Time-changed Brownian motion and option pricing

Peter Hieber
Chair of Mathematical Finance, TU Munich

6th AMaMeF Warsaw, June 13th 2013

Partially joint with
Marcos Escobar (RU Toronto),
Matthias Scherer (TU Munich).

Motivation

Stock price process $\{S_t\}_{t\geq 0}$
with **known characteristic function** $\varphi(u)$
of the log-asset price $\ln(S_T)$.

How to compute the
stock price density of S_T
efficiently?

How to compute
$f(k) := \int_{-\infty}^{\infty} \exp(-iuk)\,\varphi(u)\,du$?

Is it possible
to price more complicated products
like **barrier options**?

Overview

(1) **Time-changed geometric Brownian motion (GBM)**

(2) Pricing barrier options

(3) Pricing call options

(4) Extensions and examples

Time-changed GBM
Definition

Consider a geometric Brownian motion (**GBM**)

$$\frac{dS_t}{S_t} = r dt + \sigma dW_t, \tag{1.1}$$

where $r \in \mathbb{R}$, $\sigma > 0$, and W_t is a standard Brownian motion.

Introduce a stochastic clock $\Lambda = \{\Lambda_t\}_{t \geq 0}$ (independent of S) and consider S_{Λ_t} instead of S_t.

Definition 1.1 (Time-changed Brownian motion) *Let $\Lambda = \{\Lambda_t\}_{t \geq 0}$ be an increasing stochastic process with $\Lambda_0 = 0$, $\lim_{t \nearrow \infty} \Lambda_t = \infty$ $\mathbb{Q}$-a.s.. This stochastic time-scale is used to time-change S, i.e. we consider the process S_{Λ_t}, for $t \geq 0$.*

Denote the Laplace transform of Λ_T by $\vartheta_T(u) := \mathbb{E}[\exp(-u\Lambda_T)]$, $u \geq 0$.

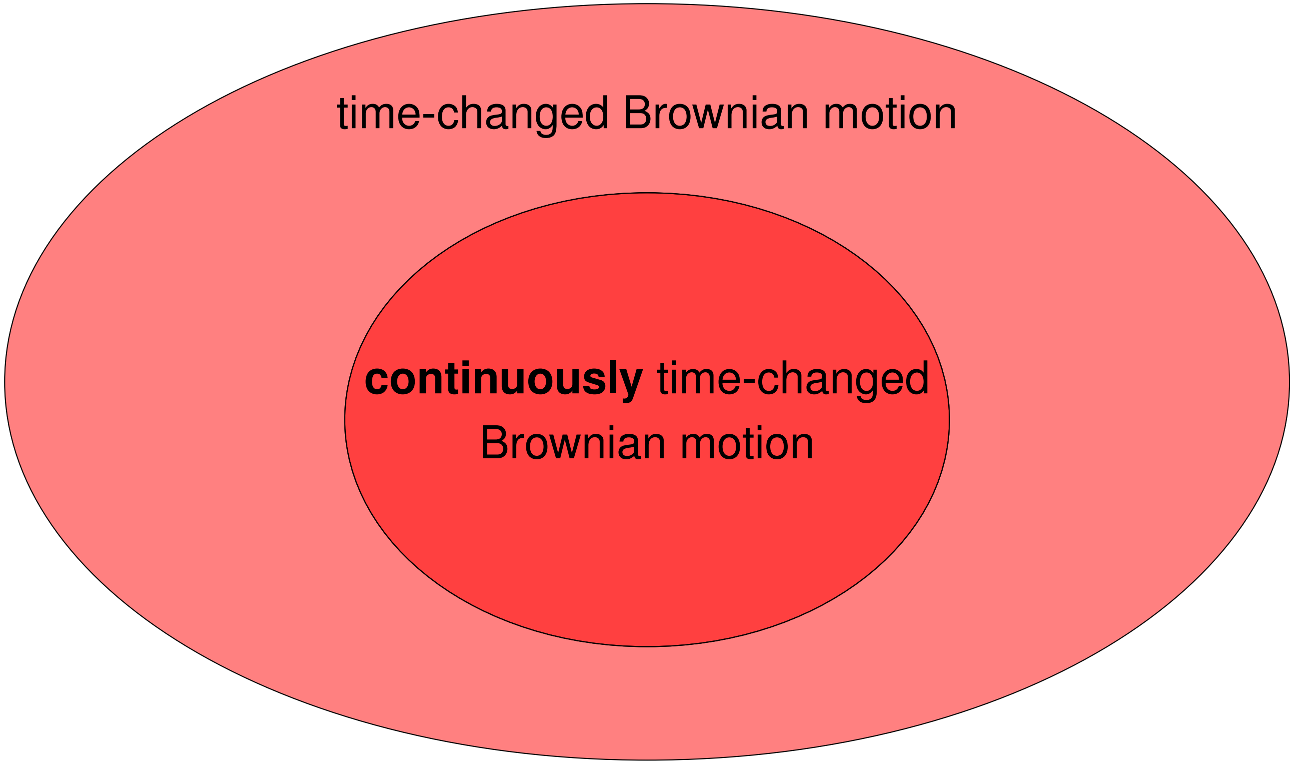
Time-changed GBM
Motivation

Time-changed Brownian motion is **convenient** since:

— Natural interpretation of time-change as measure of economic activity ('**business time scale', 'transaction clock')**.

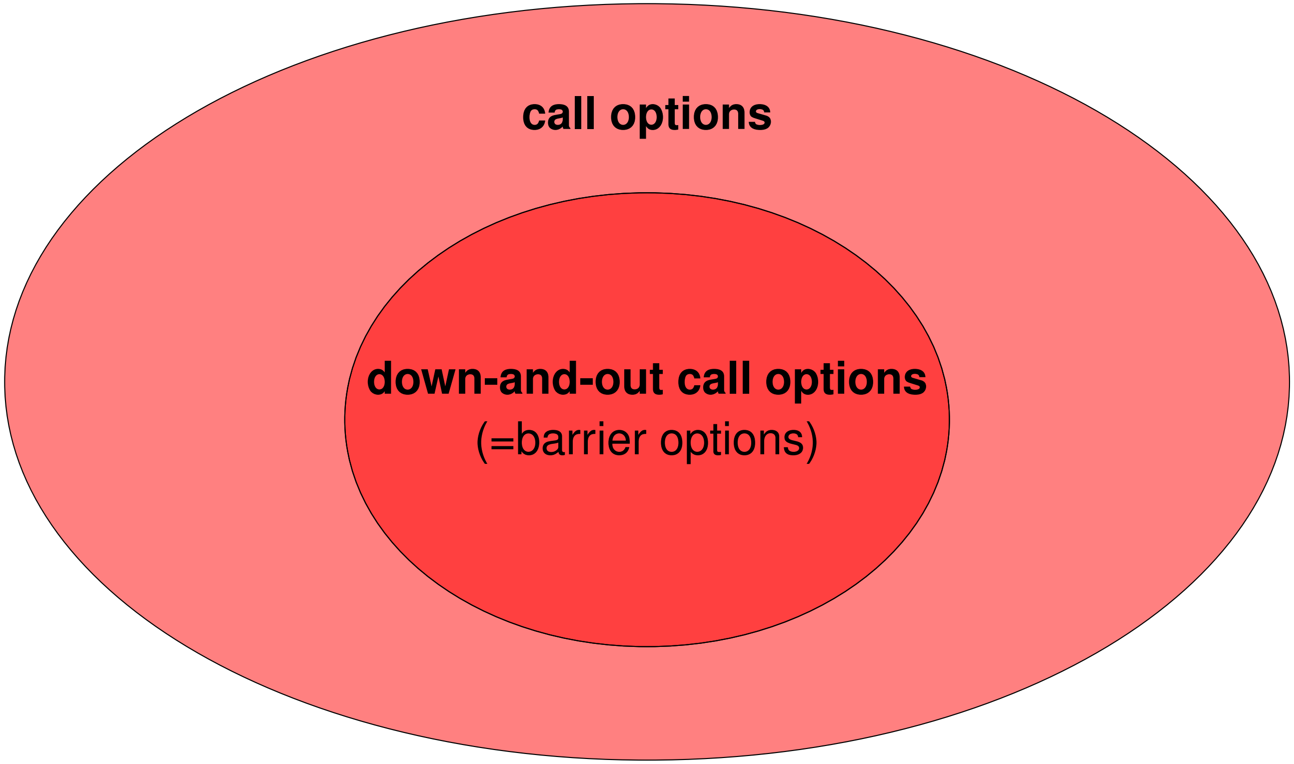
— Many well-known models can be represented as a time-changed Brownian motion (e.g. **Variance Gamma, Normal inverse Gaussian**). This covers not only Lévy-type models, but also **regime-switching, Sato**, or **stochastic volatility** models.

