On CLT type results in Wigner random matrices

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joint with

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Real Symmetric Wigner Random Matrices

$$W_N = W_N^t = (W_{ij})_{1 \le i,j \le N}$$

The entries on and above the diagonal $\{W_{ij}\}_{1 \le i \le j \le N}$ are independent random variables. In addition,

$$\mathbb{E}\, W_{ij} = 0, \qquad 1 \le i,j \le N$$

$$\mathrm{Var}\, W_{ij} = \sigma^2, \qquad i \ne j$$

$$\mathbb{E}\, W_{ij}^4 = m_4, \qquad i \ne j$$

$$\sup_{i<j,N} \mathbb{E}\, |W_{ij}|^{4+\varepsilon} < \infty$$

$$\sup_{i,N} \mathbb{E}\, |W_{ii}|^{2+\varepsilon} < \infty$$

Such a matrix W_N is called

a Wigner (real symmetric) matrix

Wigner Semicircle Law

$$X_N = \frac{1}{\sqrt{N}} W_N$$

Let $\lambda_1, \lambda_2, \dots, \lambda_N$ be the eigenval

of X_N.

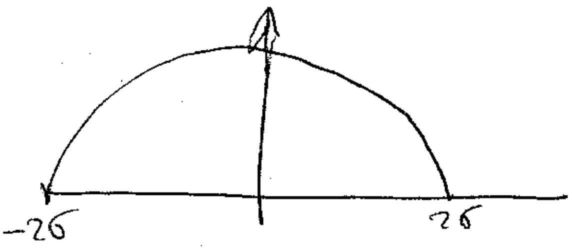
Then the empirical distribution function $F_N(x) = \frac{1}{N} \# \{ \lambda_i \leq x \}$

converges pointwise to

$$F(x) = \int_{-\infty}^{x} \rho(t)\, dt$$ with probability 1

where

$$\rho(t) = \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - t^2}\; \mathbb{1}_{[-2\sigma, 2\sigma]}(t)$$



Let A_N be a deterministic real symmetric matrix of size N and fixed rank r.

We will study the behavior of the largest eigenvalues of $M_N = X_N + A_N$

This model is an analogue of spiked sample covariance matrices introduced by Iain Johnstone in 2001. One should also mention the works by J. Baik, G. Ben Arous, and S. Péché (2005), J. Baik and J. Silverstein (2006), D. Paul (2007) and others.

We assume that the non-zero eigenvalues of A_N are fixed, together with their multiplicit

Let $\theta_1 > \theta_2 > \dots > \theta_{J_0} = 0 > \dots > \theta$

be the ordered eigenvalues of A_N. Denote the multiplicity of θ_j by k_j, $1 \leq j \leq J$. Thus,

