingent claims $f(X_T)$ (for European call options, for example). Recall that in terms of the Neumann-Pearson lemma, the problem is reduced to finding such constant $\bar{a}$ that the probability P of the set $A = A_a := \left\{\frac{dP}{dP^*} > a \cdot H\right\}$ is maximal given that $Q^*(A) \leq \alpha = \frac{\nu}{H_0}$.

Unfortunately, the probabilities $Q^*(A)$ and $P(A)$ are hardly computable, which does not permit to use directly the approach, introduced in [Foellmer, Leukert, 99] even for the simplest payoff functions $f(X_T)$ (e.g., European call option).

Using the procedure of re-discounting permits us to avoid aforementioned difficulty. Put

$$\tilde{A}_a = \left\{ a < \frac{X_T}{X_0 \cdot f(X_T)} \right\}, \tag{31}$$

and let the claim H is hedged on this set, if the initial capital is equal to ν.

The set $\tilde{A}_{\bar{a}}$ is not necessarily the maximal success probability set for initial capital ν, nevertheless $P\left(\tilde{A}_{\bar{a}}\right)$ is lower estimate for the maximal probability. Therefore, one can guarantee successful hedge of the claim with probability not smaller than $P\left(\tilde{A}_{\bar{a}}\right)$.

In the incomplete case the equivalent martingale measure is no longer unique. The following lemma states the incompleteness of the market defined by price process (32).

$$= X_0 \exp\left\{mt + \sigma_1\left(B_t^1 + \delta_1 B_t^{H_1}\right) + \sigma_2\left(B_t^2 + \delta_2 B_t^{H_2}\right)\right\}, \qquad (32)$$

where $\delta_i = \frac{\mu_i}{\sigma_i}$, $i = 1, 2$, $X_0 > 0$ is some constant.

Recall that there exists the representation

$$Y_t^i = W_t^i - \int_0^t \int_0^s r_{\delta_i}(s,u)\, dW_u^i ds, \tag{33}$$

where W_t^i is Wiener process w.r.t. P, $r = r_{\delta_i}$ is Volterra kernel, that is, the unique solution of equation

$$r(t,s) + \int_0^s r(t,x)\, r(s,x)\, dx = \delta_i^2 H_i (2H_i - 1) \cdot |t-s|^{2H_i - 2}, \tag{34}$$

which satisfies $\int_0^t \int_0^s (r_{\delta_i}(s,u))^2 \, duds < \infty$, $i = 1, 2$. Thus, Y_t^i is a semimartingale w.r.t. the natural filtration F^Y, generated by the processes $Y_t^i, i = 1, 2$. In what follows we fix this filtration and we will consider only it.

Lemma 3.1. *Let $H_i > \frac{3}{4}$, $i = 1, 2$, $\tilde{m}_1(s)$ is predictable process with respect to natural filtration F^Y , $\tilde{m}_2(s) = m - \tilde{m}_1(s)$ and the following conditions hold: $E \int_0^t m_i^2(s)ds < \infty$ and*

$$E \exp\Big(- \int_0^T \Big(\frac{\tilde{m}_i(s)}{\sigma_i} + \frac{1}{2}\sigma_i - \int_0^s r_{\delta_i}(s,u)\,dW_u^i\Big) dW_s^i -$$

$$\frac{1}{2}\int_0^T \Big(\frac{\tilde{m}_i(s)}{\sigma_i} + \frac{1}{2}\sigma_i - \int_0^s r_{\delta_i}(s,u)\,dW_u^i\Big)^2 ds\Big) = 1, \ i = 1, 2.$$

The model for the process X_t of the form (32) is incomplete since there exists a family of the martingale measures P^* depending on $\tilde{m}_1$ and $\tilde{m}_2$, with Radon-Nykodym derivatives of the form

$$\frac{dP^*}{dP}\Big|_{F_t^Y} = \prod_{i=1}^{2} \exp\left(-\int_0^t \left(\frac{\tilde{m}_i(s)}{\sigma_i} + \frac{1}{2}\sigma_i - \int_0^s r_{\delta_i}(s,u)\, dW_u^i\right) dW_s^i - \right.$$

$$\left. -\frac{1}{2}\int_0^t \left(\frac{\tilde{m}_i(s)}{\sigma_i} + \frac{1}{2}\sigma_i - \int_0^s r_{\delta_i}(s,u)\, dW_u^i\right)^2 ds\right). \qquad (35)$$

Remark. Considering another filtration than F^Y, we can obtain another martingale measures for the process X_t, not defined in Lemma 3.1.

