Time-changed Brownian motion and option pricing

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Partially joint with
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Motivation

Stock price process $\{S_t\}_{t \geq 0}$ with known characteristic function $\varphi(u)$ of the log-asset price $\ln(S_T)$.

How to compute the **stock price density** of $S_T$ efficiently?

How to compute $f(k) := \int_{-\infty}^{\infty} \exp(-iku) \varphi(u) \, du$?

Is it possible to price more complicated products like **barrier options**?
Overview

(1) **Time-changed geometric Brownian motion (GBM)**
(2) Pricing barrier options
(3) Pricing call options
(4) Extensions and examples
Time-changed GBM

Definition

Consider a geometric Brownian motion (GBM)

\[
\frac{dS_t}{S_t} = rd_t + \sigma dW_t, \quad (1.1)
\]

where \( r \in \mathbb{R}, \sigma > 0, \) and \( W_t \) is a standard Brownian motion.

Introduce a stochastic clock \( \Lambda = \{\Lambda_t\}_{t \geq 0} \) (independent of \( S \)) and consider \( S_{\Lambda_t} \) instead of \( S_t \).

Definition 1.1 (Time-changed Brownian motion) Let \( \Lambda = \{\Lambda_t\}_{t \geq 0} \) be an increasing stochastic process with \( \Lambda_0 = 0, \lim_{t \to \infty} \Lambda_t = \infty \) \( \mathbb{Q} \)-a.s.. This stochastic time-scale is used to time-change \( S \), i.e. we consider the process \( S_{\Lambda_t}, \) for \( t \geq 0 \).

Denote the Laplace transform of \( \Lambda_T \) by \( \vartheta_T(u) := \mathbb{E}[\exp(-u\Lambda_T)], \) \( u \geq 0 \).
Time-changed GBM

Motivation

Time-changed Brownian motion is convenient since:

— Natural interpretation of time-change as measure of economic activity (‘business time scale’, ’transaction clock’).

— Many well-known models can be represented as a time-changed Brownian motion (e.g. Variance Gamma, Normal inverse Gaussian). This covers not only Lévy-type models, but also regime-switching, Sato, or stochastic volatility models.
If the time change $\{\Lambda_t\}_{t \geq 0}$ is continuous, it is possible to derive the first-passage time of $\{S_{\Lambda_t}\}_{t \geq 0}$ analytically following Hieber and Scherer [2012].
Time-changed GBM

Motivation

call options

down-and-out call options
(=barrier options)
Time-changed GBM
Example 1: Variance Gamma model

The Variance Gamma process, also known as Laplace motion, is obtained if a GBM (drift $\theta$, volatility $\sigma > 0$) is time-changed by a $\text{Gamma}(t; 1/\nu, \nu)$ process, $\nu > 0$. The drift adjustment due to the jumps is $\omega := \ln \left(1 - \theta \nu - \sigma^2 \nu/2\right)/\nu$. 

![Sample paths of time-changed GBM](image)
Time-changed GBM

Example 2: Markov switching model

The Markov switching model (see, e.g., Hamilton [1989]):

\[
\frac{dS_t}{S_t} = r dt + \sigma_{Z_t} dW_t, \quad S_0 > 0,
\]

(1.2)

where \( Z = \{Z_t\}_{t \geq 0} \in \{1, 2, \ldots, M\} \) is a time-homogeneous Markov chain with intensity matrix \( Q_0 \) and \( W = \{W_t\}_{t \geq 0} \) an independent Brownian motion.
The class of time-changed GBM is rich. It also contains


— The Normal inverse Gaussian model.

— Sato models: For example extensions of the Variance Gamma model.

— The Ornstein-Uhlenbeck process.

The class is restricted by the fact that the time change $\{\Lambda_t\}_{t \geq 0}$ is independent of the stock price process $\{S_t\}_{t \geq 0}$. 
Overview

(1) Time-changed geometric Brownian motion (GBM)

(2) Pricing barrier options

(3) Pricing call options

(4) Extensions and examples
Pricing barrier options

**Barrier options** with payoff

\[
1_{\{D < S_t < P \text{ for } 0 \leq t \leq T\}} \max(S_T - K, 0).
\]

Sample path of \( \{S_t\}_{t \geq 0} \) with a lower barrier \( D \) and an upper barrier \( P \).
A **transition density** describes the probability density that the process $S$ starts at time 0 at $S_0$, stays within the corridor $[D, P]$ until time $T > 0$ and ends up at $S_T$ at time $T$.

(This of course implies that $S_0 \in (D, P)$ and $S_T \in (D, P)$.)