Time-changed GBM
Motivation



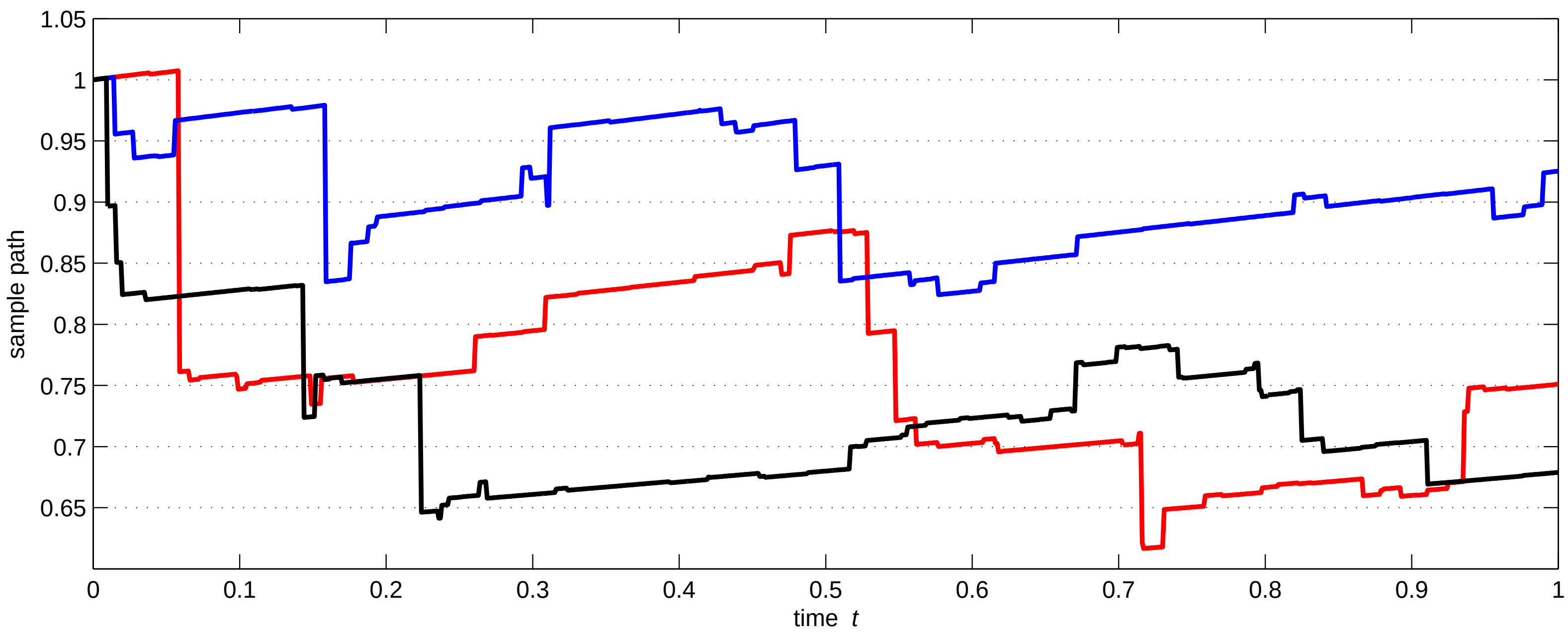
If the time change $\{\Lambda_t\}_{t\geq 0}$ is continuous, it is possible to derive the first-passage time of $\{S_{\Lambda_t}\}_{t\geq 0}$ analytically following Hieber and Scherer [2012].

Time-changed GBM
Motivation



Time-changed GBM
Example 1: Variance Gamma model

The Variance Gamma process, also known as **Laplace motion**, is obtained if a GBM (drift θ, volatility $\sigma > 0$) is time-changed by a Gamma$(t; 1/\nu, \nu)$ process, $\nu > 0$. The drift adjustment due to the jumps is $\omega := \ln\left(1 - \theta\nu - \sigma^2\nu/2\right)/\nu$.



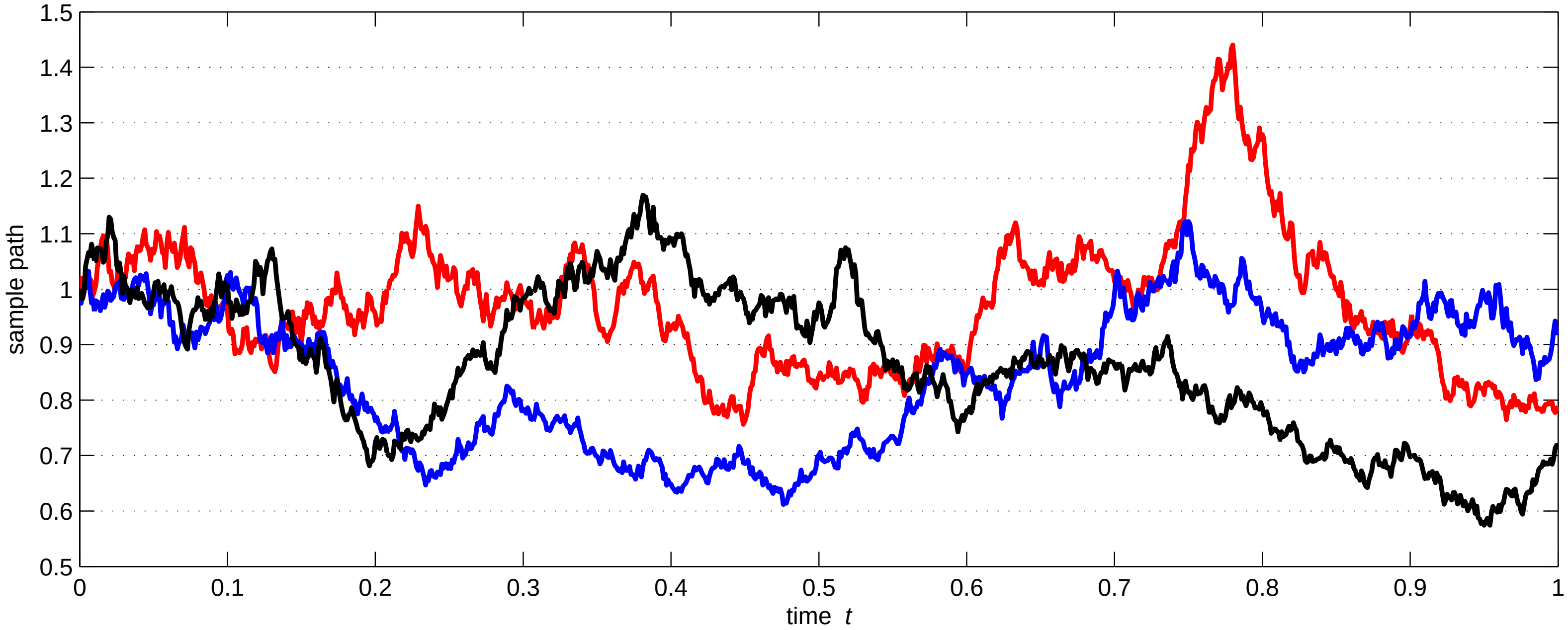
Time-changed GBM

Example 2: **Markov switching model**

The **Markov switching model** (see, e.g., **Hamilton** [**1989**]):

$$\frac{dS_t}{S_t} = r\,dt + \sigma_{Z_t} dW_t, \quad S_0 > 0, \tag{1.2}$$

where $Z = \{Z_t\}_{t \geq 0} \in \{1, 2, \ldots, M\}$ is a time-homogeneous Markov chain with intensity matrix Q_0 and $W = \{W_t\}_{t \geq 0}$ an independent Brownian motion.



