Energy Derivatives with Volume Control

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Electricity producers sell their production in open power markets.
They need hedging instruments to cover both price and volume risk.
We study financial option contracts, formulated as a stochastic control problem, with a payoff structure of a call option.
Case I: Maximal and minimal total volume constraint.
Case II: Maximal total volume constraint and penalty.

Aim
Study the optimal exercise policy under Case I and Case II.
Maximal constraint has been studied in Benth, F.E., Lempa, J., Nilssen, T. (2010), On Optimal Exercise of Swing Options in Electricity Market.
Maximal constraint

- There is a maximum amount of power that can be produced.
- Agreements in the contract.

Minimal constraint

In the contract agreement a certain amount of power is guaranteed to be delivered until maturity.

Consequence:

- The producer **have to** adapt his production to meet a minimal constraint at maturity.
- He might have to produce power even if prices are low.
Case II

Introduce a penalty if the total produced volume is below the agreement.

Consequence:

▶ The producer tries to adapt his production to meet the agreements in the contract.

▶ Have the option to take the penalty if that is more profitable.
Define the control space. 
*This will specify the total volume constraints.*

Define the value of the contract. 
*Formulated as a stochastic control problem. It will be an option paying money according to price levels and volume decisions.*

The associated HJB-equation. 
*Focus on the optimal exercise policy.*

Properties of the marginal value. 
*From the HJB-equation, the marginal value plays a key role in the optimal exercise policy.*
Define the cumulative control $Z(t)$ for $t \in [0, T]$ as

$$Z(t) = \int_0^t u(s) \, ds,$$

where $u$ is bounded and progressively measurable w.r.t the filtration generated by the underlying price process $X$. $Z$ represents the total volume and we can think of $u$ as the production rate.

**Admissible controls**

Define the set $\mathcal{U}_m(t, T)$ of admissible controls $u(s)$ for $s \in [t, T]$ as:

- (1) holds and $u(s) \in [0, \bar{u}]$.
- $Z(T) \leq M$ (Maximal constraint when $M < \bar{u}T$)
- $Z(s) \geq \ell(s) := m - \bar{u}(T - s)$ (Minimal constraint when $m > 0$)

Note that the last condition is always fullfilled for $t \in [0, \tilde{t}]$, $\tilde{t} := T - \frac{m}{\bar{u}}$, since $Z(0) = 0$ and a non-decreasing process.
The stochastic control problem

Define the price process $X$ on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ as a strong solution to

$$dX(s) = \mu(s, X(s))ds + \sigma(s, X(s))dW(s), \quad X(t) = x. \quad (2)$$

We define the value function on $S := [0, T] \times [\ell(s)\mathbf{1}(s > \tilde{t}), M] \times \mathbb{R}$.

**Value function: Case I**

$$V(t, z, x) = \sup_{u \in \mathcal{U}_m(t, T)} \mathbb{E}_{txz} \left[ \int_t^T e^{-r(s-t)}(X_s - K)u_s ds \right] \quad (3)$$

**Value function: Case II**

$$V(t, z, x) = \sup_{u \in \mathcal{U}_0(t, T)} \mathbb{E}_{txz} \left[ \int_t^T e^{-r(s-t)}(X_s - K)u_s ds + g(Z(T)) \right] \quad (4)$$
In both cases we have an option paying out the accumulated value of the difference between the price and the strike $K$. This payment is scaled by the production rate. Thus, an option paying money according to price levels and volume decisions.

**Case II**

- Increase the set of admissible controls to $\mathcal{U}_0(t, T)$.
- There is no restriction on the set of admissible controls such that we are guaranteed to have produced a minimal volume of $m$ at maturity.
- Interpret the terminal cost function $g$ as a penalty function if $Z(T) < m$. Define

  $$g(Z(T)) := \alpha(m - Z(T))^+, \quad \alpha < 0.$$  

- We can think of $m$ as an "indirect" minimal constraint, appearing as a penalty boarder.
The HJB equation

Via dynamic programing and a verification theorem, the solution to the control problems is given as a solution to the HJB equation

$$V_t(t, z, x) + \frac{1}{2}\sigma^2(t, x)V_{xx}(t, z, x) + \mu(t, x)V_x(t, z, x) - rV(t, z, x)$$

$$+ \sup_u \{u(t)(x - K + V_z(t, z, x))\} = 0,$$

with appropriate boundary conditions, which are different for the two cases.

Optimal exercise rule

The boundary $x - K + V_z(t, z, x)$ plays a key role. Define

$$\hat{u}(t) = \begin{cases} \bar{u}, & X(t) - K > -V_z(t, Z(t), X(t)), \\ 0, & X(t) - K \leq -V_z(t, Z(t), X(t)), \end{cases}$$

for all $t \in [0, T]$.

We now investigate the marginal value, $\frac{\partial V}{\partial z}$, for case I and II separately.
Case I: Marginal Value.

Define the time for which the total minimal volume constraint is reached as

\[ t_m := \inf \{ s \in [t, T] : Z_s \geq m \} . \]  \hspace{1cm} (8)

**Proposition**

Let \( 0 < m < M < \bar{u} T \), then

\[ \frac{\partial V}{\partial z} \geq 0 \quad \text{for} \quad t \in [0, t_m) \]

\[ \frac{\partial V}{\partial z} \leq 0 \quad \text{for} \quad t \in [t_m, T] \]
Recall the optimal exercise rule $X(t) - K > -V_z(t, Z(t), X(t))$.

If $z < m$:
- The usage of the option increase its value.
- Since the marginal value is non-negative, it is optimal to exercise even for a non-positive payoff.

If $z \geq m$:
- The usage of the option will lower its value.
- Since the marginal value is non-positive, it is optimal to exercise only for non-negative payoff.

If $m = 0$, i.e we have no minimal constraint, then the marginal value is non-positive.

The minimal constraint advance the optimal exercise of the option.

For the case $m \equiv 0$ it has been shown that the introduction of a maximal constraint postpone the optimal exercise.
Case II: Marginal Value.

Recall:

\[ V(t, z, x) = \sup_{u \in U_0(t, T)} E_{txz} \left[ \int_t^T e^{-r(s-t)}(X_s - K)u_s ds + \alpha(m - Z(T))^+ \right] \]  \hspace{1cm} (9)

**Proposition**

For \( M < \tilde{u} T \)

\[ \frac{\partial V}{\partial z}(t, z, x) \leq -\alpha 1(m - z > 0) \]  \hspace{1cm} (10)
Case II: Interpretation

Suppose we start below the penalty boarder, i.e \( \frac{\partial V}{\partial z}(t, z, x) \leq -\alpha \).

- Due to the penalty, the optimal exercise may be advanced.
- The producer may exercise for a negative payoff to avoid penalty, or choose to take a penalty (if that is more profitable).

Suppose we know that \( Z(T) < m \): Then it can be shown that \( \frac{\partial V}{\partial z}(t, z, x) = -\alpha \). I.e, it is optimal to exercise as long as the payoff is greater than \( \alpha < 0 \).
Summary

- Given a mathematical formulation for contracts, to hedge for price and volume risk. This is particularly useful in the powermarket.
- Deduced that the marginal value plays a key role in finding the optimal exercise strategy.
- The introduction of a total maximal volume constraint postpone the optimal exercise policy.
- The introduction of a total minimal volume constraint or a penalty advance the optimal exercise policy.
Some References

