Orthogonal martingales and Riesz transforms

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6th International Conference on Stochastic Analysis
and Its Applications
Będlewo, Poland, 12 September 2012

Motivation: inequalities for conjugate harmonic functions

Let u, v be real-valued conjugate harmonic functions on the unit disc $D \subset \mathbb{C}$, $v(0) = 0$.

Question: How the size of u controls the size of v?

Theorem (Riesz)

For $1 < p < \infty$ there is a finite absolute c_p such that

$$||v||_p \leq c_p ||u||_p,$$

where $||u||_p = \left(\int_{\partial D} |u(z)|^p dz\right)^{1/p}$.

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Probabilistic setting

Let $B = (B_t)_{t \geq 0}$ be the planar Brownian motion, stopped at ∂D.

Consider the martingales $X_t = u(B_t)$, $Y_t = v(B_t)$ for $t \geq 0$.

By Itô's formula, for $t \geq 0$ we have

$$X_t = u(0) + \int_{0+}^{t} \nabla u(B_s) \cdot \mathrm{d}B_s, \qquad Y_t = \int_{0+}^{t} \nabla v(B_s) \cdot \mathrm{d}B_s.$$

We have $|\nabla u| = |\nabla v|$ and $\nabla u \cdot \nabla v = 0$, so

$$\mathrm{d}[X, X] = \mathrm{d}[Y, Y] \qquad \text{and} \qquad \mathrm{d}[X, Y] = 0.$$

Riesz' theorem: $||Y||_p \leq c_p ||X||_p$ for $1 < p < \infty$.

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Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ - a filtered probability space,
$X = (X_t)_{t \geq 0}$, $Y = (Y_t)_{t \geq 0}$ - adapted martingales, $Y_0 = 0$.

We say that

- ⋆ Y is *differentially subordinate* to X, if $\mathrm{d}[Y, Y] \leq \mathrm{d}[X, X]$,
- ⋆ X and Y are *orthogonal*, if $\mathrm{d}[X, Y] = 0$.

Example

B - BM in $\mathbb{R}^d$, (K_s), (H_s) predictable, $\mathbb{R}^d$-valued, such that

$$|K_s| \leq |H_s|, \qquad K_s \cdot H_s = 0 \qquad \text{for all } s.$$

Then $\left(\int_0^t K_s \cdot \mathrm{d}B_s\right)_t$ is differentially subordinate to $\left(\int_0^t H_s \cdot \mathrm{d}B_s\right)_t$ and both martingales are orthogonal.

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A general problem

Let $V : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a given Borel function.

Suppose we want to show that

$$\mathbb{E}V(X_t, Y_t) \leq 0$$

for all $t \geq 0$ and any orthogonal martingales X, Y such that Y is differentially subordinate to X and $Y_0 = 0$.

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Burkholder's method

Suppose that $U : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the following.

1° $U(x, 0) \leq 0$ for all $x \in \mathbb{R}$.

2° $U(x, y) \geq V(x, y)$ for all x, $y \in \mathbb{R}$.

3° $U(\cdot, y)$ is concave for all $y \in \mathbb{R}$.

4° U is superharmonic.

The existence of such U yields the desired bound.

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Proof (sketch): Using 3° - 4° + Itô's formula gives that $(U(X_t, Y_t))_{t\geq 0}$ is a supermartingale.

Therefore, by 2° and then 1°,

$$\mathbb{E}V(X_t, Y_t) \leq \mathbb{E}U(X_t, Y_t) \leq \mathbb{E}U(X_0, Y_0) = \mathbb{E}U(X_0, 0) \leq 0$$

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Best constants in Riesz' inequality

Theorem (Pichorides (1972), Bañuelos-Wang (1995))

For $1 < p < \infty$ *we have the sharp bound*

$$||Y||_p \leq \cot\left(\frac{\pi}{2p^*}\right) ||X||_p,$$

where $p^* = \max\{p, p/(p-1)\}$.

Proof: If $1 < p \leq 2$, put $V(x,y) = |y|^p - \cot^p(\frac{\pi}{2p^*})|x|^p$ and

$$U(x,y) = -\sin\frac{\pi}{2p}\cos^{p-1}\frac{\pi}{2p} \cdot R^p \cos(p\theta).$$

For $p > 2$, a very similar function works.

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An exponential inequality

Let $\Phi(t) = e^t - 1 - t$ for $t \geq 0$, and for $K > 2/\pi$, define

$$L(K) = \frac{K}{\pi} \int_{\mathbb{R}} \frac{\Phi\left(\left|\frac{2}{\pi K} \log|t|\right|\right)}{t^2+1} \mathrm{d}t.$$

Theorem (A.O.)

Suppose that X, Y are orthogonal martingales such that $||X||_\infty \leq 1$, Y is differentially subordinate to X and $Y_0 \equiv 0$. Then for any $K > 2/\pi$ we have

$$\sup_{t \geq 0} \mathbb{E}\Phi\left(|Y_t|/K\right) \leq \frac{L(K)||X||_1}{K}.$$

The inequality is sharp.