Time-changed GBM
Further examples

The class of **time-changed GBM** is rich. It also contains

- Stochastic volatility models: **Heston model**, **Stein & Stein model**, **Hull-White model**, certain continuous limits of **GARCH models**.
- The Normal inverse Gaussian model.
- Sato models: For example extensions of the Variance Gamma model.
- The Ornstein-Uhlenbeck process.

The class is restricted by the fact that the time change $\{\Lambda_t\}_{t\geq 0}$ is independent of the stock price process $\{S_t\}_{t\geq 0}$.

Overview

(1) Time-changed geometric Brownian motion (GBM)

(2) **Pricing barrier options**

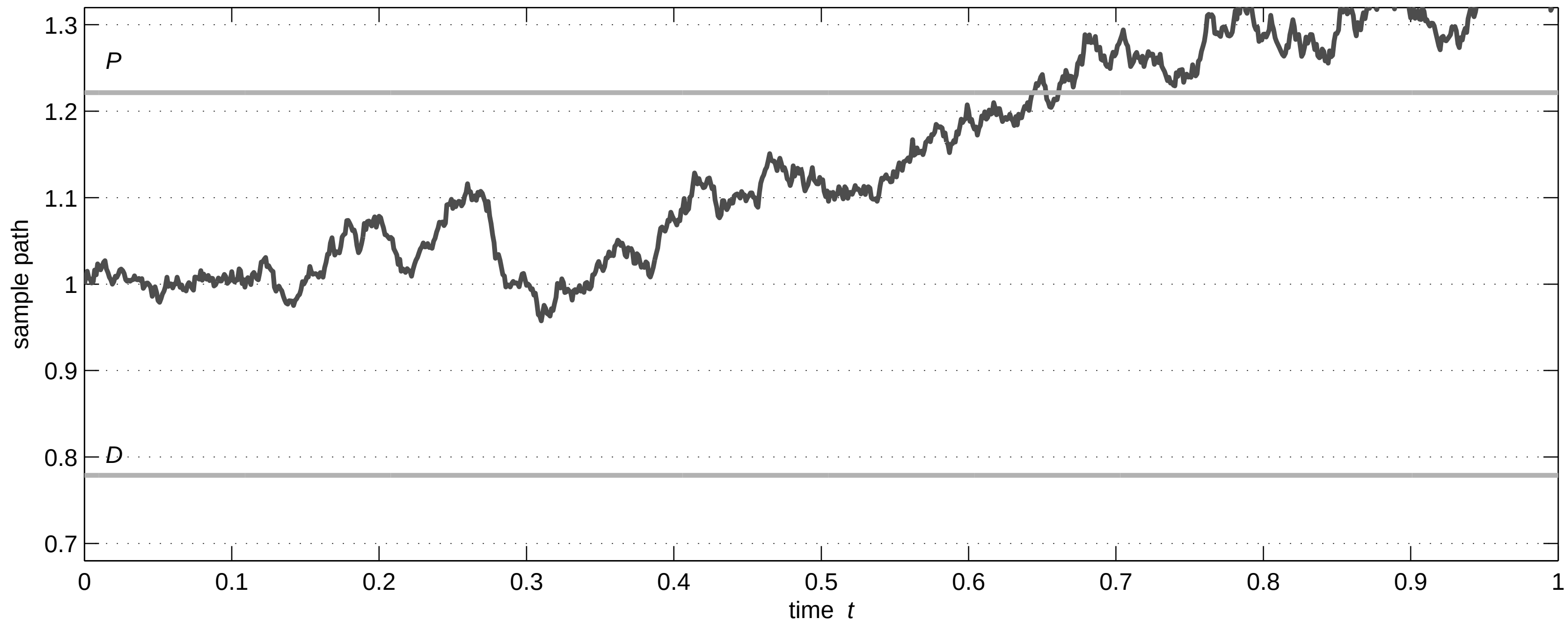
(3) Pricing call options

(4) Extensions and examples

Pricing barrier options

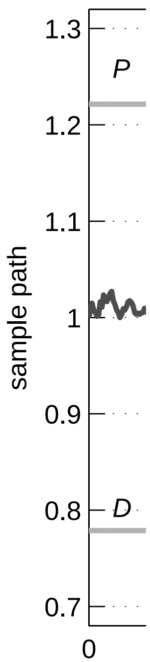
Barrier options with payoff

$$\mathbb{1}_{\{D<S_t<P \text{ for } 0\le t\le T\}} \max(S_T-K,0)\,.$$



Sample path of $\{S_t\}_{t\ge 0}$ with a lower barrier D and an upper barrier P.

Pricing barrier options
Transition density



A **transition density** describes the probability density that the process S starts at time 0 at S_0, stays within the corridor $[D, P]$ until time $T > 0$ and ends up at S_T at time T.

(This of course implies that $S_0 \in (D, P)$ and $S_T \in (D, P)$.)

More formally,

$$p(T, S_0, S_T) := \mathbb{Q}\big(S_T \in dx,\ D < S_t < P \text{ for } 0 \le t \le T \,\big|\, S_0 = s_0\big)\,.$$

Pricing barrier options
Transition density

Lemma 1.2 (Transition density GBM)

Consider $S = \{S_t\}_{t \geq 0}$ *with drift* $r \in \mathbb{R}$ *and volatility* $\sigma > 0$. S *starts at* S_0*, stays within the corridor* (D, P) *until time* T *and ends up in* S_T*. Its transition density is*

$$p(T, S_0, S_T) = \frac{2 \exp\left(\frac{\mu}{\sigma^2} \ln(S_T/S_0)\right)}{\ln(P/D)} \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi \ln(S_T)}{\ln(P/D)}\right) \exp(-\lambda_n T)\,.$$

where

$$\lambda_n := \frac{1}{2}\left(\frac{\mu^2}{\sigma^2} + \frac{n^2\pi^2\sigma^2}{\ln(P/D)^2}\right), \quad A_n := \sin\left(\frac{n\pi \ln(S_0)}{\ln(P/D)}\right), \quad \mu := r - \frac{1}{2}\sigma^2\,.$$

Proof: Cox and Miller [1965], see also Pelsser [2000].

Pricing barrier options
Transition density

Theorem 1.3 (Transition density time-changed GBM)
Consider $S = \{S_t\}_{t\geq 0}$ *with drift* $r \in \mathbb{R}$ *and volatility* $\sigma > 0$*, time-changed by independent* $\{\Lambda_t\}_{t\geq 0}$ *with Laplace transform* $\vartheta_T(u)$. S *starts at* S_0*, stays within the corridor* (D, P) *until time* T *and ends up in* S_T*. Its transition density is*

$$p(T, S_0, S_T) = \frac{2\exp\left(\frac{\mu}{\sigma^2}\ln(S_T/S_0)\right)}{\ln(P/D)} \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi\ln(S_T)}{\ln(P/D)}\right) \vartheta_T(\lambda_n)\,.$$

where

$$\lambda_n := \frac{1}{2}\left(\frac{\mu^2}{\sigma^2} + \frac{n^2\pi^2\sigma^2}{\ln(P/D)^2}\right), \quad A_n := \sin\left(\frac{n\pi\ln(S_0)}{\ln(P/D)}\right), \quad \mu := r - \frac{1}{2}\sigma^2\,.$$

Pricing barrier options
Transition density

Proof 1 (Transition density time-changed GBM)

If the time-change $\{\Lambda_t\}_{t\geq 0}$ is continuous, we are – conditional on Λ_T – back in the case of Brownian motion.