More formally,

$$p(T, S_0, S_T) := \mathbb{Q}(S_T \in dx, \ D < S_t < P \text{ for } 0 \leq t \leq T \mid S_0 = s_0).$$
Pricing barrier options
Transition density

Lemma 1.2 (Transition density GBM)
Consider $S = \{S_t\}_{t \geq 0}$ with drift $r \in \mathbb{R}$ and volatility $\sigma > 0$. $S$ starts at $S_0$, stays within the corridor $(D, P)$ until time $T$ and ends up in $S_T$. Its transition density is

$$p(T, S_0, S_T) = \frac{2 \exp\left(\frac{\mu}{\sigma^2} \ln(S_T/S_0)\right)}{\ln(P/D)} \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi \ln(S_T)}{\ln(P/D)}\right) \exp(-\lambda_n T).$$

where

$$\lambda_n := \frac{1}{2} \left(\frac{\mu^2}{\sigma^2} + \frac{n^2 \pi^2 \sigma^2}{\ln(P/D)^2}\right), \quad A_n := \sin\left(\frac{n\pi \ln(S_0)}{\ln(P/D)}\right), \quad \mu := r - \frac{1}{2} \sigma^2.$$

Proof: Cox and Miller [1965], see also Pelsser [2000].
Pricing barrier options
Transition density

Theorem 1.3 (Transition density time-changed GBM)
Consider \( S = \{S_t\}_{t \geq 0} \) with drift \( r \in \mathbb{R} \) and volatility \( \sigma > 0 \), time-changed by independent \( \{\Lambda_t\}_{t \geq 0} \) with Laplace transform \( \vartheta_T(u) \). \( S \) starts at \( S_0 \), stays within the corridor \( (D,P) \) until time \( T \) and ends up in \( S_T \). Its transition density is

\[
p(T, S_0, S_T) = \frac{2 \exp\left(\frac{\mu}{\sigma^2} \ln(S_T/S_0)\right)}{\ln(P/D)} \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi \ln(S_T)}{\ln(P/D)}\right) \vartheta_T\left(\lambda_n\right).
\]

where

\[
\lambda_n := \frac{1}{2} \left(\frac{\mu^2}{\sigma^2} + \frac{n^2 \pi^2 \sigma^2}{\ln(P/D)^2}\right), \quad A_n := \sin\left(\frac{n\pi \ln(S_0)}{\ln(P/D)}\right), \quad \mu := r - \frac{1}{2} \sigma^2.
\]
Pricing barrier options
Transition density

Proof 1 (Transition density time-changed GBM)

If the time-change \( \{\Lambda_t\}_{t \geq 0} \) is continuous, we are – conditional on \( \Lambda_T \) – back in the case of Brownian motion.

Then, by Lemma 1.2

\[
p(\Lambda_T, S_0, x) = \text{const.} \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi \ln(x)}{\ln(P/D)}\right) \exp(-\lambda_n \Lambda_T).
\]

From this,

\[
E_Q[p(\Lambda_T, S_0, x)] = \text{const.} \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi \ln(x)}{\ln(P/D)}\right) E[\exp(-\lambda_n \Lambda_T)]
\]

\[
= \text{const.} \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi \ln(x)}{\ln(P/D)}\right) \vartheta_T(\lambda_n).
\]
Pricing barrier options

**Theorem 1.4 (Barrier options, Escobar/Hieber/Scherer (2013))**

Consider \( S = \{S_t\}_{t \geq 0} \) with drift \( r \in \mathbb{R} \) and volatility \( \sigma > 0 \), continuously time-changed by independent \( \{\Lambda_t\}_{t \geq 0} \) with Laplace transform \( \vartheta_T(u) \). \( S \) starts at \( S_0 \). Conditional on \( \{D < S_t < P, \text{ for } 0 \leq t \leq T\} \), the price of a down-and-out call option with strike \( K \) and maturity \( T \) is

\[
\text{DOC}(0) = \frac{2}{\ln(P/D)} \sum_{n=1}^{\infty} \vartheta_T(\lambda_n) A_n \cdot \\
\cdot \int_{D}^{P} \max(S_T - K, 0) \sin\left(\frac{n\pi \ln(S_T)}{\ln(P/D)}\right) \exp\left(\frac{\mu}{\sigma^2} \ln(S_T/S_0)\right) dS_T,
\]

where

\[
\lambda_n := \frac{1}{2} \left( \frac{\mu^2}{\sigma^2} + \frac{n^2 \pi^2 \sigma^2}{\ln(P/D)^2} \right), \quad A_n := \sin\left(\frac{n\pi \ln(S_0)}{\ln(P/D)}\right), \quad \mu := r - \frac{1}{2} \sigma^2.
\]
### Pricing barrier options

