BSPDEs and their applications to stochastic optimal control

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Outline

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1. Introduction to BSPDEs
2. An application to stochastic optimal control
3. Another application of BSPDEs: without a convexity assumption
4. Conclusion

Notation

- K, $\mathcal{O}$ are separable Hilbert spaces.
- $M \in \mathcal{M}^{2,c}_{[0,T]}(K)$, i.e M is a continuous square integrable martingale in K.
- $< M >$, $<< M >>$ are the predictable quadratic variation, tensor quadratic variation of M, respectively.
- $<< M >>_t = \int_0^t \mathcal{Q}(s)ds$,

  for a predictable process $\mathcal{Q}(\cdot)$ s.t. $\mathcal{Q}(t) \in L_1(K)$, symmetric, positive definite, $\mathcal{Q}(t) \leq \mathcal{Q}$, where $\mathcal{Q} \in L_1(K)$ (positive definite).

Notation

- Two elements M and N of $\mathcal{M}^{2,c}_{[0,T]}(K)$ are very strongly orthogonal (VSO) if

$$\mathbb{E}\,[M(\tau)\otimes N(\tau)] = \mathbb{E}\,[M(0)\otimes N(0)],$$

for all $[0,T]$ - valued stopping times τ.

- $\Lambda^2(K;\mathcal{P},M) \rightsquigarrow$ the space of integrands Φ w.r.t. M s.t.

$$\Phi(t,\omega)\,\mathcal{Q}^{1/2}(t,\omega)\in L_2(K),$$

$(\Phi(\cdot,\cdot)\,\mathcal{Q}^{1/2}(\cdot,\cdot))(h) : [0,T]\times\Omega\to K$ is predictable $\forall\; h\in K$,

$$\mathbb{E}\,[\int_0^T ||(\Phi\circ\mathcal{Q}^{1/2})(t)||_2^2\; dt\,] < \infty.$$

BSPDEs:

$$\begin{cases} -\,dY(t) = (\,A(t)Y(t) + f(t, Y(t), Z(t)\mathcal{Q}^{1/2}(t))\,)dt - Z(t)dM(t) - dN(t), \\ \quad Y(T) = \xi. \end{cases} \tag{1}$$

Assumptions

- (A1) $f : [0, T] \times \Omega \times K \times L_2(K) \to K$ s.t.
  (i) f is $\mathcal{P} \otimes \mathcal{B}(K) \otimes \mathcal{B}(L_2(K))/\mathcal{B}(K)$ - measurable,
  (ii) $\mathbb{E}\,[\int_0^T |f(t,0,0)|_K^2\, dt] < \infty$,
  (iii) $\exists\ C_1 > 0$ s.t. $\forall\ y, y' \in K$ and $\forall\ z, z' \in L_2(K)$

$$|f(t,\omega,y,z) - f(t,\omega,y',z')|_K^2 \le C_1\,(\,|y - y'|^2 + ||z - z'||_2^2\,).$$

- (A2) $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; K)$.

Assumptions

- (A3) $A(t,\omega)$ is a predictable linear operator on K, belongs to $L(V;V')$ ((V,K,V') is a Gelfand triple),

  1. $2\,\langle A(t,\omega)\,y\,,y\rangle_{V,V'} + \alpha\,|y|_V^2 \leq \lambda\,|y|^2$ *a.e.* $t \in [0,T]$, *a.s.* $\forall\, y \in V$, for some $\alpha, \lambda > 0$,
  2. $\exists\, C_2 \geq 0$ s.t. $|\,A(t,\omega)\,y\,|_{V'} \leq C_2\,|y|_V$ $\forall\,(t,\omega)$, $\forall\, y \in V$.

Definition

A solution of the BSPDE:

$$\begin{cases} -dY(t) = \big(A(t)Y(t) + f(t, Y(t), Z(t)\mathcal{Q}^{1/2}(t))\big)dt \\ \qquad\qquad -Z(t)dM(t) - dN(t), \quad 0 \le t < T, \\ Y(T) = \xi, \end{cases}$$

is $(Y, Z, N) \in L^2_{\mathcal{F}}(0, T; V) \times \Lambda^2(K; \mathcal{P}, M) \times \mathcal{M}^{2,c}_{[0,T]}(K)$ s.t. $\forall\, t \in [0, T]$:

$$\begin{aligned} Y(t) &= \xi + \int_t^T \big(A(s)Y(s) + f(s, Y(s), Z(s)\mathcal{Q}^{1/2}(s))\big)ds \\ &\quad - \int_t^T Z(s)dM(s) - \int_t^T dN(s), \end{aligned}$$

$N(0) = 0$, N is VSO to M.

