Stochastic variational inequalities driven by Poisson random measures

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Introduction

We consider the following equation

$$dX_t + \partial\varphi(X_t)(dt) \ni b(X_t)\,dt + \sigma(X_t)\,dW_t + \int_{\mathbb{R}^d} \gamma(X_{t-}, z)\,d\tilde{N}_t(dz),$$

where

- $\partial\varphi$ is the subdifferential of proper, l.s.c., convex function φ;
- W is a Brownian motion;
- $\tilde{N}$ is the compensated measure of a homogeneous Poisson random measure with intensity ν;
- W and $\tilde{N}$ are independent.

Subdifferentials

Let $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper, l.s.c., convex function with $\operatorname{int}(\operatorname{Dom}\varphi) \neq \emptyset$.

The *subdifferential* of φ is defined by

$$\partial\varphi(x) := \{x^* \in \mathbb{R}^n \mid \langle x^*, y - x\rangle + \varphi(x) \leq \varphi(y), \ \forall y \in \mathbb{R}^n\}.$$

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Examples:

- $\varphi(x) := |x|$, $x \in \mathbb{R}^n$:

$$\partial\varphi(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0; \\ \overline{B}(0;1), & x = 0. \end{cases}$$

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- $\varphi \equiv I_{\overline{O}} : x \mapsto \begin{cases} 0, & x \in \overline{O}; \\ +\infty, & x \notin \overline{O}, \end{cases}$ the subdifferential is given by

$$\partial I_{\overline{O}}(x) = \begin{cases} \{0\}, & x \in O; \\ N_{\overline{O}}(x), & x \in \operatorname{bd} O; \\ \emptyset, & x \notin \overline{O}. \end{cases}$$

  This corresponds to the reflected jump-diffusions case: [Menaldi, Robin, 1985]: $x + \gamma(x,z) \in \overline{O}$, $\forall x \in \overline{O}$.

Diffusing particles with electrostatic repulsion

[Cépa, Lepingle, 1997]: continuous case

Let $\varphi : \mathbb{R}^N \to \overline{\mathbb{R}}$ be the proper, l.s.c., convex function defined by

$$\varphi(x) := \begin{cases} -c \sum_{1\le i<j\le N} \ln\left(x^{(j)} - x^{(i)}\right), & x^{(1)} < x^{(2)} < \cdots < x^{(N)}; \\ +\infty & \text{otherwise.} \end{cases}$$

Then $\operatorname{Dom}\varphi = \left\{ x \in \mathbb{R}^N \mid x^{(1)} < x^{(2)} < \cdots < x^{(N)} \right\}$ and, for $x \in \operatorname{Dom}\varphi$

$$\partial\varphi(x) = \left(c \sum_{1\le j\le N,\ j\ne i} \frac{1}{x^{(j)} - x^{(i)}} \right)_{1\le i\le N}$$

Stochastic variational inequalities

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where $b : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times d'}$, $\gamma : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$ are measurable functions.
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We say that $(X, K) \in L^0_{\mathrm{ad}}(\Omega; D([0, T]; \mathbb{R}^n)) \times L^0_{\mathrm{ad}}(\Omega; C([0, T]; \mathbb{R}^n))$ is a (strong) *solution* of (SVI) if:

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[Asiminoaiei, Răşcanu, 1997]: existence and uniqueness in case $\gamma \equiv 0$.

Assumptions

We suppose that the coefficients satisfy the following assumptions:

(H1) $|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq L|x - y|$;

(H2) $\gamma(0, \cdot) \in L^p(\nu)$ and $\|\gamma(x, \cdot) - \gamma(y, \cdot)\|_{L^p(\nu)} \leq L|x - y|$ for $p \in \{2, 4\}$;

(H3) $\varphi(x + \gamma(x, z)) \leq \varphi(x) + \psi(x, \gamma(x, z))$, $\forall x \in \overline{\mathrm{Dom}\,\varphi}$, where

$$\left(\int_{\mathbb{R}^d} \psi(x, \gamma(x, z))^2 \nu(dz)\right)^{1/2} \leq L(1 + |x|^\alpha)\left(1 + |(\partial\varphi)_0(x)|^\beta\right)$$

for some $\alpha > 0$ and $\beta < \frac{4}{3}$. Here, $(\partial\varphi)_0(x) := \mathrm{proj}_{\partial\varphi(x)}(0)$.