**Proof 2 (Barrier options)**

\[
DOC(0) = \int_D \max(S_T - K, 0) \ p(T, S_0, S_T) \ dS_T
\]

\[
= \text{const.} \sum_{n=1}^{\infty} A_n \varphi_T(\lambda_n) \int_D \max(S_T - K, 0) \ \sin\left(\frac{n\pi \ln(S_T)}{\ln(P/D)}\right) \ \exp\left(\frac{\mu}{\sigma^2} \ln(S_T/S_0)\right) \ dS_T.
\]

The integral \( \int_D \max(S_T - K, 0) \ \sin\left(\frac{n\pi \ln(S_T)}{\ln(P/D)}\right) \ dS_T \) can be computed explicitly.

**The same ideas apply to any other down-and-out contract**

(e.g. **bonus certificates**, **digital options**).
Pricing barrier options
Numerical example

Implementation:

\[
DOC(0) = \text{const.} \sum_{n=1}^{\infty} f_n(K) \vartheta_T(\lambda_n) \\
\approx \text{const.} \sum_{n=1}^{N} f_n(K) \vartheta_T(\lambda_n).
\]

Error bounds for the truncation parameter \( N \) are available for many models.
Overview

call options

down-and-out call options
(=barrier options)
Pricing call options

Sample path of $\{S_t\}_{t \geq 0}$ with a lower barrier $D$ and an upper barrier $P$. 
Pricing call options

Sample path of $\{S_t\}_{t \geq 0}$ with a lower barrier $D$ and an upper barrier $P$. 
Pricing call options

Sample path of \( \{ S_t \}_{t \geq 0} \) with a lower barrier \( D \) and an upper barrier \( P \).

A barrier option can **approximate** a call option, i.e.

\[
\mathbb{1}_{\{ D < S_t < P \text{ for } 0 \leq t \leq 1 \}} \max(S_1 - K, 0) \approx \max(S_1 - K, 0).
\]
Pricing call options
Numerical example

(Vanilla) **Call options** can be approximated by **barrier options**.
Again: **Black-Scholes** model \((r = 0, \sigma = 0.2), \, T = 1, \, K = 80\).

<table>
<thead>
<tr>
<th>((D; P))</th>
<th><strong>barrier price</strong></th>
<th>(N)</th>
<th>comp. time</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.7; 1.3)</td>
<td>12.21580525385</td>
<td>7</td>
<td>0.1ms</td>
</tr>
<tr>
<td>(0.6; 1.4)</td>
<td>13.08137347245</td>
<td>9</td>
<td>0.1ms</td>
</tr>
<tr>
<td>(0.4; 2.7)</td>
<td>21.18586311986</td>
<td>22</td>
<td>0.1ms</td>
</tr>
<tr>
<td>(0.1; 7.4)</td>
<td><strong>21.18592951321</strong></td>
<td>44</td>
<td>0.1ms</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>call price</strong></th>
<th>comp. time</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>21.18592951321</strong></td>
<td>1.2ms</td>
</tr>
</tbody>
</table>

— Computation of barrier options faster than Black-Scholes formula\(^a\).

— Accuracy of approximation is very high.

\(^a\)The call option was priced using `blaprice.m` in Matlab (version 2009a).
Overview

(1) Time-changed geometric Brownian motion (GBM)
(2) Pricing barrier options
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(4) Extensions and examples
Numerical example

Stock price process $\{S_t\}_{t \geq 0}$
with known characteristic function $\varphi(u)$
of the log-asset price $\ln(S_T)$.

How to compute
$$\mathbb{E}[(S_T - K)^+] := \int_0^\infty \exp(-iuk) \rho(\varphi(u), u) \, du.$$
Numerical example
Alternatives

— **Fast Fourier pricing**: Most popular approach, see Carr and Madan [1999]. Many extensions, e.g., Raible [2000], Chourdakis [2004].

\[
\mathbb{E}[(S_T - K)^+] \approx \text{const.} \sum_{n=1}^{N} B_n \, C_{BS}(\mu_n, \sigma_n, K).
\]

— **Black-Scholes (BS) approximation**: Works for time-changed Brownian motion, see Albrecher et al. [2013].

\[
\mathbb{E}[(S_T - K)^+] \approx \text{const.} \sum_{n=1}^{N} B_n \, C_{BS}(\mu_n, \sigma_n, K).
\]

— **COS Method**: Closest to our approach, see Fang and Oosterlee [2008].