Then, by Lemma 1.2

$$p(\Lambda_T, S_0, x) = const. \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi \ln(x)}{\ln(P/D)}\right) \exp(-\lambda_n \Lambda_T) \,.$$

From this,

$$\mathbb{E}_{\mathbb{Q}}\left[p(\Lambda_T, S_0, x)\right] = const. \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi \ln(x)}{\ln(P/D)}\right) \mathbb{E}\left[\exp(-\lambda_n \Lambda_T)\right]$$

$$= const. \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi \ln(x)}{\ln(P/D)}\right) \vartheta_T(\lambda_n) \,.$$

Pricing barrier options

Theorem 1.4 (Barrier options, Escobar/Hieber/Scherer (2013))
Consider $S = \{S_t\}_{t\geq 0}$ *with drift* $r \in \mathbb{R}$ *and volatility* $\sigma > 0$*, continuously time-changed by independent* $\{\Lambda_t\}_{t\geq 0}$ *with Laplace transform* $\vartheta_T(u)$*.* S *starts at* S_0*. Conditional on* $\{D < S_t < P,$ *for* $0 \leq t \leq T\}$*, the price of a down-and-out call option with strike* K *and maturity* T *is*

$$DOC(0) = \frac{2}{\ln(P/D)} \sum_{n=1}^{\infty} \vartheta_T(\lambda_n)\, A_n \cdot$$

$$\cdot \int_D^P \max(S_T - K, 0)\, \sin\left(\frac{n\pi \ln(S_T)}{\ln(P/D)}\right) \exp\left(\frac{\mu}{\sigma^2} \ln(S_T/S_0)\right) dS_T\,,$$

where

$$\lambda_n := \frac{1}{2}\left(\frac{\mu^2}{\sigma^2} + \frac{n^2\pi^2\sigma^2}{\ln(P/D)^2}\right), \quad A_n := \sin\left(\frac{n\pi \ln(S_0)}{\ln(P/D)}\right), \quad \mu := r - \frac{1}{2}\sigma^2\,.$$

Pricing barrier options

Proof 2 (Barrier options)

$$DOC(0) = \int_D^P \max\big(S_T - K, 0\big)\, p(T, S_0, S_T)\, dS_T$$

$$= const. \sum_{n=1}^{\infty} A_n\, \vartheta_T(\lambda_n) \int_D^P \max\big(S_T - K, 0\big)\, \sin\left(\frac{n\pi \ln(S_T)}{\ln(P/D)}\right) \exp\left(\frac{\mu}{\sigma^2} \ln(S_T/S_0)\right) dS_T\,.$$

The integral $\int_D^P \max\big(S_T - K, 0\big)\, \sin\left(\frac{n\pi \ln(S_T)}{\ln(P/D)}\right) dS_T$ can be computed explicitly.

*The same ideas apply to any other down-and-out contract (e.g. **bonus certificates, digital options**).*

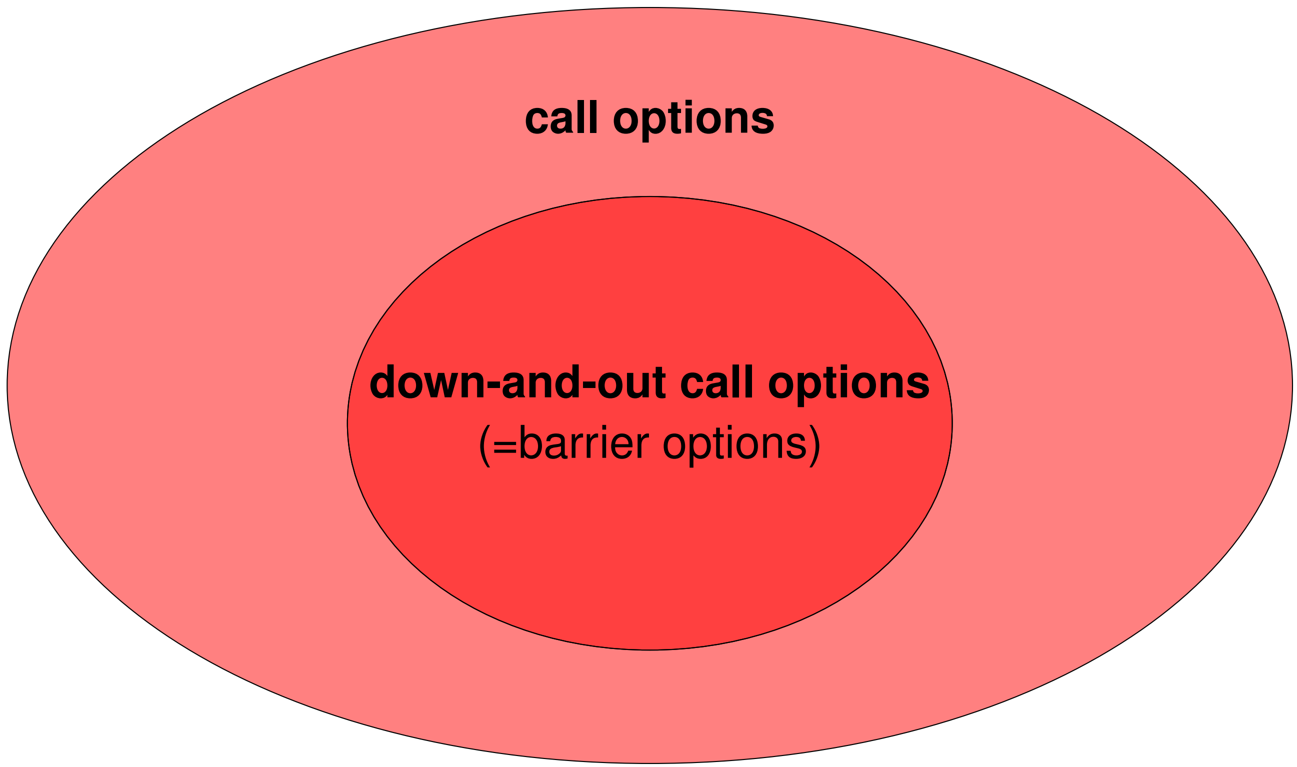
Pricing barrier options
Numerical example

Implementation:

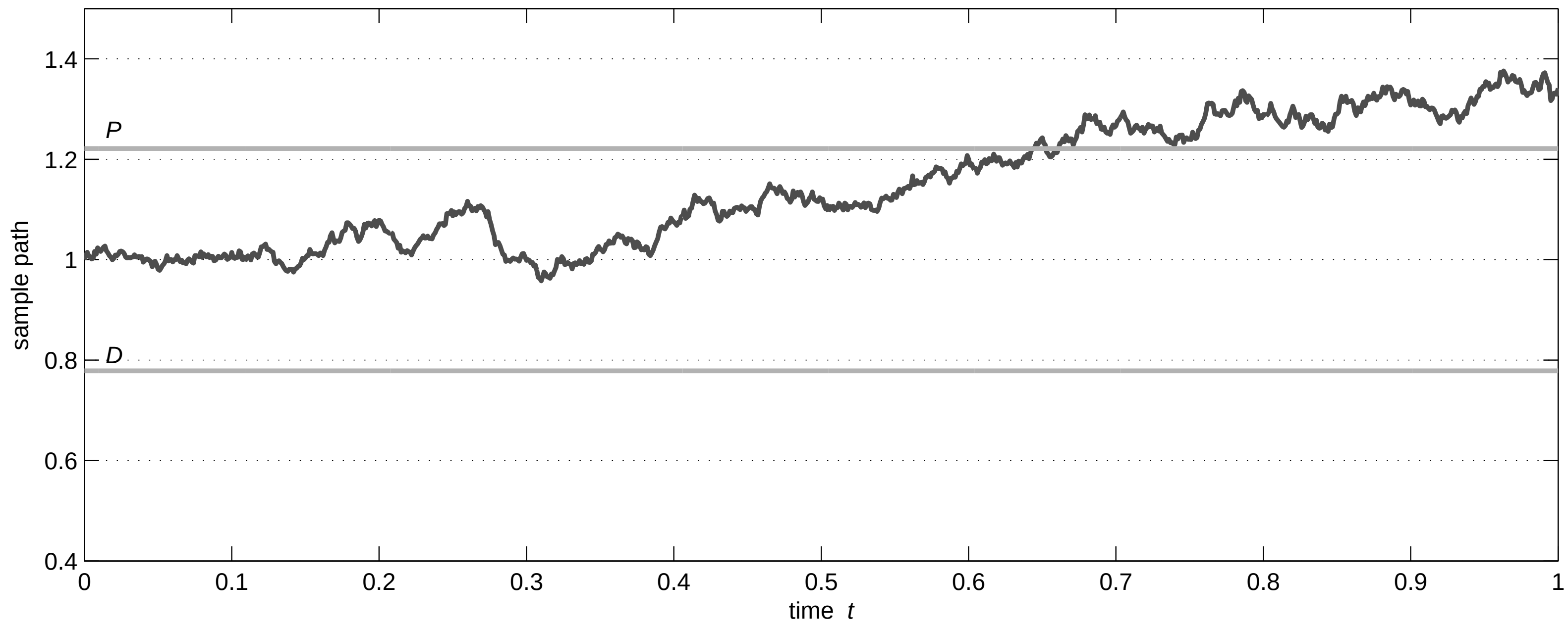
$$DOC(0) = const. \sum_{n=1}^{\infty} f_n(K)\, \vartheta_T(\lambda_n)$$
$$\approx const. \sum_{n=1}^{\mathbf{N}} f_n(K)\, \vartheta_T(\lambda_n) \ .$$

Error bounds for the truncation parameter N are available for many models.