Uniqueness

Theorem

Under assumptions (H1)-(H2), equation (SVI) has at most one solution starting from $x_0 \in \overline{\mathrm{Dom}\,\varphi}$.

For the proof, we consider two solutions (X, K) and $(\tilde{X}, \tilde{K})$ and apply Itô's formula to $\left|X_t - \tilde{X}_t\right|^2$:

$$|X_t - \tilde{X}_t|^2 + \int_0^t \langle X_s - \tilde{X}_s, d(K_s - \tilde{K}_s)\rangle = 2\int_0^t \langle X_s - \tilde{X}_s, b(X_s) - b(\tilde{X}_s)\rangle\, ds$$
$$+ 2\int_0^t \langle X_s - \tilde{X}_s, [\sigma(X_s) - \sigma(\tilde{X}_s)]\, dW_s\rangle + \int_0^t \left|\sigma(X_s) - \sigma(\tilde{X}_s)\right|^2 ds$$
$$+ 2\int_0^t \int_{\mathbb{R}^d} \left\{\langle X_{s-} - \tilde{X}_{s-}, \gamma(X_{s-}, z) - \gamma(\tilde{X}_{s-}, z)\rangle + \left|\gamma(X_{s-}, z) - \gamma(\tilde{X}_{s-}, z)\right|^2\right\} d\tilde{N}_s(dz)$$
$$+ \int_0^t \int_{\mathbb{R}^d} \langle X_{s-} - \tilde{X}_{s-}, \gamma(X_{s-}, z) - \gamma(\tilde{X}_{s-}, z)\rangle\, \nu(dz)\, ds.$$

Existence

Theorem

Under assumptions (H1)-(H3), equation (SVI) has a unique solution starting from $x_0 \in \overline{\mathrm{Dom}\,\varphi}$*.*

The proof uses the penalization method. We consider Yosida's regularization of φ

$$\varphi_\varepsilon(x) := \inf\left\{ \tfrac{1}{2\varepsilon}|x-y|^2 + \varphi(y) \mid y \in \mathbb{R}^n \right\}, \ \varepsilon > 0,$$

which is a C^1, convex function on $\mathbb{R}^n$, with $\nabla\varphi_\varepsilon$ a Lipschitz function with Lipschitz constant equal to $1/\varepsilon$. Moreover, by (H3),

$$\begin{aligned}\varphi_\varepsilon(x+\gamma(t,x,z)) \leq\ & \varphi_\varepsilon(x) + |\nabla\varphi_\varepsilon(x)|\,|\gamma(t,J_\varepsilon x,z)-\gamma(t,x,z)| \\ & + \frac{1}{2\varepsilon}|\gamma(t,J_\varepsilon x,z)-\gamma(t,x,z)|^2 + \psi(J_\varepsilon x,\gamma(t,J_\varepsilon x,z)),\end{aligned}$$

where $J_\varepsilon x := x - \varepsilon\nabla\varphi_\varepsilon(x)$ satisfies

$$\varphi_\varepsilon(x) = \frac{1}{2\varepsilon}|J_\varepsilon x - x|^2 + \varphi(J_\varepsilon x) = \frac{\varepsilon}{2}|\nabla\varphi_\varepsilon(x)|^2 + \varphi(J_\varepsilon x).$$

Approximation

We consider the jump-diffusion X^ε given by

$$dX_t^\varepsilon + \nabla\varphi_\varepsilon\left(X_t^\varepsilon\right)dt = b\left(X_t^\varepsilon\right)dt + \sigma\left(X_t^\varepsilon\right)dW_t + \int_{\mathbb{R}^d}\gamma\left(X_{t-}^\varepsilon, z\right)d\tilde{N}_t\left(dz\right).$$

Existence and uniqueness:

- [Gihman, Skorohod, 1972]
- [Jacod, 1979]

We will show that X^ε and $K_t^\varepsilon := \int_0^t \nabla\varphi_\varepsilon\left(X_s^\varepsilon\right)ds$ converge to some X and K.