\[
\mathbb{E}[(S_T - K)^+] \approx \text{const.} \sum_{n=1}^{N} C_n(K) \, \text{Re}\left(\varphi\left(\frac{n\pi}{a-b}\right) e^{-in\pi\frac{b}{a-b}}\right).
\]


\[
\mathbb{E}[(S_T - K)^+] \approx \text{const.} \sum_{n=1}^{N} D_n(K) \left(\sum_{m=1}^{M} c_m \left(\frac{e^{x_n}}{x_n + d_m}\right) \vartheta_T(x_n)\right).
\]
Numerical example
Parameter set

**Variance Gamma model**

<table>
<thead>
<tr>
<th>⊖ parameter set</th>
<th>⊖ parameter set</th>
<th>⊖ parameter set</th>
</tr>
</thead>
<tbody>
<tr>
<td>θ</td>
<td>-0.10</td>
<td>-0.20</td>
</tr>
<tr>
<td>ν</td>
<td>0.10</td>
<td>0.20</td>
</tr>
<tr>
<td>σ</td>
<td>0.15</td>
<td>0.30</td>
</tr>
<tr>
<td>T</td>
<td>0.10</td>
<td>0.25</td>
</tr>
</tbody>
</table>

**Markov switching model**

<table>
<thead>
<tr>
<th>⊖ parameter set</th>
<th>⊖ parameter set</th>
<th>⊖ parameter set</th>
</tr>
</thead>
<tbody>
<tr>
<td>σ₁</td>
<td>0.10</td>
<td>0.20</td>
</tr>
<tr>
<td>σ₂</td>
<td>0.10</td>
<td>0.15</td>
</tr>
<tr>
<td>λ₁</td>
<td>0.10</td>
<td>0.50</td>
</tr>
<tr>
<td>λ₂</td>
<td>0.10</td>
<td>1.00</td>
</tr>
<tr>
<td>T</td>
<td>0.10</td>
<td>0.25</td>
</tr>
</tbody>
</table>

The parameters sets were obtained from Chourdakis [2004].

We use 31 equidistant strikes \( K \) out of \([85, 115]\), the current price is \( S_0 = 100 \).

The rows ⊖ and ⊖ allow us to test many different parameter sets to adequately compare the different numerical techniques.
Numerical example
Results I: Pricing call options

**Variance Gamma model** (char. fct. decays hyperbolically)

<table>
<thead>
<tr>
<th></th>
<th>our approach</th>
<th>FFT</th>
<th>COS method</th>
<th>BS approx.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N )</td>
<td>100</td>
<td>4096</td>
<td>200</td>
<td>10</td>
</tr>
<tr>
<td>average comp. time</td>
<td>0.5ms</td>
<td>4.9ms</td>
<td>1.4ms</td>
<td>0.3ms</td>
</tr>
<tr>
<td>average rel. error</td>
<td>4.5e-08</td>
<td>2.0e-07</td>
<td>3.5e-07</td>
<td>5.4e-05</td>
</tr>
<tr>
<td>max. rel. error</td>
<td>2.7e-07</td>
<td>5.8e-07</td>
<td>2.6e-06</td>
<td>3.0e-04</td>
</tr>
<tr>
<td>sample price</td>
<td>20.76524</td>
<td>20.76523</td>
<td>20.76524</td>
<td>20.76105</td>
</tr>
</tbody>
</table>

Numerical comparison on different parameter sets following [Chourdakis 2004](#).
A sample price was obtained using \( K = 80 \) and the average parameter set from slide 26. The barriers \((D; P)\) were set to \((\exp(-3); \exp(3))\).
Numerical example
Results II: Pricing call options

Absolute error vs. number of terms $N$: Variance Gamma model.
Numerical example
Results III: Pricing call options

Absolute error vs. number of terms $N$: Markov switching model.
Numerical example
Results IV: Pricing call options

Logarithmic error vs. number of terms $N$: Markov switching model.
Numerical example

Discussion

— Our approach and the Fang and Oosterlee [2008] results are extremely fast for quickly (e.g. exponentially) decaying characteristic functions.

— High accuracy (e.g. 1e–10) is possible since one avoids any kind of discretization. Error bounds are available.

— Evaluation of several strikes comes at almost no cost.

— Apart from option pricing, one is able to evaluate densities or distributions with known characteristic function.
Summary

continuously time-changed Brownian motion

continuously time-changed Brownian motion
Summary

call options

down-and-out call options
(=barrier options)


Discontinuous time-change

Example of a discontinuous time-change. While the original process $\{B_t\}_{t \geq 0}$ (black) hits the barrier, the time-changed process $\{B_{\Lambda_t}\}_{t \geq 0}$ (grey) does not. This is not possible if the time-change is continuous; then all barrier crossings are observed until time $\Lambda_T$. 

![Diagram showing Brownian motion and time change]

Brownian motion

calendar time $t$

Time change $\Lambda_t$

calendar time $t$