Super-replication under liquidity constraints

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6th General AMaMef and Banach Centre Conference, Warsaw, 13th June 2013

Super-replication problem for the seller of an option

An agent ("seller") promises to pay £W to another agent ("buyer") at time $T > 0$.

The amount £W is random, depending on stock prices at (or up to) T.

Buyer pays deterministic amount c to seller at time 0.

Seller wishes to make sure that (s)he can meet her/his obligations at T (w. p. 1).

What c should (s)he charge to the buyer (at least) ?

Relevance of the above question I

Unless the market is complete, no perfect replication in general.

In practice, c is unrealistically high. Seller should be less stringent and contend with meeting payment obligations with a probability close to 1 (quantile hedging, Föllmer-Leukert).

Alternatively, one may try to control the seller's loss in case of failure: mean-variance hedging, minimising expected shortfall or another risk-measure.

Relevance of the above question II

Natural question.

Superhedging theorems usually rely on the closedness of the set of attainable positions in some topological vector space of random variables.

At the level of mathematics, many approaches to hedging are linked to this closedness result.

Intimate relationship with fundamental questions: existence of arbitrage opportunities, characterization of (dual) pricing functionals, utility maximisation.

Setting

Fix a finite time horizon $T > 0$.

Consider a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\mathcal{F}_t$, $t \in [0, T]$, $\mathcal{F}_T = \mathcal{F}$.

$\mathcal{F}_0$ is trivial, saturated with sets of measure 0, filtration right-continuous.

We will use the optional sigma-algebra $\mathcal{O}$ over $\Omega \times [0, T]$, generated by the class of right-continuous, measurable and adapted processes which have left limits ("càdlàg").

Assets in the economy

Càdlàg, adapted process $S_t, t \in [0, T]$, representing the price of a risky asset.

There is a riskless asset ("bank account") with constant 1 unit price.

Portfolio position at time t is Φ_t in risky asset, Ψ_t in the riskless one.

Frictionless dynamics of the wealth process I

We assume self-financing: the overall value of the portfolio should change only due to price fluctuations, no money is invested or withdrawn.

This means that the wealth increment on a small interval $[t, t+dt]$ should be

$$V_{t+dt} - V_t = \Phi_t(S_{t+dt} - S_t) + \Psi_t(1-1) = \Phi_t(S_{t+dt} - S_t).$$

Hence the wealth dynamics is determined by Φ_t alone. Assuming continuous trading this leads to

Dynamics of the wealth process II

$$V_t = V_0 + \int_0^T \Phi_t dS_t,$$

and we need to stipulate that S_t is a semimartingale and Φ_t is S-integrable (in particular, predictable).

Large class of candidate processes for describing the asset price, but still restrictive. Larger class when frictions are present.

Dual characterisation of superhedging

Theorem. Assume $\mathcal{M}(S) \neq \emptyset$. There is Φ superhedging W from c if and only if

$$c \geq \sup_{Q \in \mathcal{M}(S)} E_Q[W].$$

El Karoui, Quenez, Kramkov, Schäl, Föllmer, Kabanov, Delbaen, Schachermayer.

Succint and logical: $\mathcal{M}(S)$ is the family of equivalent risk-neutral measures, i.e. arbitrage-free pricing rules.

Markets with liquidity constraints

Trades of large volumes may influence prices, at a certain moment it may be difficult to sell or buy a given asset. Small investor approach: their activities do not move prices in a permanent way. The extra trading costs are not necessarily proportional to the trading volume. Change of portfolio value over $[t, t + dt]$ is

$$V_{t+dt} - V_t = S_t[\Phi_{t+dt} - \Phi_t] - G_t(\Phi_{t+dt} - \Phi_t)$$

where G represents the additional cost of trading a large volume, S_t would be the “hypothetical” price if no liquidity issue arose.

Discussion

Such models are around since Kyle (1985). More recently: Schied, Schöneborn, Rogers, Singh, Almgren, Chriss and many others.

Illiquidity loss related to quadratic variation: Cetin, Soner, Touzi. Protter, Jarrow, etc.

Large investor models. Temporal dimension of illiquidity (Cretarola, Gozzi, Pham, Tankov).

Continuous-time model I

Bookkeeping: position in risky asset is V_t^1 at t. Cash position is V_t^0.

The infinitesimal dynamics above leads to the evolution of our cash position as

$$V_t^0 = V_0^0 - \int_0^T S_t d\Phi_t - \int_0^T G_t(\Phi_t')dt.$$

I.e. we assume $\Phi_t = \Phi_0 + \int_0^t f_u du$ with f_u representing trading speed.

Continuous-time model II

Then we get

$$V_t^0 = V_0^0 - \int_0^T S_t f_t dt - \int_0^T G_t(f_t) dt.$$

We assume $f \in \mathcal{A}$ where

$$\mathcal{A} := \{f : f \text{ is } \mathcal{O}\text{-measurable}, \int_0^T |f_t| dt < \infty\}.$$

Continuous-time model III

Position in the risky asset is

$$V_t^1 = V_0^1 + \int_0^T f_t dt.$$

We assume that G is $\mathcal{O} \otimes \mathcal{B}(\mathbb{R})$-measurable, $G_t(\cdot)$ is convex for all t, a.s.

