Robust consumption-investment problem over infinite horizon

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Introduction

- The aim of the investor is to maximize total expected discounted utility of consumption.
  Literature:
  Karatzas et al (1989), Fleming and Hernandez (2003), Fleming and Pang (2004), Hata and Sheu (2013)
- The investor believes that model is misspecified and he tries to protect against the worst scenario (the worst model) by looking for robust investment and consumption.
  Literature:
  Schied (2008), Trojani and Vanini (2004), Hansen et al (2006), Gagliardini et al (2009), Faria and Coreia-da-Silva (2012)

Overview

Financial market-reference model

- A bank account

$$dS_t^0 = r(Y_t)S_t^0 \, dt,$$

- An asset

$$dS_t = b(Y_t)\, dt + \sigma(Y_t)\, dB_t^1,$$

Market price of risk: $\lambda(y) = \frac{b(y)-r(y)}{\sigma(y)}$

- A non-tradable economic factor

$$dY_t = g(Y_t)\, dt + a(Y_t)\, (\rho \, dB_t^1 + \sqrt{1-\rho^2}\, dB_t^2).$$

Robust portfolio optimization (finite horizon)

- Model risk is described by a set of probability measures $\mathcal{Q}$
- The investor tries to maximize a functional

$$X \to \inf_{Q \in \mathcal{Q}} \mathbb{E}^Q U(X),$$

where U is a utility function and X is a terminal wealth.

Model misspecification - finite horizon

To describe model uncertainty many authors use

$$\mathcal{Q} := \left\{ Q \sim P \mid \frac{dQ}{dP} = \mathcal{E}\left(\int \eta_{1,t} dB_t^1 + \eta_{2,t} dB_t^2 \right)_T \quad (\eta_1, \eta_2) \in \mathcal{M} \right\},$$

where $\mathcal{E}(\cdot)_T$ denotes the Doleans-Dade exponential $\mathcal{M}$ denotes the set of all progressively measurable processes $\eta = (\eta_1, \eta_2)$ taking values in a fixed compact convex set $\Gamma \subset \mathbb{R}^2$.

Model misspecification – infinite horizon

Instead of modeling $\mathcal{Q}$ we consider a set of alternative market dynamics with uncertain drift:

$$\begin{cases} dS_t^0 &= r(Y_t)S_t^0 dt, \\ dS_t &= (b(Y_t) + \eta_{1,t}\sigma(Y_t))S_t dt + \sigma(Y_t)S_t dB_t^1, \\ dY_t &= (g(Y_t) + (\eta_{1,t}\rho + \eta_{2,t}\bar{\rho})a(Y_t))\, dt + a(Y_t)(\rho dB_t^1 + \bar{\rho} dB_t^2). \end{cases}$$

Investor's wealth dynamics $(X_t^{\pi,c,\eta}, 0 \le t < +\infty)$ is given by:

$$dX_t = \left[r(Y_t)X_t + \pi_t(b(Y_t) - r(Y_t) + \eta_{1,t}\sigma(Y_t))\right]dt + \pi_t\sigma(Y_t)dB_t^1 - c_t dt,$$

π - capital invested in risky asset,
c - consumption per unit of time.

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Main problem

- Objective function

$$\mathcal{J}^{\pi,c,\eta}(x,y) := \mathbb{E}_{x,y} \int_0^{\tau^{\pi,c,\eta}} e^{-wt} \frac{(c_t)^\gamma}{\gamma} dt. \quad 0 < \gamma < 1,$$

$\tau^{\pi,c,\eta} = \inf\{t \geq 0 : X_t^{\pi,c,\eta} \leq 0\}$

- Investor, who doubts his model, uses maxmin criterion:

$$\text{maximize} \ \inf_{\eta \in \mathcal{M}} \mathcal{J}^{\pi,c,\eta}(x,y) \text{ over } (\pi, c) \in \mathcal{A}_{x,y}.$$

- This can be considered as a zero sum stochastic differential game, where the investor is looking for a saddle point (π^*, c^*, η^*), that is

$$\mathcal{J}^{\pi,c,\eta^*} \leqslant \mathcal{J}^{\pi^*,c^*,\eta^*}(x,y) \leqslant \mathcal{J}^{\pi^*,c^*,\eta}(x,y)$$

HJBI equation

- Differential operator

$$\mathcal{L}^{\pi,c,\eta}V(x,y)=\frac{1}{2}a^2(y)V_{yy}+\frac{1}{2}\pi^2\sigma^2(y)V_{xx}+\rho\pi\sigma(y)a(y)V_{xy}$$
$$+(\rho\eta_1+\bar{\rho}\eta_2)a(y)V_y+g(y)V_y+\pi\big(b(y)-r(y)+\eta_1\sigma(y)\big)V_x+r(y)xV_x-cV_x.$$