$$\sum_{j \neq J_0} k_j = r,$$

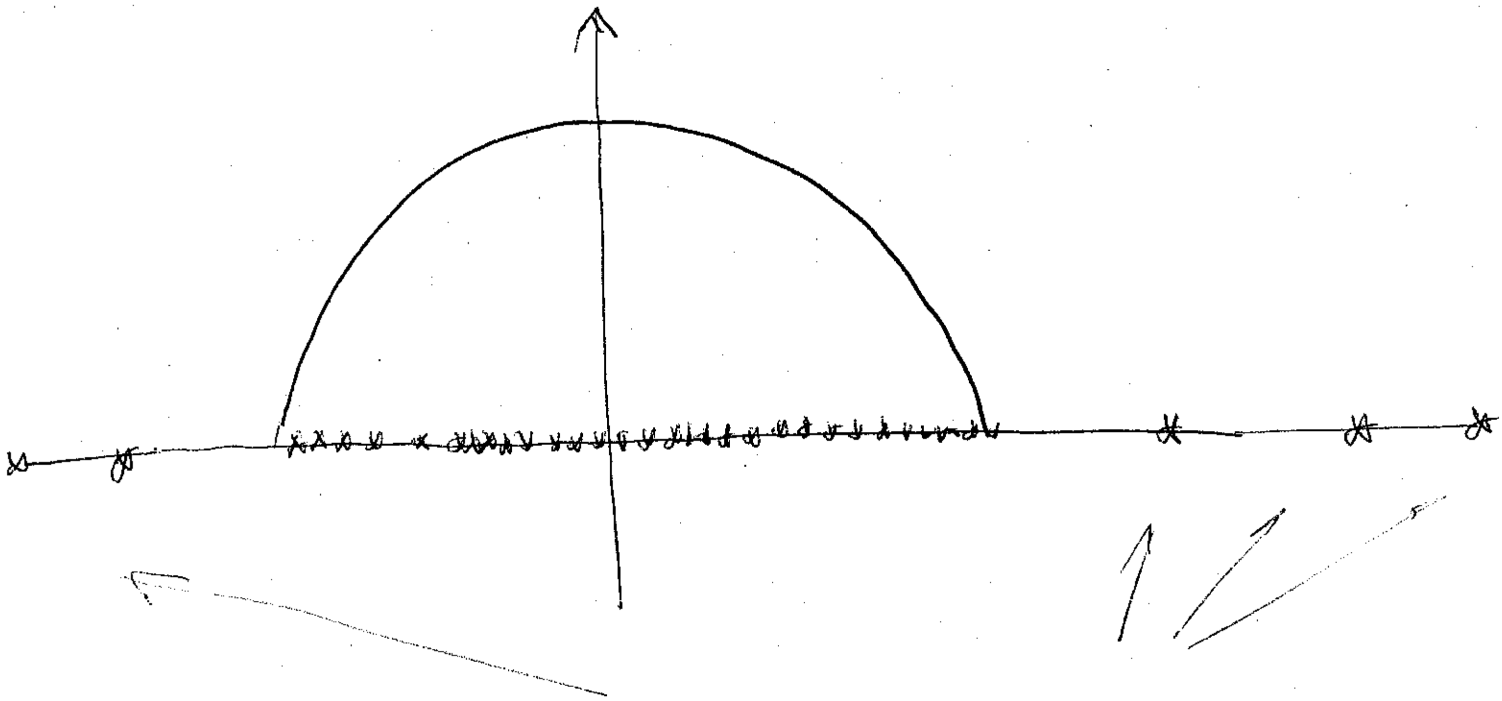
where r is the rank of A_N.

It is known that $\|X_N\| \to 2\sigma$ as $N \to \infty$, so $\forall \varepsilon > 0$

$$\text{Prob}\left(\begin{array}{l}\text{there is an eigenvalue}\\ \text{outside } [-2\sigma-\varepsilon, 2\sigma+\varepsilon]\end{array}\right) \xrightarrow[N \to \infty]{} 0$$

What can be said about $X_N + A_N$?

We would like to study the outliers in the spectrum of $X_N + A_N$ i.e. the eigenvalues that lie outside of $[-2\sigma-\varepsilon, 2\sigma+\varepsilon]$ for sufficiently small ε and all but finitely many N.



outliers in the spectrum of $X_N + A_N$

The case of a special rank one matrix $(A_N)_{ij} = \frac{\theta}{N}$, $1 \le i, j \le N$, has been studied by S. Péché (GUE) and D. Féral and S. Péc

(general Wigner matrices).

For arbitrary A_N important contributions have been made by many mathematicians, including

M. Capitaine, C. Donati-Martin & D. Féra
M. Février

A. Knowles & J. Yin,

J. Baik & D. Wang,

F. Benaych-Georges, A. Guionnet and M. Ma

A. Bloemendal and B. Virag,

and some others.

Question 1

How many outliers are in the spectrum of $M_N = X_N + A_N$?

Let θ_j be a non-zero eigenval of A_N. If $|\theta_j| > \sigma$ then there are k_j corresponding outliers in the spectrum of M_N all converging to $\rho_j = \theta_j + \frac{\sigma^2}{\theta_j}$.

Otherwise, there are no outliers corresponding to θ_j.

Thus, the number of outliers is equal to

$$\boxed{\sum_{j : |\theta_j| > \sigma} k_j}$$

Remark By interlacing property for eigenvalues of rank one perturbatio the number of outliers can not exceed r.