Overview

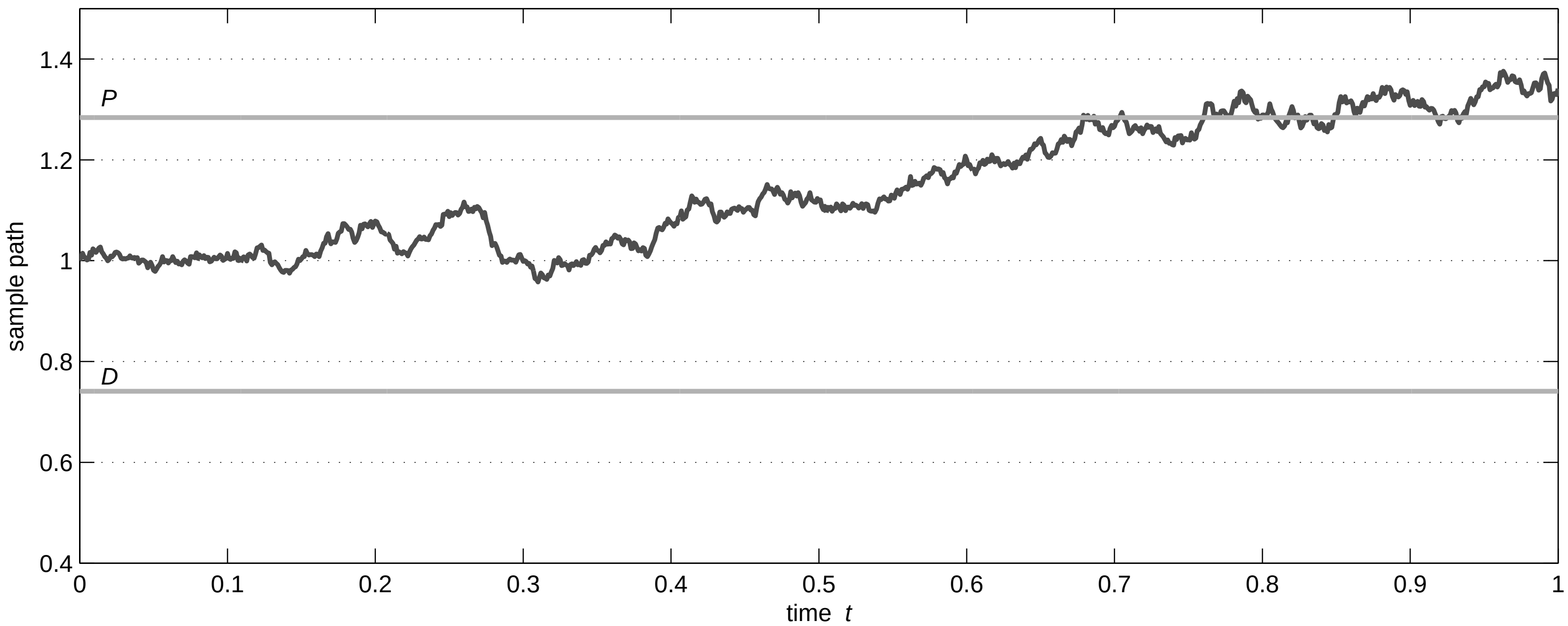


Pricing call options



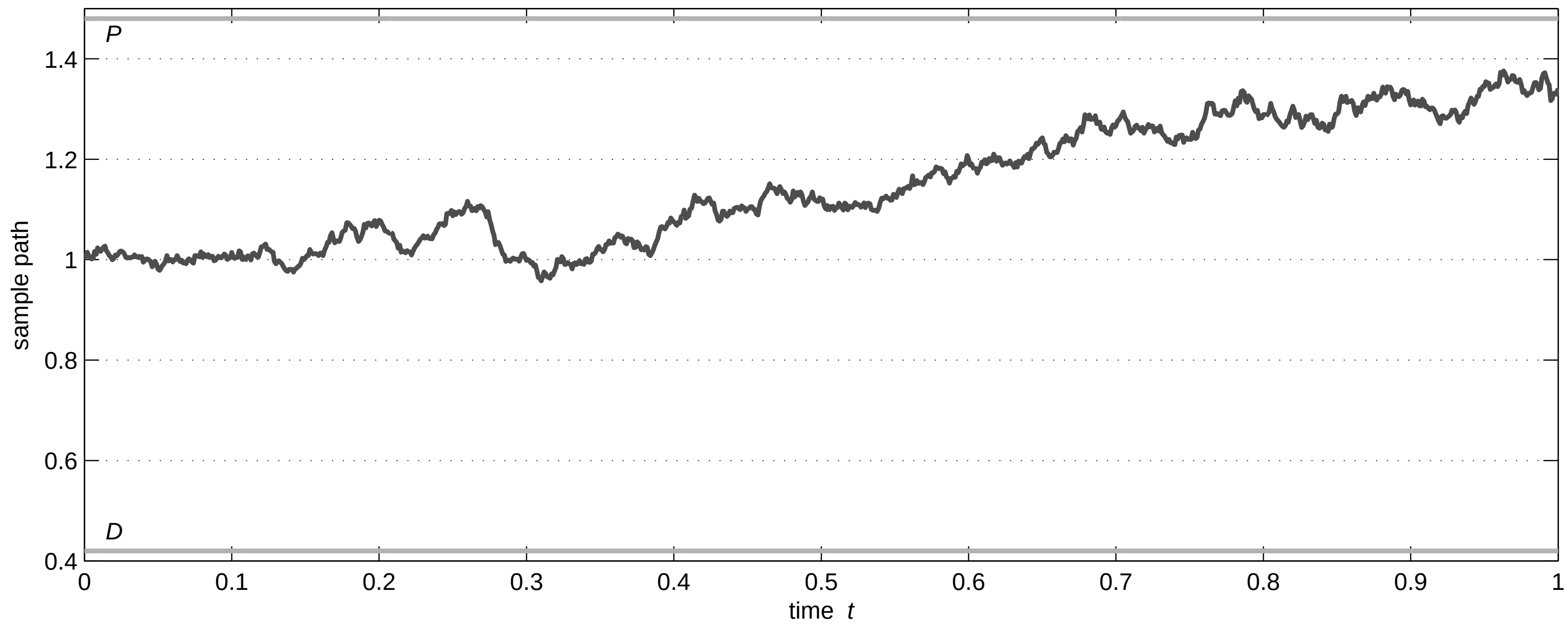
Sample path of $\{S_t\}_{t\geq 0}$ with a lower barrier D and an upper barrier P.

Pricing call options



Sample path of $\{S_t\}_{t\geq 0}$ with a lower barrier D and an upper barrier P.

Pricing call options



Sample path of $\{S_t\}_{t\geq 0}$ with a lower barrier D and an upper barrier P.