First, we obtain uniform boundedness for X^ε and K^ε:

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}\left|X_t^\varepsilon\right|^4 + \mathbb{E}\left(\int_0^T \varphi_\varepsilon\left(X_s^\varepsilon\right)ds\right)^2 &\leq C\left(1+\left|x_0\right|^4\right);\\ \mathbb{E}\left\|K^\varepsilon\right\|^2_{BV([0,T];\mathbb{R}^n)} &\leq C\left(1+\left|x_0\right|^4\right).\end{aligned}$$

Cauchy estimates

$$
\begin{aligned}
|X_t^\varepsilon - X_t^\delta|^2 = & -2\int_0^t \left\langle X_s^\varepsilon - X_s^\delta, \nabla\varphi_\varepsilon(X_s^\varepsilon) - \nabla\varphi_\delta(X_s^\delta)\right\rangle ds + \int_0^t \left|\sigma(s,X_s^\varepsilon) - \sigma(s,X_s^\delta)\right|^2 ds \\
& + 2\int_0^t \left\langle X_t^\varepsilon - X_t^\delta, b(s,X_s^\varepsilon) - b(s,X_s^\delta)\right\rangle ds + 2\int_0^t \left\langle X_t^\varepsilon - X_t^\delta, \left[\sigma(s,X_s^\varepsilon) - \sigma(s,X_s^\delta)\right] dW_s\right\rangle \\
& + 2\int_0^t\int_{\mathbb{R}^d} \left\langle X_{s-}^\varepsilon - X_{s-}^\delta, \gamma(s,X_{s-}^\varepsilon,z) - \gamma(s,X_{s-}^\delta,z)\right\rangle d\tilde{N}_s(dz) \\
& + \int_0^t\int_{\mathbb{R}^d} \left|\gamma(s,X_{s-}^\varepsilon,z) - \gamma(s,X_{s-}^\delta,z)\right|^2 dN_s(dz).
\end{aligned}
$$

Since (we can suppose that $\varphi(x) \geq \varphi(0) = 0,\ \forall x \in \mathbb{R}^n$ and $0 \in \operatorname{int}(\operatorname{Dom}\varphi)$).

$$
\langle x - y, \nabla\varphi_\varepsilon(x) - \nabla\varphi_\delta(y)\rangle \geq -(\varepsilon+\delta)\langle \nabla\varphi_\varepsilon(x), \nabla\varphi_\delta(y)\rangle,
$$

we get

$$
\mathbb{E}\sup_{s\in[0,t]} |X_s^\varepsilon - X_s^\delta|^2 \leq 4(\varepsilon+\delta)\,\mathbb{E}\int_0^t \left\langle \nabla\varphi_\varepsilon(X_s^\varepsilon), \nabla\varphi_\delta(X_s^\delta)\right\rangle ds
$$

$$
+ C\int_0^t \mathbb{E}\sup_{r\in[0,s]} |X_r^\varepsilon - X_r^\delta|^2 ds.
$$

It remains to estimate the term $\mathbb{E}\sup_{t\in[0,T]}\left|\nabla\varphi_\varepsilon\left(X_t^\varepsilon\right)\right|^2$:

$$\begin{aligned}
&\varphi_\varepsilon^2(X_t^\varepsilon)+2\int_0^t\varphi_\varepsilon(X_s^\varepsilon)\left|\nabla\varphi_\varepsilon(X_s^\varepsilon)\right|^2ds\le\varphi_\varepsilon^2(x_0)+2\int_0^t\varphi_\varepsilon(X_s^\varepsilon)\langle\nabla\varphi_\varepsilon(X_s^\varepsilon),b\left(s,X_s^\varepsilon\right)\rangle ds\\
&+2\int_0^t\varphi_\varepsilon(X_s^\varepsilon)\langle\nabla\varphi_\varepsilon(X_s^\varepsilon),\sigma\left(s,X_s^\varepsilon\right)dW_s\rangle\\
&+\int_0^t\left|\nabla\varphi_\varepsilon(X_s^\varepsilon)\right|^2\left|\sigma\left(s,X_s^\varepsilon\right)\right|^2ds+\frac{1}{\varepsilon}\int_0^t\varphi_\varepsilon(X_s^\varepsilon)\left|\sigma\left(s,X_s^\varepsilon\right)\right|^2ds\\
&+\int_0^t\int_{\mathbb{R}^d}\varphi_\varepsilon^2(X_{s-}^\varepsilon+\gamma\left(s,X_{s-}^\varepsilon,z\right))-\varphi_\varepsilon^2\left(X_{s-}^\varepsilon\right)-2\varphi_\varepsilon(X_{s-}^\varepsilon)\langle\nabla\varphi_\varepsilon(X_{s-}^\varepsilon),\gamma\left(s,X_{s-}^\varepsilon,z\right)\rangle dN_s(dz)\\
&+2\int_0^t\int_{\mathbb{R}^d}\varphi_\varepsilon(X_{s-}^\varepsilon)\langle\nabla\varphi_\varepsilon(X_{s-}^\varepsilon),\gamma\left(s,X_{s-}^\varepsilon,z\right)\rangle d\tilde{N}_s\left(dz\right)
\end{aligned}$$