Question 2

What can be said about the fluctuation of the outliers?

The fluctuation is non-universa and depends on localization / delocalization properties of the eigenvectors of A_N.

Consider two special cases of A_N when $r = 1$.

<u>Example 1</u> (<u>Delocalized Case</u>)

$(A_N)_{ij} = \frac{\theta}{N}$ for all $1 \le i,j \le N$,

Then (Féral, Péché) assuming $\theta > \sigma$

$$\sqrt{N}\left(\lambda_{\max}(M_N) - \rho_\theta\right) \xrightarrow[N\to\infty]{d} \mathcal{N}\left(0, \sigma^2\left(1-\frac{\sigma}{\theta}\right.\right.$$

Note that $\|v\|_\infty \to 0$ for $v = \frac{1}{\sqrt{N}}(1,1,1,\dots,1)$

Example 2 (Localized Case)

Let $A_N = \mathrm{diag}(\theta, 0, 0, 0, \dots 0)$

Then if $\theta > \sigma$, we have

$$c_\theta \sqrt{N} \left(\lambda_{\max}(M_N) - \rho_\theta\right) \xrightarrow[N \to \infty]{d} \mu * \mathcal{N}(0,$$

where

$$c_\theta = \frac{\theta^2}{\theta^2 - \sigma^2},$$

μ is the marginal distribution of Wi

and

$$v_\theta = \frac{1}{2}\left(\frac{m_4 - 3\sigma^4}{\theta^2}\right) + \frac{\sigma^4}{\theta^2 - \sigma^2}$$

Note that the convolution of μ and a Gaussian distribution is Gaussian iff μ is Gaussian itself.

Similar situation holds for arbitrary r case. If the ℓ^∞ norms $\left(\|v\|_\infty = \max_{1 \le i \le N} |v_i| \right)$ of the eigenvectors of A_N vanish in the limit $N \to \infty$ then the fluctuation of the outliers is universal. Otherwise, it is not universal.

To simplify the notations, consider the largest eigenvalu of A_N, namely θ_1.

Denote by $v_1, v_2, \ldots, v_{k_1}$ corresponding orthonormal eigenvectors of A_N (recall that θ_1 has multiplicity k_1).

Denote by $R_N(z)$ the resolvent of a standard Wigner matrix X_N,

$$R_N(z) = (zI_N - X_N)^{-1}$$

As before, $\rho_1 = \theta_1 + \frac{\sigma^2}{\theta_1}$ is the limit of the first k_1 outli

Denote the standard Eucledian inner product by $\langle a, b\rangle = \sum_{i=1}^{N} a_i b$

To study the fluctuation of these k_1 outliers around ρ_1 consider a $k_1 \times k_1$ matrix

Ξ_N with the entries

$$\Xi_{ij} = \sqrt{N}\left[\langle v_i, R_N(\rho_1)\, v_j\rangle - \frac{1}{\theta_1}\delta_{ij}\right],$$

$1 \leq i, j \leq k_1$.

Note that $\frac{1}{\theta_1} = g_\sigma(\rho_1)$,

where $g_\sigma(z) = \frac{z - \sqrt{z^2 - 4\sigma^2}}{2\sigma^2} = \int_{-2\sigma}^{2\sigma} \frac{1}{z-x}\, d\mu$

is the Stieltjes transform of the Wigner semicircle law

Denote by $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{k_1}$ the first k_1 largest eigenvalues of $X_N + A_N$ and by $\eta_1 \geq \ldots \geq \eta_{k_1}$ the ordered eigenvalues of Ξ_N.