- We have to solve

$$\max_{\pi\in\mathbb{R}}\max_{c>0}\min_{\eta\in\Gamma}(\mathcal{L}^{\pi,c,\eta}V-wV+\frac{c^\gamma}{\gamma})$$
$$=\min_{\eta\in\Gamma}\max_{\pi\in\mathbb{R}}\max_{c>0}(\mathcal{L}^{\pi,c,\eta}-wV+\frac{c^\gamma}{\gamma})=0.$$

HJBI equation

- Differential operator

$$\mathcal{L}^{\pi,c,\eta}V(x,y) = \frac{1}{2}a^2(y)V_{yy} + \frac{1}{2}\pi^2\sigma^2(y)V_{xx} + \rho\pi\sigma(y)a(y)V_{xy}$$
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$$= \min_{\eta\in\Gamma}\max_{\pi\in\mathbb{R}}\max_{c>0}(\mathcal{L}^{\pi,c,\eta} - wV + \frac{c^\gamma}{\gamma}) = 0.$$

Verification theorem, part I

Suppose there exists a function $V \in \mathcal{C}^{2,2}((0,+\infty)\times\mathbb{R}) \cap \mathcal{C}([0,+\infty)\times\mathbb{R})$ an admissible Markov control $(\pi^*(x,y), c^*(x,y), \eta^*(x,y))$ and constants $D_1, D_2 > 0$ such that

$$\mathcal{L}^{\pi^*(x,y),c^*(x,y),\eta} V(x,y) - wV(x,y) + \frac{(c^*(x,y))^\gamma}{\gamma} \geq 0,$$

$$\mathcal{L}^{\pi,c,\eta^*(x,y)} V(x,y) - wV(x,y) + \frac{c^\gamma}{\gamma} \leq 0,$$

$$\mathcal{L}^{\pi^*(x,y),c^*(x,y),\eta^*(x,y)} V(x,y) - wV(x,y) + \frac{(c^*(x,y))^\gamma}{\gamma} = 0,$$

$$D_1 x^\gamma \leq (c^*(x,y))^\gamma,$$

$$V(x,y) \leq D_2 x^\gamma$$

for all $\eta \in \Gamma$, $(\pi, c) \in \mathbb{R}\times(0,+\infty)$, $(x,y) \in (0,+\infty)\times\mathbb{R}$

Verification theorem, part II

and

$$\tau_{x,y}^{\pi^*,c^*,\eta} = +\infty, ,$$

$$\mathbb{E}_{x,y}\left(\sup_{0 \le s \le t \wedge \tau} e^{-ws} |V(X_s^{\pi,c,\eta}, Y_s)| \right) < +\infty$$

for all $(x, y) \in (0, +\infty) \times \mathbb{R}$, $t \in [0, +\infty)$, $(\pi, c) \in \mathcal{A}$, $\eta \in \mathcal{M}$.

Then

$$\mathcal{J}^{\pi,c,\eta^*}(x, y) \le V(x, y) \le \mathcal{J}^{\pi^*,c^*,\eta}(x, y)$$

for all $\pi \in \mathcal{A}$, $\eta \in \mathcal{M}$,

and

$$V(x, y) = \mathcal{J}^{\pi^*,c^*,\eta^*}(x, y).$$

Saddle point derivation

- Applying standard minimax results we can reduce the task to solving only one equation:

$$\min_{\eta \in \Gamma} \max_{\pi \in \mathbb{R}} \max_{c>0} (\mathcal{L}^{\pi,c,\eta} V - wV + \frac{c^\gamma}{\gamma}) = 0,$$

- The maximum with respect to π and c is achieved at

$$\pi^*(x,y,\eta) = -\frac{\rho a(y)}{\sigma(y)} \frac{V_{xy}}{V_{xx}} - \frac{(b(y) + \eta_1 \sigma(y))}{\sigma^2(y)} \frac{V_x}{V_{xx}},$$

$$c^*(x,y) = \left(\frac{V_x}{\gamma}\right)^{\frac{1}{\gamma-1}}.$$

- The following ansatz is made:

$$V(x,y) = \frac{x^\gamma}{\gamma} F(y).$$

Transformation of HJBI equation

After substitution and dividing by $\frac{x^\gamma}{\gamma}$ we get

$$\frac{1}{2}a^2(y)F_{yy} + \boxed{\frac{\rho^2\gamma}{2(1-\gamma)}a^2(y)\frac{F_y^2}{F}} + \left(g(y) + \frac{\rho\gamma}{1-\gamma}a(y)\lambda(y)\right)F_y$$
$$+ \min_{(\eta_1,\eta_2)\in\Gamma}\left(\bar{\rho}\eta_2 a(y)F_y + \frac{\rho}{(1-\gamma)}a(y)\eta_1 F_y + \frac{\gamma}{2(1-\gamma)}(\lambda(y)+\eta_1)^2 F\right)$$
$$+ \gamma r(y)F + \boxed{(1-\gamma)F^{\frac{-\gamma}{1-\gamma}}} - wF = 0, \tag{2.1}$$