Example 3.1. Let we define the processes

$$\tilde{B}_t^1(\alpha) = B_t^1 \cos\alpha - B_t^2 \sin\alpha,$$

$$\tilde{B}_t^2(\alpha) = B_t^1 \sin\alpha + B_t^2 \cos\alpha.$$

These processes are uncorrelated and, thus, independent. We have

$$B_t^1(\alpha) = \tilde{B}_t^1 \cos\alpha + \tilde{B}_t^2 \sin\alpha,$$

$$B_t^2(\alpha) = -\tilde{B}_t^1 \sin\alpha + \tilde{B}_t^2 \cos\alpha,$$

and

$$X_t = X_0 \exp\left\{mt + \sigma_1\left(B_t^1 + \delta_1 B_t^{H_1}\right) + \sigma_2\left(B_t^2 + \delta_2 B_t^{H_2}\right)\right\} =$$

$$= X_0 \exp\left\{mt + \sigma_1\left(\tilde{B}_t^1(\alpha)\cos\alpha + \tilde{B}_t^2(\alpha)\sin\alpha + \delta_1 B_t^{H_1}\right) + \right.$$

$$\left. + \sigma_2\left(-\tilde{B}_t^1(\alpha)\sin\alpha + \tilde{B}_t^2(\alpha)\cos\alpha + \delta_2 B_t^{H_2}\right)\right\} =$$

$$= X_0 \exp\left\{mt + (\sigma_1\cos\alpha - \sigma_2\sin\alpha)\tilde{B}_t^1(\alpha) + (\sigma_1\sin\alpha + \sigma_2\cos\alpha)\tilde{B}_t^2(\alpha) + \right.$$

$$\left. + \mu_1 B_t^{H_1} + \mu_2 B_t^{H_2}\right\} =$$

$$= X_0 \exp\left\{mt + \tilde{\sigma}_1\tilde{B}_t^1(\alpha) + \tilde{\sigma}_2\tilde{B}_t^2(\alpha) + \mu_1 B_t^{H_1} + \mu_2 B_t^{H_2}\right\} =$$

$$= X_0 \exp\left\{mt + \tilde{\sigma}_1\left(\tilde{B}_t^1(\alpha) + \frac{\mu_1}{\tilde{\sigma}_1} B_t^{H_1}\right) + \tilde{\sigma}_2\left(\tilde{B}_t^2(\alpha) + \frac{\mu_2}{\tilde{\sigma}_2} B_t^{H_2}\right)\right\} =$$

$$= X_0 \exp\left\{mt + \tilde{\sigma}_1\left(\tilde{B}_t^1(\alpha) + \tilde{\delta}_1 B_t^{H_1}\right) + \tilde{\sigma}_2\left(\tilde{B}_t^2(\alpha) + \tilde{\delta}_2 B_t^{H_2}\right)\right\} =$$

$$= X_0 \exp\left\{mt + \tilde{\sigma}_1 \tilde{Y}_t^1 + \tilde{\sigma}_2 \tilde{Y}_t^2\right\}, \tag{36}$$

where $\tilde{\sigma}_1 = \sigma_1 \cos\alpha - \sigma_2 \sin\alpha$, $\tilde{\sigma}_2 = \sigma_1 \sin\alpha + \sigma_2 \cos\alpha$, $\tilde{\delta}_i = \frac{\mu_i}{\tilde{\sigma}_i}$, $\tilde{Y}_t^i = \tilde{B}_t^i(\alpha) + \tilde{\delta}_i B_t^{H_i}$, $i = 1, 2$.

Note that in general case we have $\tilde{\delta}_i \neq \delta_i$, $i = 1, 2$. Thus, the processes $\tilde{Y}_t^i$ and Y_t^i have different Volterra kernels, which are the solutions of (34). It means that corresponding martingale measures for the process X_t, which satisfy (35), are different.

Remark. Although the martingale measures, mentioned above, are different, the following equality holds:

$$\sigma_1^2 + \sigma_2^2 = \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2. \tag{37}$$

If the contingent claim in incomplete model is not attainable, it is possible to hedge the claim almost surely by means of superhedging strategy. As it is mentioned in [4], in such case the least amount of capital, which is needed to be on the safe side, is given by

$$\inf \left\{ V_0 | \quad \exists \xi : (V_0, \xi) \quad \textit{admissible}, \quad V_0 + \int_0^T \xi_s dX_s \geq H \quad P - a.s. \right\}. \tag{38}$$

Thus, the least needed amount is equal to the largest arbitrage-free price:

$$U_0 := \sup_{P^* \in \mathcal{P}} E^*[H] < \infty \tag{39}$$

where $\mathcal{P}$ is the set of all equivalent martingale measures P^* satisfying the conditions of lemma 3.1.