Also, $G_t(\cdot) \geq G_t(0) \geq 0$ and $G_\cdot(\cdot)$ is a.s. locally bounded (slightly less suffices).

$G_t(0) > 0$ is possible: just being present in the market may be costly. We also need that $\sup_t G_t(0)$ is a bounded random variable.

Key assumption: superlinearity

We assume there is an optional process H_t such that $\inf_{t \in [0,T]} H_t$ is an a.s. positive random variable and

$$G_t(x) \geq H_t |x|^a$$

almost surely for some $a > 1$.

Typical specification: $G_t(x) := l\, x^2$ for some $l > 0$.

If we took $G_t := l\, S_t |x|$ we would just get proportional transaction costs.

We fix some $1 < b < a$.

Dual liquidity function

It seems reasonable to suppose that for large volumes trading costs grow in a superlinear way. Our results do not cover (though formally coincide with) the case of transaction costs which is rather "singular".

Fenchel-Legendre conjugate:

$$G_t^*(y) := \sup_{x \in \mathbb{R}} [xy - G_t(x)].$$

If $G_t(x) := l\,x^2$ then $G_t^*(x) := x^2/(4l)$.

Hedging theorem - idealized version

The following result holds for finite Ω:

Theorem. There exists f such that $V_0^0 = c$, $V_0^1 = 0$ and at the terminal date $V_T^0 \geq W$ and $V_T^1 \geq 0$ if and only if one has

$$c \geq E_Q[W] - E_Q[\int_0^T G_t^*(Z_t - S_t)],$$

for all $Q \sim P$ and all non-negative Q-martingales Z.

Usual term + penalty term. In the transaction cost/classical cases this formula reduces to the superhedging theorem in the respective settings (though our result does not subsume those theorems).

Hedging theorem

In general we need to fix a reference probability $Q \in \tilde{\mathcal{P}}(W)$ - this set of probabilities will be defined later.

Theorem. There exists f such that $V_0^0 = c$, $V_0^1 = 0$ and at the terminal date $V_T^0 \geq W$ and $V_T^1 \geq 0$ if and only if one has

$$c \geq E_{Q'}[W] - E_{Q'}[\int_0^T G_t^*(Z_t - S_t)],$$

for all $Q' \ll P$ and positive Q'-martingales Z such that $Z_T(dQ'/dQ)$, dQ'/dQ are bounded.

Integrability condition

Let $\tilde{\mathcal{P}}$ denote the set of $Q \sim P$ such that the random variables

$$\int_0^T H_t^{b/(b-a)}(1+|S_t|)^{ba/(a-b)}dt,$$

$$\int_0^T \sup_{|x|\le N} G_t(x)dt, N \in \mathbb{N}, \quad \int_0^T |S_t|dt,$$

are all Q-integrable. For a (possibly multidimensional) random variable W we define

$$\tilde{\mathcal{P}}(W) := \{Q \in \tilde{\mathcal{P}} : E_Q|W| < \infty\}.$$

Literature

Discrete-time market models: Astic and Touzi, Pennanen and Penner, Dolinsky and Soner.

Continuous-time models: more specific settings, different trading mechanism, Cetin, Soner and Touzi. Limiting procedure: Dolinsky and Soner.

General, intuitive expression. Works without specific assumptions on the model. There is no assumption on S (apart from being càdlàg adapted).

Line of argument - under liquidity constraints

Absence of arbitrage is not necessary for a dual characterisation of the superhedging price.

Moreover, absence of arbitrage can be characterised using the superhedging theorem.

The superhedging price is the supremum of possible (shadow) prices. The formula in the present case suggests that it is actually a supremum of all possible expectations minus a penalization term.

Utility maximisation I

Let $u : \mathbb{R} \to \mathbb{R}$ be concave and nondecreasing. Consider a random endowment W and fix an initial capital c. Let $E|u(c+B+W)| < \infty$ hold where

$$B := \int_0^T G_t^*(-S_t)dt < \infty \text{ a.s.}$$

under our assumptions.

We restrict the set of admissible strategies to

$$\mathcal{A}'(u) = \{f \in \mathcal{A} : V_T^1(c,f) = 0,\ Eu_-(V_T^0(c,f) + W) < \infty\}.$$

Utility maximisation II

Theorem. There is $f^* \in \mathcal{A}'(u)$ such that

$$Eu(V_T^0(c, f^*) + W) = \sup_{f \in \mathcal{A}'(u)} Eu(V_T^0(c, f) + W).$$

General existence theorem. Just like in Cetin and Rogers, there is no need to assume absence of arbitrage, only a concrete integrability condition (ensuring well-posedness).

Conclusion

We derive benefits from the superhedging theorem under liquidity constraints which are similar to the ones we enjoy in the frictionless case.

No scalable arbitrage. Superhedging result holds regardless of having arbitrage or not.

Extensions: general superlinear functions ? (Works in discrete time.) Calculable bounds (certainly trivial) ? Indifference pricing ?