This gives

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]}\varphi_\varepsilon^2(X_t^\varepsilon)&\le2\varphi^2(x_0)+\frac{C}{\varepsilon}\mathbb{E}\int_0^T\left(1+|X_s^\varepsilon|^{2+\alpha}\right)ds+\frac{C}{\varepsilon^{3/2}}\mathbb{E}\int_0^T\left(1+|X_s^\varepsilon|^4\right)ds\\
&+\frac{C}{\varepsilon^{\frac{4+\beta}{4-\beta}}}\mathbb{E}\int_0^T\left(1+|X_s^\varepsilon|^{\frac{4(2+\alpha)}{4-\beta}}\right)ds.
\end{aligned}$$

Finally, we obtain

$$\mathbb{E}\sup_{t\in[0,T]}|X_t^\varepsilon - X_t^\delta|^2 \le C\varepsilon^{\frac{4-3\beta}{4(4-\beta)}}\left(\mathbb{E}\,\|K^\delta\|^2\right)^{1/2} + C\delta^{\frac{4-3\beta}{4(4-\beta)}}\left(\mathbb{E}\,\|K^\varepsilon\|^2\right)^{1/2},$$

from which we conclude the existence of (X, K) as the limit of $(X^\varepsilon, K^\varepsilon)$ in $L^2_{\mathrm{ad}}(\Omega; D([0,T];\mathbb{R}^n)) \times L^2_{\mathrm{ad}}(\Omega; C([0,T];\mathbb{R}^n))$.

It remains only to verify that (X, K) is a solution of equation (SVI), which is done by passing to the limit in the approximating equation and in relation

$$\int_0^T \langle Y_r - X_r^\varepsilon, dK_r^\varepsilon\rangle + \int_0^T \varphi_\varepsilon(X_r^\varepsilon)\,dr \le \int_0^T \varphi_\varepsilon(Y_r)\,dr,\ \forall Y \in L^0_{\mathrm{ad}}(\Omega; D([0,T];\mathbb{R}^n)).$$

Weak Solutions of SVIs

Tightness

The coefficients b, σ and $\gamma(\cdot, z)$ are only *continuous*, satisfying the *growth condition*:

(H4) $|b(x)| + |\sigma(x)| + \|\gamma(x, \cdot)\|_{L^p(\nu)} \leq c\,(1 + |x|)$ for $p \in \{2, p_0\}$ with $p_0 \geq 4$.

Theorem

Let I be an arbitrary set of indexes. For each $i \in I$, suppose that $(\Omega^i, \mathcal{F}^i, P^i, \mathbb{F}^i, W^i, N^i, X^i, K^i)$ is a weak solution of the equation

$$dX_t^i + \partial\varphi(X_t^i)dt \ni b^i(X_t^i)dt + \sigma^i(X_t^i)dW_t^i + \int_{\mathbb{R}^d} \gamma^i(X_{t-}^i, z)\,d\tilde{N}_t^i(dz)\,, \;\; t \in [0, T],$$

where b^i, σ^i and γ^i satisfy (H4) uniformly and $\sup_{i \in I} \mathbb{E}^i |X_0^i|^2 < +\infty$. Then $(X^i, K^i)_{i \in I}$ is tight in $D([0, T]; \mathbb{R}^n) \times C([0, T]; \mathbb{R}^n)$.

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Notations:

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- $\bar{\Omega}_0$ is the set of $(x,\eta) \in \mathbf{D} \times \mathbf{C}_{BV}$ such that

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- $\bar{\Omega}_a := \{(x, \eta) \in \bar{\Omega}_0 \mid \|\eta\|_{BV} \leq a\}$.

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$$\int_0^T \langle y(t) - x(t), d\eta(t) \rangle + \int_0^T \varphi(x(t))\, dt \leq \int_0^T \varphi(y(t))\, dt, \ \forall y \in \mathbf{D};$$

- $\bar{\Omega}_a := \{(x,\eta) \in \bar{\Omega}_0 \mid \|\eta\|_{BV} \leq a\}$.

As a consequence of Helly-Bray theorem, $\bar{\Omega}_a$ is closed.