If investor is not able to use whole amount U_0 and is willing to use only amount $\nu < U_0$, then he cannot hedge the claim H with probability 1 because of the absence of arbitrage. But if the following inequality holds:

$$\sup_{P^* \in \mathcal{P}} E_{P^*} \left[H I_{\tilde{A}_a} \right] \leq \nu, \tag{40}$$

where $\tilde{A}_a$ has the form introduced in (31), then an investor with initial capital ν is able to hedge $H I_{\tilde{A}_a}$ almost surely, i.e., to hedge H with probability $P\left(\tilde{A}_a\right)$.

We make use of the form (31) of the set $\tilde{A}_a$.
Remark. When a increases, the sets $\tilde{A}_a$ decrease.
So, we look for minimal a, for which the inequality (40) holds. Let consider any of martingale measures P^* of the form of (35) and find the minimal a, for which the following inequality holds:

$$E_{P^*}\left[HI_{\tilde{A}_a}\right] \leq \nu. \tag{41}$$

In this case an investor with initial capital ν is able to hedge $HI_{\tilde{A}_a}$ almost surely, i.e., to hedge H with probability $P\left(\tilde{A}_a\right)$.

Theorem 3.1. *Let the function $f(x)$ satisfy the condition: for any $z \in R$:*

$$\lambda\left(\left\{\frac{x}{f(x)} = z\right\}\right) = 0, \tag{42}$$

where λ is the Lebesgue measure.
Then the probability of successful hedge of the claim $H = f(X_T)$ is at least $P\left(\tilde{A}_{\bar{a}}\right)$, where $\bar{a}$ is determined by the equation

$$\int_{C_{\bar{a}}} f\left(X_0 e^{\sqrt{T}\sigma y - \frac{1}{2}\sigma^2 T}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = \nu \tag{43}$$

and

$$C_a = \left\{ y \,\middle|\, a < \frac{\exp\left(\sqrt{T}\sigma y - \frac{1}{2}\sigma^2 T\right)}{f\left(X_0 \exp\left(\sqrt{T}\sigma y - \frac{1}{2}\sigma^2 T\right)\right)} \right\}, \tag{44}$$

where

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}. \tag{45}$$

Let consider European call option

$$f\left(X_T\right) = \left(X_T - K\right)_+ , \tag{46}$$

where $K > 0$, and find the representation for the set $\tilde{A}_a = \left\{ a < \frac{X_T}{X_0 \cdot f(X_T)} \right\}$ in this case.

Note that for $X_T \le K$ the inequality $a < \frac{X_T}{X_0 \cdot f(X_T)} = \infty$ holds.
For $X_T \ge K$, the inequality $a < \frac{X_T}{X_0 \cdot f(X_T)}$ takes the form:

$$\frac{X_T}{X_0 (X_T - K)} > a,$$

$$X_T > (X_T - K)\, aX_0,$$

$$KaX_0 > X_T (aX_0 - 1)\,. \tag{47}$$

If $aX_0 \le 1$, then (47) is always true, i.e. $\tilde{A}_a = \Omega$. Thus, when $a \le \frac{1}{X_0}$ we have that $\tilde{A}_a = \Omega$, but this cannot be true, because by the problem formulation it is impossible to hedge the claim almost surely, but on $\tilde{A}_a$ it will be hedged.
If $aX_0 > 1$, i.e. $a > \frac{1}{X_0}$, then $\tilde{A}_a = \left\{ X_T < \frac{KaX_0}{aX_0 - 1} \right\}$, at that $K < \frac{KaX_0}{aX_0 - 1}$.

That is, the set $\tilde{A}_a$ can be presented in the following form:

$$\tilde{A}_a = \begin{cases} \Omega, & a \le \frac{1}{X_0}, \\ \left\{X_T < \frac{KaX_0}{aX_0 - 1}\right\}, & a > \frac{1}{X_0}. \end{cases} \quad (48)$$

Then inequality (??) takes the form:

$$\int\limits_{\left\{X_T < \frac{KaX_0}{aX_0-1}\right\}} \frac{(X_T - K)_+}{X_T} \cdot X_0 dP = \int\limits_{\left\{K < X_T < \frac{KaX_0}{aX_0-1}\right\}} \frac{X_T - K}{X_T} \cdot X_0 dP \le \nu \quad .$$