Proposition

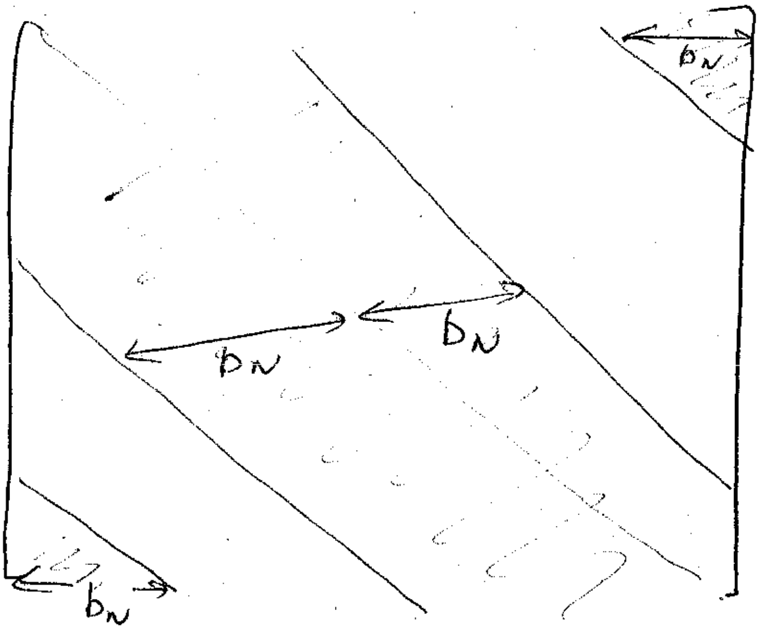
$$\sqrt{N}\left(\lambda_i - \rho_1\right) - \left(\theta_1^2 - \sigma^2\right) y_i \xrightarrow[N \to \infty]{} 0$$

$$1 \leq i \leq k_1.$$

Thus, the question about the fluctuation of the outliers is reduced to the limiting behavior of the resolvent bilinear form for a standard Wigner matrix

Band Random Matrices

$W_N =$



Let $W_{ij} = 0$ if

$$\min\{|i-j|, N-|i-j|\} > b_N,$$

where $b_N \to +\infty$ as $N \to \infty$

For the remaining (non-zero) entries of W_N we still assume that they are independent (up from the diagonal), centered, $Var(W_{ij}) = \sigma^2(1+\delta_{ij}$

We will also need some additi
conditions on higher moments.
In the simplest case, one can
assume that marginal distributio
of non-zero entries satisfy a Poinca
inequality.

Consider $X_N = \frac{1}{\sqrt{2b_N+1}} W_N$.

It is not difficult to show
that X_N satisfies Wigner
semicircle law.

Our goal is to study linear
statistics of the eigenvalues

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a sufficiently smooth test function (it is enough to assume $\int_{-\infty}^{\infty} (1+t^4)\, |\hat{\varphi}(t)|\, dt < \infty$).

Denote $S_N(\varphi) = \sum_{i=1}^{N} \varphi(\lambda_i)$, where $\lambda_1, \lambda_2, .., \lambda_N$ are the eigenvalues of X_N.

Assume $b_N >> \sqrt{N}$ and $b_N/N \to 0$. Then

$$\sqrt{\frac{b_N}{N}} \left(S_N(\varphi) - \mathbb{E} S_N(\varphi) \right) \xrightarrow[N\to\infty]{d} \mathcal{N}\left(0,\ V_{band}(\varphi)\right),$$

where

$$V_{band}(\varphi) =$$

$$\int_{-2\sigma,2\sigma]^3} \frac{(\varphi(x) - \varphi(u))\, \varphi'(y)}{4\pi^4 (x-u) \sqrt{4\sigma^2 - u^2}} \sqrt{4\sigma^2 - x^2}\, \sqrt{4\sigma^2 - y^2}\, F(x,y)\, dx\, dy$$

$$+ \frac{K_4}{4\pi \sigma^8} \left(\int_{-2\sigma}^{2\sigma} \frac{\varphi(x)\,(2\sigma^2 - x^2)}{\sqrt{4\sigma^2 - x^2}}\, dx \right)^2,$$

$$F(x,y) = \int_{-\infty}^{\infty} \frac{(t \sin^3 t - t^3 \sin t)\, dt}{2\sigma^2 (t^2 - \sin^2 t)^2 - (t^3 \sin t + t \sin^3 t)\, xy + t^2 \sin^2 t\, (x^2 +}$$

Main Ingredients of the Proof

Let $T_N(\varphi) = \sqrt{\frac{b_N}{N}}\left(S_N(\varphi) - \mathbb{E}\, S_N(\varphi)\right).$