Martingale problem

Let $\bar{X}$ and $\bar{K}$ be the canonical processes on $\bar{\Omega}$:

$$\bar{X}_t(\mathbf{x},\boldsymbol{\eta}) := \mathbf{x}(t), \;\; \bar{K}_t(\mathbf{x},\boldsymbol{\eta}) := \boldsymbol{\eta}(t).$$

Let $\mathcal{L}$ be the *integro-differential operator* defined by

$$\begin{aligned}\mathcal{L}f(x) := {} & \frac{1}{2}\operatorname{tr}\sigma\sigma^*(x)D^2 f(x) + \langle b(x), Df(x)\rangle \\ & + \int_{\mathbb{R}^d}\left[f(x+\gamma(x,z)) - f(x) - \langle Df(x), \gamma(x,z)\rangle\right]\nu(dz).\end{aligned}$$

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We say that a probability measure $\mathbf{P}$ on $\bar{\Omega}$ is a *solution of the martingale problem* for (SVI) if

1. $\mathbf{P}(\bar{\Omega}_0) = 1$;
2. for each $f \in C_c^2(\mathbb{R}^d)$, the process

$$\bar{M}_t^f := f(\bar{X}_t) - f(\bar{X}_0) - \int_0^t \mathcal{L}f(\bar{X}_s) ds + \int_0^t \langle Df(\bar{X}_s), d\bar{K}_s \rangle, \quad t \in [0, T],$$

is a $\mathbf{P}$-martingale.

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The two formulations are equivalent:

- If $(\Omega, \mathcal{F}, P, \mathbb{F}, W, N, X, K)$ is a weak solution of (SVI), then $P \circ (X, K)^{-1}$ solves the martingale problem:

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  [Lepeletier, Marchal, 1976], [Ikeda, Watanabe, 1981]

Existence

In addition, we impose the conditions

(H5) $\left|(\partial\varphi)_0(x)\right| \leq L\left(1+|x|^{p_0-2}\right), \ \forall x \in \overline{\mathrm{Dom}\,\varphi}$ and

(H6) $x+\gamma(x,z) \in \overline{\mathrm{Dom}\,\varphi}, \ \forall x \in \overline{\mathrm{Dom}\,\varphi}$.

Theorem

Let μ be a probability measure on $\overline{\mathrm{Dom}\,\varphi}$ such that $\int |x|^2\mu(dx) < +\infty$. If the coefficients b, σ and γ satisfy conditions (H4)-(H5), then there exists a weak solution of equation (SVI) with μ as initial distribution.

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Sketch of the proof. Several steps:
Smoothing: the coefficients b, σ and γ are approximated by Lipschitz functions b_n, σ_n and γ_n. We consider the corresponding SVI with strong solution (X^n, K^n).
Then $\mathbf{P}_n := P\circ(X^n, K^n)^{-1}$ solves the associated martingale problem.
By the tightness result, $\{\mathbf{P}_n\}_{n\geq 1}$ is a tight family of distributions on $\bar{\Omega}$. By Prohorov's theorem, we can suppose that $\mathbf{P}_n$ converges weakly to some probability measure $\mathbf{P}$ on $\bar{\Omega}$.

Existence

Passing to the limit: Let, for $f \in C_c^2(\mathbb{R}^d)$

$$\bar{M}_t^{f,n} := f(\bar{X}_t) - f(\bar{X}_0) - \int_0^t \mathcal{L}^n f(\bar{X}_s) ds + \int_0^t \langle Df(\bar{X}_s), d\bar{K}_s \rangle, \ t \in [0, T],$$

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Then:

- $\mathbf{P}_n(\bar{\Omega}_0) = 1$;
- $\mathbf{P}_n \circ \bar{X}_0^{-1} = \mu$;
- $\mathbb{E}_{\mathbf{P}_n}(\bar{M}_t^{f,n} - \bar{M}_s^{f,n})\Psi = 0, \;\; \forall f \in C^2(\mathbb{R}^n), \;\; \forall \Psi \in C_b(\bar{\Omega}_0) \cap L^0(\hat{\mathcal{F}}_s)$, since $\bar{M}^{f,n}$ is a $\mathbf{P}_n$-martingale.

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We can pass to the limit in those relations, using the uniform estimates on (X^n, K^n) and the approximation of b, σ and γ by b_n, σ_n, respectively γ_n. We obtain that $\mathbf{P}$ is a solution of the martingale problem and hence there exists a weak solution to equation (SVI).

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Thank you for your attention!