A barrier option can **approximate** a call option, i.e.

$$\mathbb{1}_{\{D<S_t<P \text{ for } 0\leq t\leq 1\}} \max(S_1 - K, 0) \approx \max(S_1 - K, 0)\,.$$

Pricing call options
Numerical example

(Vanilla) **Call options** can be approximated by **barrier options**.

Again: **Black-Scholes** model ($r = 0$, $\sigma = 0.2$), $T = 1$, $K = 80$.

$(D; P)$	**barrier price**	N	comp. time	**call price**	comp. time
(0.7; 1.3)	12.21580525385	7	0.1ms		
(0.6; 1.4)	13.08137347245	9	0.1ms		
(0.4; 2.7)	21.18586311986	22	0.1ms		
(0.1; 7.4)	**21.18592951321**	44	0.1ms	**21.18592951321**	1.2ms

— Computation of barrier options faster than Black-Scholes formula[a].

— Accuracy of approximation is very high.

[a] The call option was priced using `blsprice.m` in Matlab (version 2009a).

Overview

(1) Time-changed geometric Brownian motion (GBM)

(2) Pricing barrier options

(3) Pricing call options

(4) **Extensions and examples**

Numerical example

Stock price process $\{S_t\}_{t\geq 0}$

with **known characteristic function** $\varphi(u)$

of the log-asset price $\ln(S_T)$.

How to compute

$\mathbb{E}\big[(S_T - K)^+\big] := \int_0^\infty \exp(-iuk)\,\rho\big(\varphi(u), u\big)\,du$?

Numerical example
Alternatives

— **Fast Fourier pricing**: Most popular approach, see Carr and Madan [1999]. Many extensions, e.g., Raible [2000], Chourdakis [2004].

— **Black-Scholes (BS) approximation**: Works for time-changed Brownian motion, see Albrecher et al. [2013].

$$\mathbb{E}\left[(S_T - K)^+\right] \approx const. \sum_{n=1}^{N} B_n \, C^{BS}\big(\mu_n,\, \sigma_n,\, K\big).$$

— **COS Method**: Closest to our approach, see Fang and Oosterlee [2008].

$$\mathbb{E}\left[(S_T - K)^+\right] \approx const. \sum_{n=1}^{N} C_n(K) \, \mathrm{Re}\left(\varphi\left(\frac{n\pi}{a-b}\right) \mathrm{e}^{-in\pi\frac{b}{a-b}}\right).$$

— **Rational approximations**: Works for time-changed Brownian motion, see Pistorius and Stolte [2012]. Uses Gauss-Legendre quadrature.

$$\mathbb{E}\left[(S_T - K)^+\right] \approx const. \sum_{n=1}^{N} D_n(K) \left(\sum_{m=1}^{M} \frac{c_m}{x_n + d_m}\right) \vartheta_T(x_n) \; .$$

Numerical example
Parameter set

Variance Gamma model

	$\ominus$	parameter set	$\oplus$
θ	-0.10	**-0.20**	-0.30
ν	0.10	**0.20**	0.30
σ	0.15	**0.30**	0.45
T	0.10	**0.25**	1.00

Markov switching model

	$\ominus$	parameter set	$\oplus$
σ_1	0.10	**0.20**	0.30
σ_2	0.10	**0.15**	0.20
λ_1	0.10	**0.50**	1.00
λ_2	0.10	**1.00**	2.00
T	0.10	**0.25**	1.00

The parameters sets were obtained from Chourdakis [2004].

We use 31 equidistant strikes K out of $[85, 115]$, the current price is $S_0 = 100$.

The rows $\ominus$ and $\oplus$ allow us to test many different parameter sets to adequately compare the different numerical techniques.

Numerical example
Results I: Pricing call options

Variance Gamma model (char. fct. decays hyperbolically)

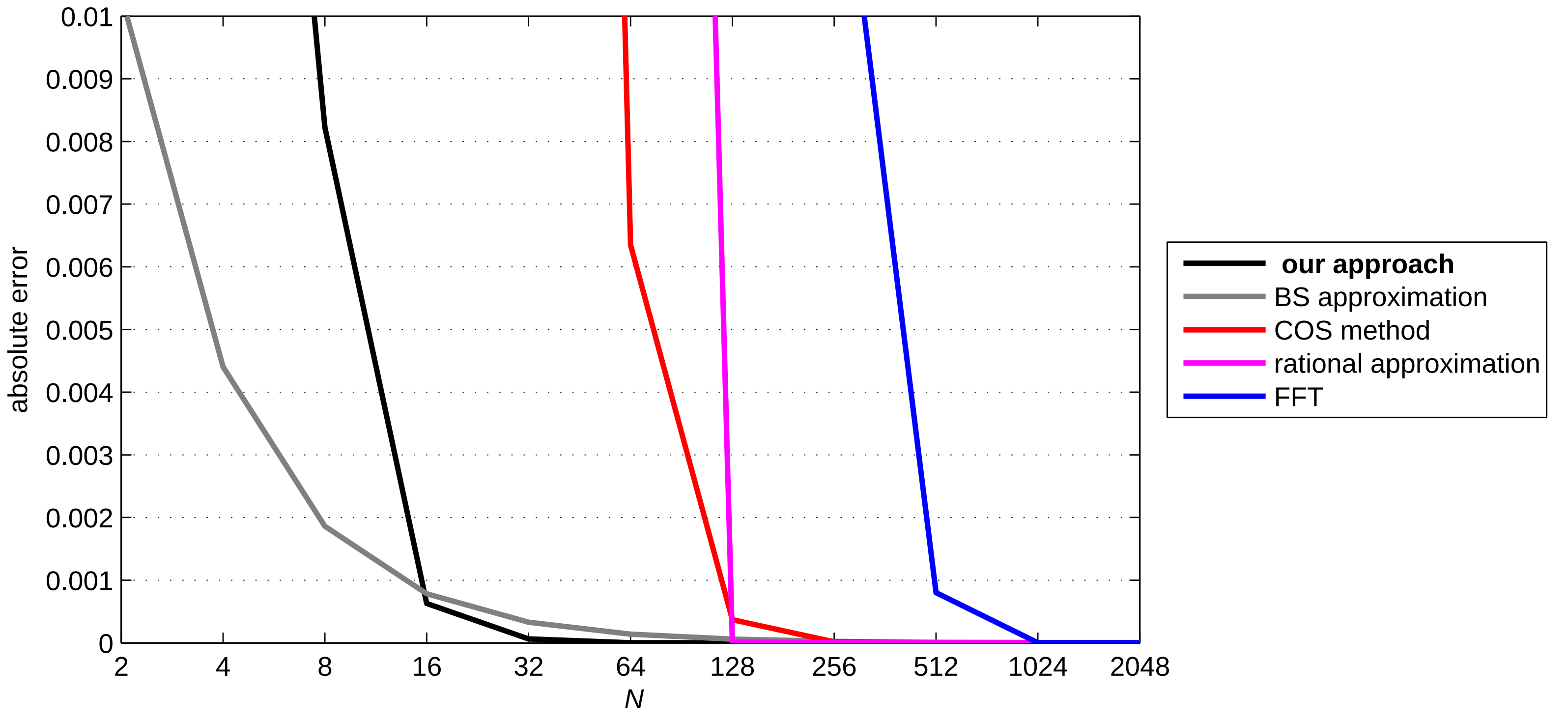
	our approach	**FFT**	**COS method**	**BS approx.**
N	**100**	**4096**	**200**	**10**
average comp. time	**0.5ms**	**4.9ms**	**1.4ms**	**0.3ms**
average rel. error	4.5e-08	2.0e-07	3.5e-07	5.4e-05
max. rel. error	2.7e-07	5.8e-07	2.6e-06	3.0e-04
sample price	20.76524	20.76523	20.76524	20.76105

Numerical comparison on different parameter sets following Chourdakis [2004]. A sample price was obtained using $K = 80$ and the average parameter set from slide 26. The barriers $(D; P)$ were set to $(\exp(-3); \exp(3))$.