$L^2_{\mathcal{F}}(0, T; E) := \{\psi : [0, T] \times \Omega \to E, \text{predictable}, \mathbb{E}\,[\int_0^T |\psi(t)|^2_E dt\,] < \infty\,\}.$

Theorem 1 (Existence & uniqueness of the solution of BSPDE (1))

Assume (A1)(on f), (A2)(on ξ), (A3)(on A). There exists a unique solution (Y, Z, N) to BSPDE (1).

Outline

An application to stochastic optimal control

Consider the SPDE on K :

$$\begin{cases} dX^{u(\cdot)}(t) = (A(t)\,X^{u(\cdot)}(t) + F(X^{u(\cdot)}(t), u(t))\,)dt + G(X^{u(\cdot)}(t))dM(t), \\ X^{u(\cdot)}(0) = x_0. \end{cases} \tag{2}$$

$$F : K \times \mathcal{O} \to K, \quad G : K \to L_{\mathcal{Q}}(K)$$
$$(L_{\mathcal{Q}}(K) = L_2(\mathcal{Q}^{-1/2}(K); K)).$$

- $u(\cdot) : [0, T] \times \Omega \to \mathcal{O}$ is admissible if $u(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathcal{O})$ and $u(t) \in U$ *a.e.*, *a.s.* (U is a nonempty convex subset of $\mathcal{O}$) .

  The set of admissible controls $\rightsquigarrow \mathcal{U}_{ad}$.

The control problem for this SPDE is to find a control $u^*(\cdot)$ and the corresponding solution $X^{u^*(\cdot)} \equiv X^*$ of (2) s.t.

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$$J^* = J(u^*(\cdot)),$$

$$J^* := \inf\{J(u(\cdot)) : \; u(\cdot) \in \mathcal{U}_{ad}\}.$$

$$J(u(\cdot)) = \mathbb{E}\left[\int_0^T \ell(X^{u(\cdot)}(t), u(t))dt + h(X^{u(\cdot)}(T)) \right],$$

$\ell : K \times \mathcal{O} \to \mathbb{R}, \; h : K \to \mathbb{R}$ are measurable mappings.

$\Longrightarrow$ $(X^*, u^*(\cdot))$ is called an optimal pair.

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Assumptions

$F : K \times \mathcal{O} \to K, \quad G : K \to L_{\mathcal{Q}}(K)$ satisfy:

- (H1) F, G, ℓ, h are C^1 w.r.t. x, F, ℓ is C^1 w.r.t. u,
  the derivatives $F_x,\ F_u,\ G_x, \ell_x,\ \ell_u$ are uniformly bounded,

  $|h_x|_K \leq C_3\,(1 + |x|_K)$, some constant $C_3 > 0$.

- (H2) ℓ_x satisfies Lipschitz condition with respect to u uniformly in x.

- (H3)=(A3) (conditions on A).

Hamiltonian & the adjoint equation

The **Hamiltonian:**

$$H : [0, T] \times \Omega \times K \times \mathcal{O} \times K \times L_2(K) \to \mathbb{R},$$

$$H(t, x, u, y, z) := \ell(x, u) + \langle F(x, u), y \rangle + \langle G(x) \mathcal{Q}^{1/2}(t), z \rangle_2 .$$

The **(adjoint) BSPDE:**

$$\begin{aligned} -dY^{u(\cdot)}(t) &= [\, A^*(t)\, Y^{u(\cdot)}(t) + \nabla_x H(X^{u(\cdot)}(t), u(t), Y^{u(\cdot)}(t), Z^{u(\cdot)}(t) \mathcal{Q}^{1/2}(t)) \,] dt \\ &\qquad - Z^{u(\cdot)}(t) dM(t) - dN^{u(\cdot)}(t), \quad 0 \leq t < T, \\ Y^{u(\cdot)}(T) &= \nabla h(X^{u(\cdot)}(T)), \end{aligned}$$

$A^*(t)$ is the adjoint operator of $A(t)$.