Transformation of HJBI equation

If there exist F solution to (2.1) and $m_1, m_2, R > 0$ such that

$$\left|\frac{F_y}{F}\right| \leq R \quad \text{and} \quad m_1 \leq F^{\frac{1}{\gamma-1}} \leq m_2,$$

then

$$\max_{q \in [-R,R]} (-Fq^2 + 2F_y q) = \frac{F_y^2}{F},$$

$$\max_{m_1 \leq c \leq m_2} \left(-\gamma c F + c^\gamma\right) = (1-\gamma) F^{\frac{-\gamma}{1-\gamma}}.$$

Transformation of HJBI equation

Therefore it is worth to consider

$$\frac{1}{2}a^2(y)F_{yy}+\max_{q\in[-R,R]}(-Fq^2+2F_yq)+\min_{\eta\in\Gamma}\Big(i(y,\eta)F_y+h(y,\eta)F\Big)$$
$$+\max_{m_1\le c\le m_2}\Big(-\gamma cF+c^\gamma\Big)-wF=0,$$

HJB equation

This equation can be generalized to

$$\frac{1}{2}a^2(y)u_{yy}+\max_{\eta\in D}\Big(i(y,\eta)u_y+h(y,\eta)u\Big)+\max_{c>0}\Big(-\gamma cu+c^\gamma\Big)-wu=0,$$

where $D\subset\mathbb{R}^n$ is a compact set.

Assumption 1

a, h, i are continuous functions, $a(y)>\varepsilon>0$ and there exist constants $L_1>0$, $L_2\geq 0$ such that

$$|h(y,\eta)-h(\bar{y},\eta)|+|i(y,\eta)-i(\bar{y},\eta)|\leq L_1|y-\bar{y}|,$$

$$|h(y,\eta)|+|a(y)|^2\leq L_1,\quad |i(y,\eta)|\leq L_1(1+|y|),$$

$$(y-\bar{y})(i(y,\eta)-i(\bar{y},\eta))+|a(y)-a(\bar{y})|^2\leq L_2|y-\bar{y}|^2.$$

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$$(y-\bar{y})(i(y,\eta)-i(\bar{y},\eta))+|a(y)-a(\bar{y})|^2\leq L_2|y-\bar{y}|^2.$$

Existence and uniqueness

Theorem

Suppose that **Assumption 1** *is satisfied with* $L_1 > 0$, $L_2 \geq 0$ *and* $w > \sup_{\eta,y} h(y,\eta) + L_2$. *Then there exists a unique bounded solution to*

$$\frac{1}{2}a^2(y)u_{yy} + \max_{\eta \in D}\big(i(y,\eta)u_y + h(y,\eta)u\big) + \max_{m_1 \leq c \leq m_2}\left(-\gamma c u + c^{\gamma}\right) - wu = 0,$$

which, in addition, has bounded y-derivative and is bounded away from zero.

Sketch of the proof

- Solution to infinite time HJB is approximated by solution to finite time horizon HJB of the form

$$u_t + \frac{1}{2}a^2(y)u_{yy} + \max_{\eta \in D}\big(i(y,\eta)u_y + h(y,\eta)u\big) + \max_{m_1 \le c \le m_2}\big(-\gamma c u + c^\gamma\big) - wu = 0,$$

  with terminal condition $u(y,T) = 0$.