Numerical example

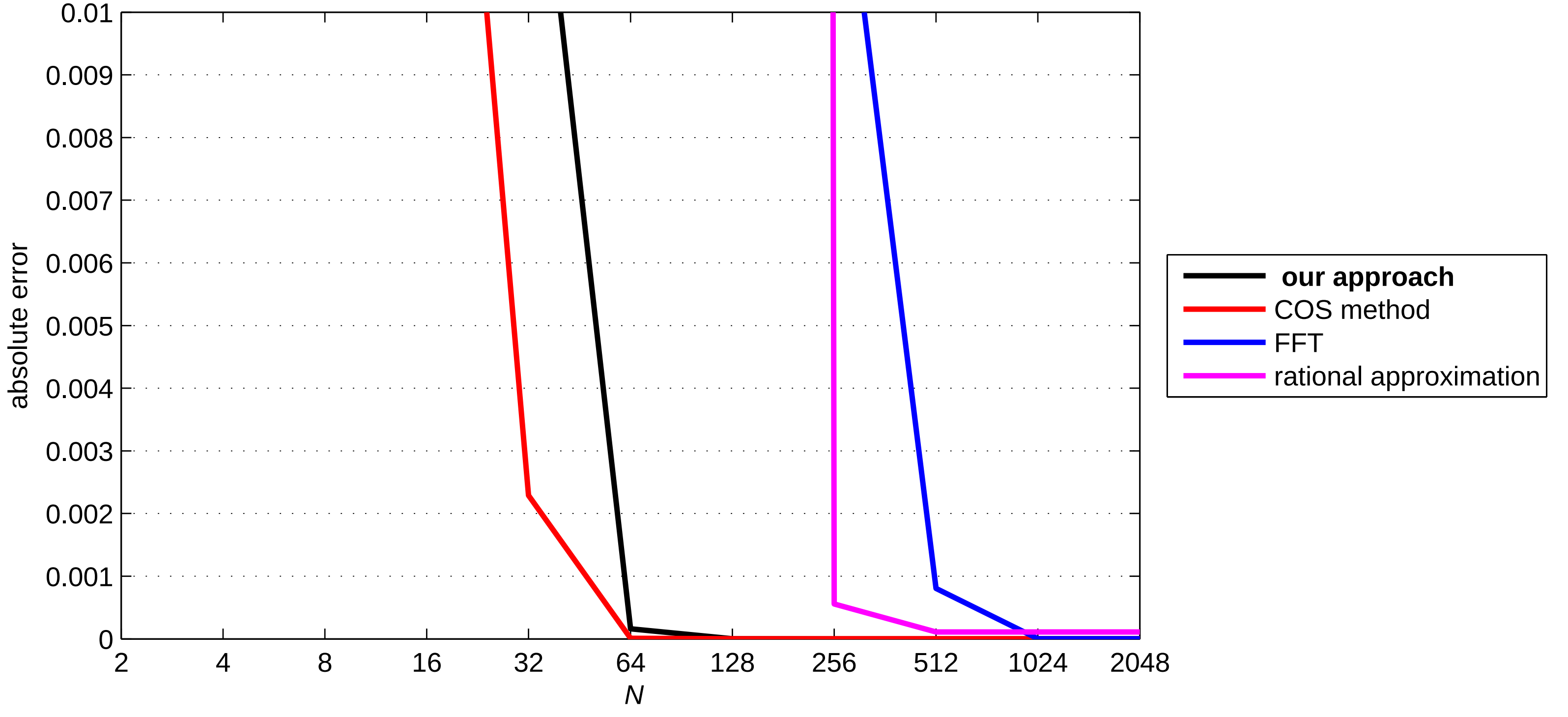
Results II: Pricing call options

Absolute error vs. number of terms N: **Variance Gamma model**.



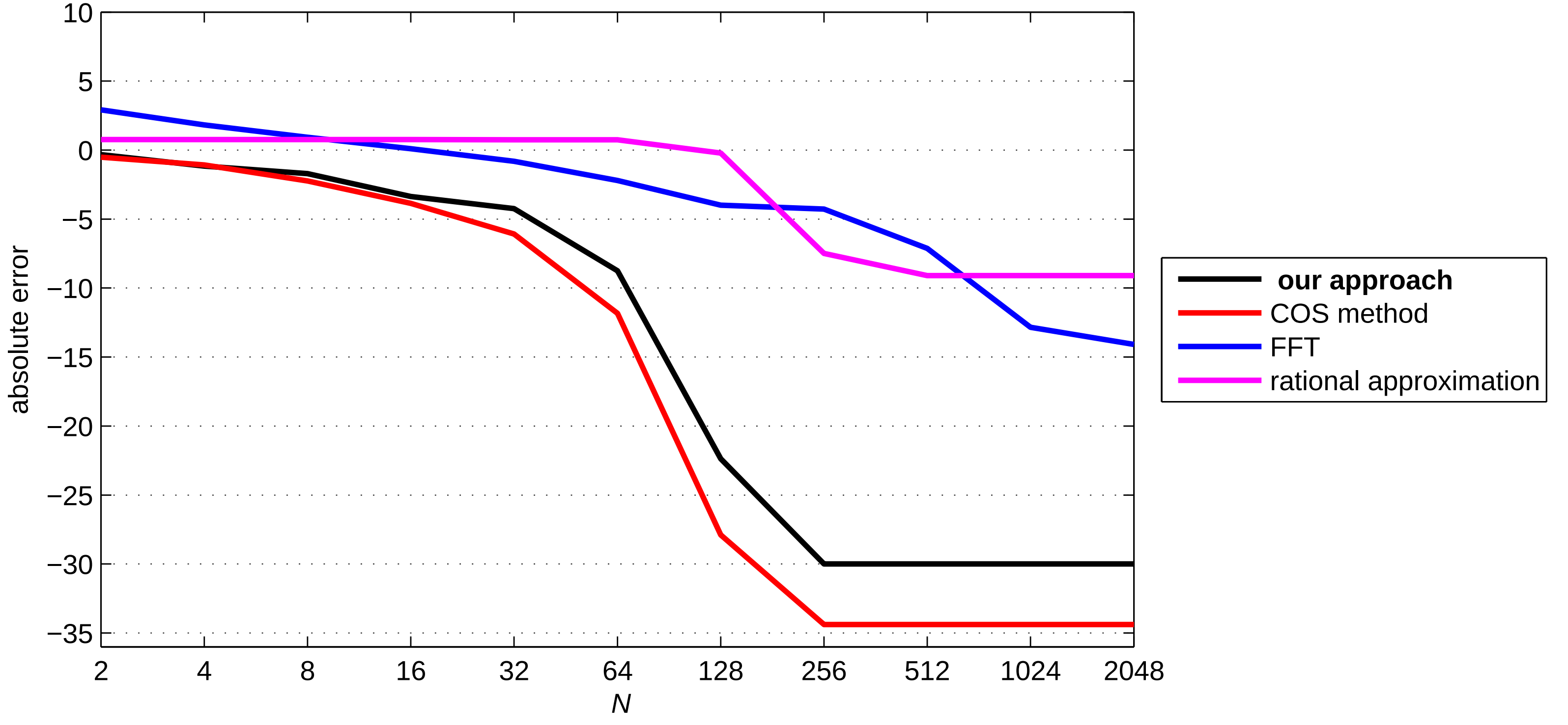
Numerical example
Results III: Pricing call options

Absolute error vs. number of terms N: Markov switching model.