Since the expression $\frac{KaX_0}{aX_0-1}$ is strictly decreasing in a for $a > \frac{1}{X_0}$ and is continuous in a, then for $\bar{a}$ the following equation must hold:

$$\int\limits_{\left\{K < X_T < \frac{K\bar{a}X_0}{\bar{a}X_0-1}\right\}} \frac{X_T - K}{X_T} \cdot X_0 dP = \nu. \quad (49)$$

Theorem 4.1. *For European call option (46) the maximal probability of successful hedge is at least* $\Phi(U)$, *where* U *is given by the equation*

$$X_0\left(\Phi\left(U-\sigma\sqrt{T}\right)-\Phi\left(L-\sigma\sqrt{T}\right)\right)-K\left(\Phi(U)-\Phi(L)\right)=\nu, \quad (50)$$

where

$$L=\frac{\ln\frac{K}{X_0}+\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}. \quad (51)$$

and Φ *is distribution function for standard Gaussian distribution.*

Since in this case the set $\tilde{A}_a$ has form (48), then the left-hand side (41) takes the form:

$$E_{P^*}\left[HI_{\tilde{A}}\right] = \int\limits_{\tilde{A}} (X_T - K)_+ \, dP^* = \int\limits_{\left\{K < X_T < \frac{KaX_0}{aX_0 - 1}\right\}} (X_T - K) \, dP^*. \quad (52)$$

The inequality $K < X_T < \frac{KaX_0}{aX_0-1}$ can be rewritten as:

$$K < X_0 e^{\sigma\sqrt{T}\xi - \frac{\sigma^2}{2}T} < K\frac{aX_0}{aX_0 - 1};$$

$$\frac{\ln\frac{K}{X_0} + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}} < \xi < \frac{\ln\frac{Ka}{aX_0-1} + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}. \quad (53)$$

Denote:

$$U = U(a) = \frac{\ln\frac{Ka}{aX_0-1} + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}. \quad (54)$$

The inequality (53) takes the form $L < \xi < U(a)$, and equation (43) can be re-written as:

$U(\bar{a})$ …

Now compute $P\left(\tilde{A}_{\bar{a}}\right)$:

$$P\left(\tilde{A}_{\bar{a}}\right) = P\left(X_0 e^{\sigma\sqrt{T}\xi - \frac{\sigma^2}{2}T} < \frac{K\bar{a}X_0}{\bar{a}X_0 - 1}\right) = \tag{55}$$

$$= P\left(\xi < \frac{\ln\frac{K\bar{a}}{\bar{a}X_0 - 1} + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}\right) = \Phi\left(U\left(\bar{a}\right)\right)$$

If we find $U\left(\bar{a}\right)$ from equation (50), we can find $P\left(\tilde{A}_{\bar{a}}\right)$.

Note that to compute $P\left(\tilde{A}_{\bar{a}}\right)$ it is enough to find $U = U(\bar{a})$, and it is not needed to evaluate $\bar{a}$. $\square$

Example 4.1. For European call option (46) formulae (50), (55) allow for wishful value of $P\left(\tilde{A}_a\right)$ to compute $U = U_a$ and, correspondingly, the necessary capital ν, or vice versa: having ν to compute the success probability $P\left(\tilde{A}_a\right)$. Note that hedge always succeeds with probability at least $P\left(X_T < K\right) = \Phi\left(L\right)$, where L is given by formula (51), thus for $X_T < K$ there is nothing to pay for the claim, which means that it will be hedged with the void strategy. On the other hand, when the success probability $P\left(\tilde{A}_a\right)$ increases from $\Phi\left(L\right)$ to 1 the necessary capital ν increases from 0 to option's fair value H_0.
Let $H = 0,8,\ X_0 = 1,\ T = 10,\ m = 2,\ \sigma_1 = \sigma_2 = \mu_2 = \mu_2 = \frac{1}{2\sqrt{2}}$, then for different (depending on K) European calls, fixing $P\left(\tilde{A}_a\right)$, we can compute corresponding values ν.

K	$\Phi(L)$	$P\left(\tilde{A}_a\right)$	ν	H_0
5	0,964733	0,99	0,055677	0,23375
2	0,890456	0,9	0,000807	0,418561
		0,99	0,210488	
1	0,785402	0,9	0,053051	0,570805
		0,99	0,352732	
0,5	0,63765	0,9	0,141527	0,709281
		0,99	0,486208	
0,1	0,252797	0,9	0,305201	0,912955
		0,99	0,685882	
0,01	0,016919	0,9	0,373309	0,990063
		0,99	0,76209	

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