- Rubio (2012) result ensures that under Assumtion 1 a unique bounded solution exists (u^T)
  Related results: Friedman (1973), Pham (2002)

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- Stochastic control representation:

$$u^T(y,t) = \max_{\eta\in\mathcal{M},\ c\in\mathcal{C}_{m_1,m_2}} \mathbb{E}_{y,t}\left(\int_t^T e^{\int_t^s (h(Y_k,\eta_k)-\gamma c_k - w)\,dk}\, c_s^{\gamma}\, ds\right),$$

$$dY_t = i(Y_t,\eta_t)\,dt + a(Y_t)dB_t,$$

Sketch of the proof

- Our equation can be rewritten as

$$u_t + \frac{1}{2}a^2(y)u_{yy} + H(y, u, u_y) = 0,$$

where

$$H(y,u,p) = \max_{\eta\in\Gamma}\Big(i(y,\eta)p + h(y,\eta)u\Big) + \max_{m_1\le c\le m_2}\Big(-\gamma cu + c^\gamma\Big)$$

-

$$u^T(y,t) = \mathbb{E}_{y,t}\int_t^T e^{-w(s-t))}H(Y_s, u^T(Y_s), u_y^T(Y_s))\,ds$$
$$= \mathbb{E}_{y,t}\left(\int_t^T e^{-w(s-t)}H(Y_s, u^T(Y_s), u_y^T(Y_s))\,ds\right),$$

where $dY_t = a(Y_t)dB_t$.

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-

$$\begin{aligned} u^T(y, t) &= \mathbb{E}_{y,t}\int_t^T e^{-w(s-t))}H(Y_s, u^T(Y_s), u_y^T(Y_s))\,ds \\ &= \mathbb{E}_{y,t}\left(\int_t^T e^{-w(s-t)}H(Y_s, u^T(Y_s), u_y^T(Y_s))\,ds\right), \end{aligned}$$

where $dY_t = a(Y_t)dB_t$.

Sketch of the proof

- $v(y,t) := u^T(y, T-t)$
- v satisfies

$$v_t - \frac{1}{2}a^2(y)v_{yy} - H(y,u,u_y) + wv = 0$$

- passing $t \to \infty$ and using stochastic representation to estimate uniform bounds for v, v_t, v_y, we may define the solution as

$$\hat{v}(y) = \lim_{t\to\infty} v(y,t) \quad \text{(Arzel-Ascoli Lemma)}$$

Non-linearity recovering

Let $F^{m_1,m_2,R}$ be the unique solution to

$$\frac{1}{2}a^2(y)F_{yy} + \max_{q\in[-R,R]}\left(-\theta a^2(y)Fq^2 + 2\theta a^2(y)F_y q\right) + \max_{\eta\in D}\left(i(y,\eta)F_y + h(y,\eta)F\right) + \max_{m_1\leq c\leq m_2}\left(-\gamma cF + c^\gamma\right) - wF = 0,$$

We can use infinite horizon stochastic control representation

$$F^{m_1,m_2,R}(y) = \max_{c\in C_{m_1,m_2},q\in Q_R} \mathbb{E}_{y,0}\left(\int_0^{+\infty} e^{\int_0^s (h(Y_k,\eta_k^*)-\theta q_k^2-\gamma c_k - w)\,dk}\left(c_s^*\right)^\gamma ds\right)$$

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Non-linearity recovering

- Using stochastic representation we can find $m_1^*, m_2^*, R^* > 0$ that

$$\max_{q\in[-R^*,R^*]}(-F^*q^2 + 2F_y^*q) = \frac{(F_y^*)^2}{F^*},$$

$$\max_{m_1\le c\le m_2}\left(-\gamma cF^* + c^\gamma\right) = (1-\gamma)\left(F^*\right)^{\frac{-\gamma}{1-\gamma}},$$

where $F^* = F^{m_1^*, m_2^*, R^*}$

- This implies that F^* is a solution to

$$\frac{1}{2}a^2(y)F_{yy} + \frac{\rho^2\gamma}{2(1-\gamma)}a^2(y)\frac{F_y^2}{F}$$

$$+\max_{\eta\in D}\left(i(y,\eta)F_y + h(y,\eta)F\right) + (1-\gamma)F^{\frac{-\gamma}{1-\gamma}} - wF = 0. \quad (4.1)$$

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final result

Theorem

Suppose that a, g, r, λ are Lipschitz continuous functions, a, λ, r are bounded and $a(y) > \varepsilon > 0$, g is of linear growth, $w > \sup_{\eta,y} h(y,\eta) + L_2$. Then there exists a saddle point $(\pi^*(x,y), c^*(x,y), \eta^*(x,y))$ such that

$$\pi^*(x,y) = \frac{\rho a(y)x}{(1-\gamma)\sigma(y)}\frac{F_y}{F} + \frac{(\lambda(y)+\eta_1^*(y))x}{(1-\gamma)\sigma(y)}, \quad c^*(x,y) := \left(\frac{F}{\gamma}\right)^{\frac{1}{\gamma-1}} x$$

where F is a bounded together with y-derivative and bounded away from zero solution to (4.1). η^* is a Borel measurable function which realizes maximum in (4.1).

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Thank you for your attention.