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Proof

We define $V : [-1,1] \times \mathbb{R} \to \mathbb{R}$ by

$$V(x,y) = \Phi(|y|/K) - L(K)K^{-1}|x|$$

and let $U : [-1,1] \times \mathbb{R} \to \mathbb{R}$ be the harmonic lift of V:

$$U(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos\left(\frac{\pi}{2}x\right) \Phi\left(\left|\frac{2}{\pi K}\log|s| + \frac{y}{K}\right|\right)}{s^2 - 2s\sin\left(\frac{\pi}{2}x\right) + 1} ds - L(K)K^{-1}.$$

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Some definitions

Let $d \geq 1$ be a fixed integer.

Riesz transforms in $\mathbb{R}^d$: a collection $R_1, R_2, \ldots, R_d$ of singular integral operators given by

$$R_j f(x) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{(d+1)/2}} \text{ p.v.} \int_{\mathbb{R}^d} \frac{x_j - y_j}{|x-y|^{d+1}} f(y) \mathrm{d}y,$$

acting on $f : \mathbb{R}^d \to \mathbb{R}$.

Alternative definition: R_j is a Fourier multiplier,

$$\widehat{R_j f}(\xi) = -i \frac{\xi_j}{|\xi|} \hat{f}(\xi), \qquad \text{for } \xi \in \mathbb{R}^d \setminus \{0\}.$$

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Relation to orthogonal martingales

Question: How to prove (sharp) inequalities for R_j? For example, what are the L^p-norms of R_j?

Probabilistic representation of Riesz transforms

Let $(X, Y) \in \mathbb{R}^d \times \mathbb{R}$ be a Brownian motion.

For $y > 0$, let $\tau(y) = \inf\{t \geq 0 : Y_t \in \{-y\}\}$.

For $f : \mathbb{R}^d \to \mathbb{R}$, let $U_f : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ be the Poisson extension.

For $j \in \{1, 2, \ldots, d\}$, let $A^j = [a^j_{\ell m}]$ be a $(d+1) \times (d+1)$ matrix,

$$a^j_{\ell m} = \begin{cases} 1 & \text{if } \ell = d+1,\ m = j, \\ -1 & \text{if } \ell = j,\ m = d+1, \\ 0 & \text{otherwise.} \end{cases} \qquad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Note that $Ax \cdot x = 0$ for all $x \in \mathbb{R}^{d+1}$.

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Probabilistic representation of Riesz transforms

Let $(X, Y) \in \mathbb{R}^d \times \mathbb{R}$ be a Brownian motion.

For $y > 0$, let $\tau(y) = \inf\{t \geq 0 : Y_t \in \{-y\}\}$.

For $f : \mathbb{R}^d \to \mathbb{R}$, let $U_f : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ be the Poisson extension.

For $j \in \{1, 2, \ldots, d\}$, let $A^j = [a^j_{\ell m}]$ be a $(d+1) \times (d+1)$ matrix,

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They are orthogonal and ζ is differentially subordinate to ξ.

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How to get an estimate for R_j?

Let $\Phi : [0,\infty) \to \mathbb{R}$ - a convex increasing function. How to get the bound for $\int_{\mathbb{R}^d} \Phi(|R_j f(x)|)\mathrm{d}x$?

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$$\int_{\mathbb{R}^d} \Phi(|T^y_{A^j} f(z)|)\mathrm{d}z = \int_{\mathbb{R}^d} \Phi(|\tilde{\mathbb{E}}\left[\zeta^{x,y}_\infty | x + X_{\tau(y)} = z\right]|)\mathrm{d}z$$

$$\leq \int_{\mathbb{R}^d} \mathbb{E}\Phi(|\zeta^{x,y}_\infty|)\mathrm{d}x.$$

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L^p estimates

Theorem (Iwaniec-Martin (1996), Bañuelos-Wang (1995))

For $1 < p < \infty$ we have the sharp bound

$$||R_j f||_p \leq \cot\left(\frac{\pi}{2p^*}\right) ||f||_p.$$

Proof. Take $\Phi(x) = x^p$. We get

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Exponential inequality for Riesz transforms

Now take $\Phi(t) = e^t - 1 - t$ and recall that

$$L(K) = \frac{K}{\pi} \int_{\mathbb{R}} \frac{\Phi\left(\left|\frac{2}{\pi K} \log|t|\right|\right)}{t^2+1} \mathrm{d}t.$$

Theorem (A.O.)

For any f with $||f||_{L^\infty(\mathbb{R}^d)} \leq 1$,

$$\int_{\mathbb{R}^d} \Phi\left(|R_j f(x)|/K\right)\, dx \leq \frac{L(K)||f||_{L^1(\mathbb{R}^d)}}{K}$$

and the constant is the best possible.

Exponential inequality for Riesz transforms

Proof. We have

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LlogL estimate

Let $\Psi(t) = (t+1)\log(t+1) - t$ - a conjugate to Φ.

Theorem (A.O., JFA (2012))

For any integer d, $K > 2/\pi$ and $A \subset \mathbb{R}^d$,

$$\int_A |R_j f(x)| dx \leq K \int_{\mathbb{R}^d} \Psi(|f(x)|) dx + |A| \cdot L(K).$$

For each K and d the constant $L(K)$ is the best possible.

Proof: Use duality and the previous bound.

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Some questions

1. Let $1 \leq p < \infty$. What is the best constant c_p in the estimate

$$|\{x : |R_j f(x)| \geq 1\}| \leq c_p \int_{\mathbb{R}^d} |f(x)|^p \mathrm{d}x \ ?$$

2. Does the inequality

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holds with some constant c_1 not depending on the dimension?

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