Numerical example
Results IV: Pricing call options

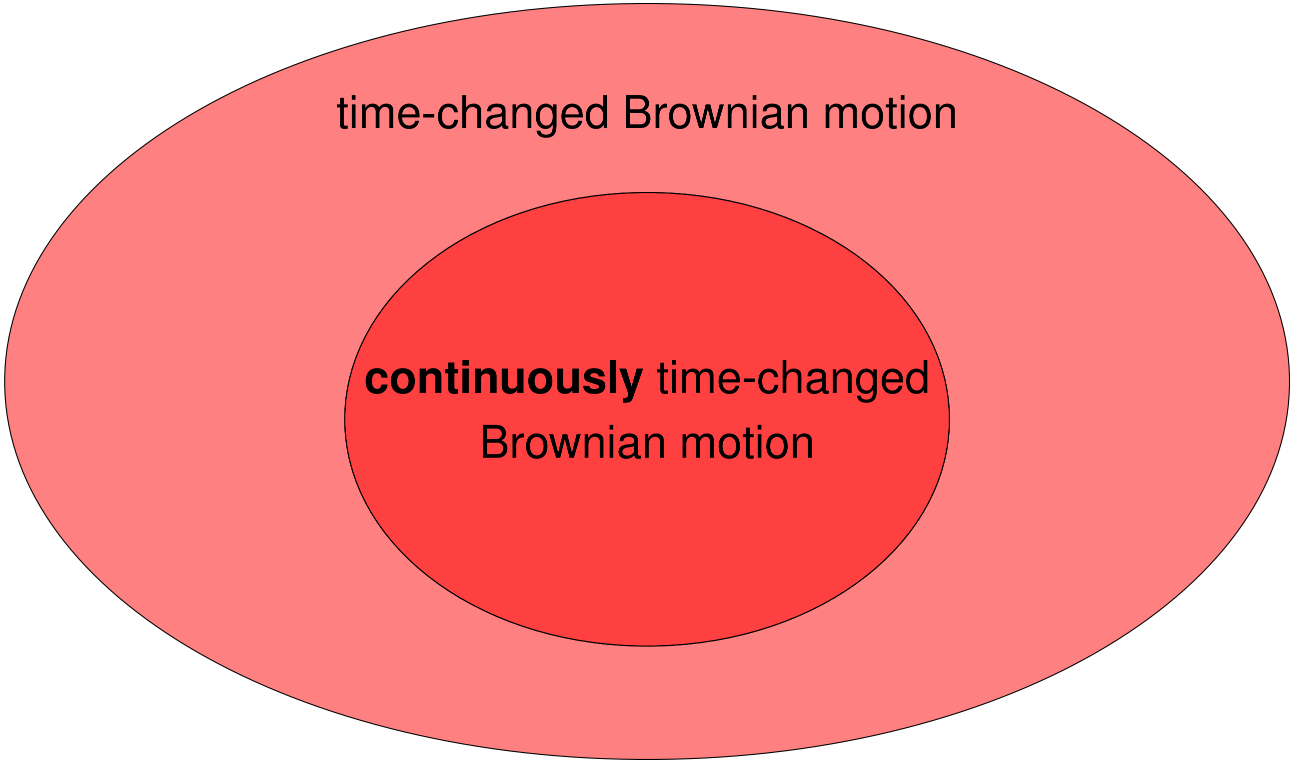
Logarithmic error vs. number of terms N: Markov switching model.



Numerical example
Discussion

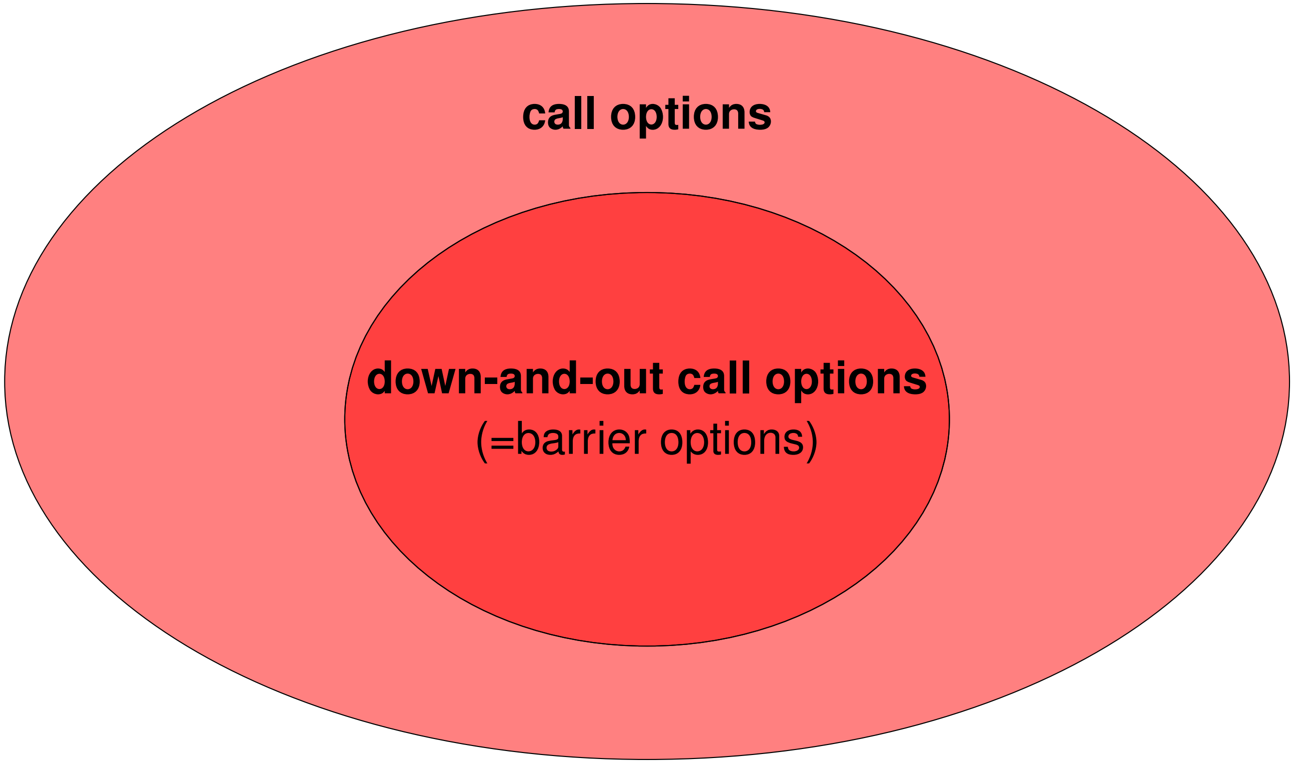
- Our approach and the Fang and Oosterlee [2008] results are extremely fast for **quickly (e.g. exponentially) decaying characteristic functions**.
- High accuracy (e.g. 1e–10) is possible since one avoids any kind of discretization. Error bounds are available.
- Evaluation of **several strikes** comes at almost no cost.
- Apart from option pricing, one is able to evaluate densities or distributions with known characteristic function.

Summary



time-changed Brownian motion
continuously time-changed
Brownian motion

Summary



Literature

H.-J. Albrecher, F. Guillaume, and W. Schoutens. Implied liquidity: Model sensitivity. **Journal of Empirical Finance**, Vol. 23:pp. 48–67, 2013.

P. Carr and D. B. Madan. Option valuation using the fast Fourier transform. **Journal of Computational Finance**, Vol. 2:pp. 61–73, 1999.

K. Chourdakis. Option pricing using the fractional FFT. **Journal of Computational Finance**, Vol. 8, No. 2:pp. 1–18, 2004.

M. Escobar, P. Hieber, and M. Scherer. Efficiently pricing barrier derivatives in stochastic volatility models. **Working paper**, 2013.

F. Fang and K. Oosterlee. A novel pricing method for European options based on Fourier-Cosine series expansions. **SIAM Journal on Scientific Computing**, Vol. 31, No. 2:pp. 826–848, 2008.

P. Hieber and M. Scherer. A note on first-passage times of continuously time-changed Brownian motion. **Statistics & Probability Letters**, Vol. 82, No. 1:pp. 165–172, 2012.

A. Pelsser. Pricing double barrier options using Laplace transforms. **Finance and Stochastics**, Vol. 4:pp. 95–104, 2000.

M. Pistorius and J. Stolte. Fast computation of vanilla prices in time-changed models and implied volatilities using rational approximations. **International Journal of Theoretical and Applied Finance**, Vol. 14, No. 4:pp. 1–34, 2012.

S. Raible. **Lévy processes in finance: Theory, numerics, and empirical facts**. PhD thesis, Freiburg University, 2000.

Discontinuous time-change

Example of a discontinuous time-change. While the original process $\{B_t\}_{t\geq 0}$ (black) hits the barrier, the time-changed process $\{B_{\Lambda_t}\}_{t\geq 0}$ (grey) does not. This is not possible if the time-change is continuous; then all barrier crossings are observed until time Λ_T.