Theorem 2 (Stochastic maximum principle)

Suppose (H1)–(H3). Assume $(X^*, u^*(\cdot))$ is an optimal pair for the control problem associated with (2). Then there exists a unique solution (Y^*, Z^*, N^*) to the corresponding adjoint BSPDE s.t. the following inequality holds:

$$\left\langle \nabla_u H(t, X^*(t), u^*(t), Y^*(t), Z^*(t)\mathcal{Q}^{1/2}(t)), u^*(t) - u \right\rangle_{\mathcal{O}} \leq 0,$$

$$\forall\, u \in U, \text{ a.e. } t \in [0, T], \text{ a.s.}$$

Outline

Another application of BSPDEs: without a convexity assumption

Consider the following SPDE:

$$\begin{cases} dX^{u(\cdot)}(t) = \big(A(t)X^{u(\cdot)}(t) + a(t,u(t))X^{u(\cdot)}(t) + b(t,u(t))\big)dt \\ \qquad\qquad + \big[\langle \sigma(t,u(t)), X^{u(\cdot)}(t)\rangle_K + G(t,u(t))\big]dM(t), \\ X^{u(\cdot)}(0) = x_0 \in K. \end{cases} \tag{3}$$

Here

- (H4) $a : \Omega \times [0,T] \times \mathcal{O} \to \mathbb{R}, b : \Omega \times [0,T] \times \mathcal{O} \to K,$
  $\sigma : \Omega \times [0,T] \times \mathcal{O} \to K, G : \Omega \times [0,T] \times \mathcal{O} \to L_Q(K)$

  are predictable and bounded mappings.

$u(t) \in U$ *a.e.*, *a.s.* (U is a subset of $\mathcal{O}$ not necessarily convex).

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  are predictable and bounded mappings.

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Let the cost functional:

$$J(u(\cdot)) := \mathbb{E}\left[\int_0^T \langle \rho(t, u(t)), X^{u(\cdot)}(t) \rangle_{V,V'} \, dt + \langle \theta, X^{u(\cdot)}(T) \rangle_K \right], \; u(\cdot) \in \mathcal{U}_{ad},$$

$\rho : [0, T] \times \mathcal{O} \to V'$ is a bounded measurable mapping,

θ is a fixed element of K.

Again: we would like to minimize this cost functional over $\mathcal{U}_{ad}$.

The Hamiltonian $H : [0,T] \times \Omega \times K \times \mathcal{O} \times K \times L_2(K) \to \mathbb{R}$

$$H(t,\omega,x,u,y,z) = \langle \rho(t,u)\,, x \rangle_{V,V'} + a(t,\omega,u) \langle x\,, y \rangle_K + \langle b(t,\omega,u)\,, y \rangle_K + \langle \tilde{\sigma}(t,\omega,x,u)\mathcal{Q}^{1/2}(t,\omega)\,, z \rangle_2\,,$$

$\tilde{\sigma} : [0,T] \times \Omega \times K \times \mathcal{O} \to L_{\mathcal{Q}}(K)$ s.t.

$$\tilde{\sigma}(t,\omega,x,u) = \langle \sigma(t,u)\,, x \rangle_K \,\mathrm{id}_K + G(t,\omega,u).$$

Similarly, the adjoint equation of (3) is the BSPDE:

$$-dY^{u(\cdot)}(t) = \Big[A^*(t) Y^{u(\cdot)}(t) + \nabla_x H(t, X^{u(\cdot)}(t), u(t), X^{u(\cdot)}(t), z^{u(\cdot)}(t)\mathcal{Q}^{1/2}(t)) \Big] dt - Z^{u(\cdot)}(t) dM(t) - dN^{u(\cdot)}(t), \quad 0 \le t < T,$$

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Theorem 3 (A global stochastic maximum principle)

Suppose (H3) and (H4) hold, and $(X^*, u^*(\cdot))$ is an optimal pair for the problem (3). Then there exists a unique solution (Y^*, Z^*, N^*) to the corresponding (adjoint) BSPDE s.t.

$$H(t, X^*(t), u^*(t), Y^*(t), Z^*(t)\mathcal{Q}^{1/2}(t)) \\ \leq H(t, X^*(t), u, Y^*(t), Z^*(t)\mathcal{Q}^{1/2}(t))$$

$$\text{a.e. } t \in [0, T], \text{ a.s. } \forall\, u \in U.$$

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- We have handled SPDEs with convex control domain (linear variation): necessary conditions (local maximum principle, cf. Theorem 2) (Proofs are in Al-Hussein, AMO 2011).
- Can handle: SPDEs with a convex control domain + G depends on the control process (linear variation): necessary conditions (local maximum principle). Probably sufficient conditions are OK.
- Problem 1: SPDEs non-convex control domain (spike variation): necessary conditions, global maximum principle.

  This problem has been solved for SDEs, see [2] in the abstract.
- This problem is much easier than Problem 2: SPDEs (even SDEs) non-convex control domain + G depends on the control variable (spike variation): necessary conditions, global maximum principle.
- Partial answer: Linear SDEs and SPDEs convex or non-convex control domain + G depends on the control variable (spike variation): necessary conditions (global maximum principle), sufficient conditions. (Application 2 - Theorem 3). (Al-Hussein 2012 - to appear soon.)

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Thank you