Consider

$$Z_N(x) = \mathbb{E}\; e^{ix\, T_N(\varphi)}$$

Our goal is to show that

$$\lim_{N\to\infty} Z_N(x) = Z(x) = \exp\left(-\tfrac{1}{2}x^2 V_{band}(\varphi)\right)$$

This can be proved by showing that any limit of $Z_N(x)$ satisfies the equation

$$Z(x) = 1 - V_{band}(\varphi)\int_0^x y\; Z(y)\, dy$$

One writes

$$Z_N(x) = 1 + \int_0^x Z_N'(y)\,dy,$$

where

$$Z_N'(y) = i\,\mathbb{E}\left\{T_N(\varphi)\, e^{iyT_N(\varphi)}\right\}$$

Using the Fourier inversion formula, ~~this is equal to~~ we can write $Z_N'(y)$

$$i\int_{-\infty}^{\infty} \hat{\varphi}(t)\, Y_N(y,t)\,dt \quad , \text{ where}$$

$$Y_N(y,t) = \sqrt{\frac{b_N}{N}}\;\mathbb{E}\left\{\mathrm{Tr}\, e^{itX_N}\left(e^{iyT_N(\varphi)} - \mathbb{E}\, e^{iyT_N}\right.\right.$$

It is not difficult to show that $Y_N(x,t)$ and its first derivatives are bounded by Const $\sqrt{1+t^2}$ in absolute value

Applying the Duhamel formula (2

$$\left(e^{(M_1+M_2)t} = e^{M_1 t} + \int_0^t e^{M_1(t-s)} M_2 e^{(M_1+M_2)s} \, ds \right)$$

$(M_1 = 0, \ M_2 = i X_N)$ so $e^{itX} = \mathrm{Id} + i \int_0^t X e^{isX} ds$

one has

$$\mathrm{Tr}\, e^{itX_N} = N + i \int_0^t \sum_{j,k} X_{jk} \left(e^{it_1 X_N}\right)_{jk} dt$$

Now one employs the decoupling formula

$$\mathbb{E}\{\xi g(\xi)\} = \sum_{l=0}^{p} \frac{\varkappa_{l+1}}{l!} \mathbb{E}\{g^{(l)}(\xi)\} + \varepsilon_p$$

where

$$|\varepsilon_p| \le C_p \, \mathbb{E}\{|\xi|^{p+2}\} \sup_{t \in \mathbb{R}} |g^{(p+1)}(t)|$$

In particular, one has to study

$$\langle f,g\rangle_N := \frac{1}{N}\,\mathbb{E}\sum^{*} f(M)_{jk}\; g(M)_{kj}$$

(for $f(x)= e^{itx}$, $g(x)=\varphi'(x)$)

where the summation in $\sum^{*}$ is over all pairs (j,k) such th

$1\le j,k\le N$ and

$\min\left(|j-k|,\ N-|j-k|\right)\le b_N$

In the case of full Wigner matrix, of course, one has

$$\langle f,g\rangle_N = \frac{1}{N}\,\mathbb{E}\ \mathrm{Tr}\,(fg)(M)$$

$$\xrightarrow[N\to\infty]{} \int_{-2\sigma}^{2\sigma} f(x)g(x)\,\frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2-x^2}\,dx$$

For Band Wigner matrices the situation is more complicated

Lemma

For all $f, g \in C_0(\mathbb{R})$

$$\langle f, g\rangle := \lim_{N\to\infty} \langle f, g\rangle_N \text{ exists.}$$

The limiting bilinear form is diagonalized by the Tchebyshev polynomials of the second kind $\left(U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta} \right)$ so that

$$\langle U_n, U_m\rangle = \delta_n \delta_{nm} \text{ where}$$

$$\delta_n = \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^{n+1} dx$$

$= \mathrm{Vol}$ of $|y_1| \le \frac{1}{2}, |y_2| \le \frac{1}{2}, \dots, |y_n| \le \frac{1}{2}$ $|y_1 + y_2 + \dots + y_n| \le \frac{1}{2}\}$

Combinatorial Identity

$$\sum_{k=0}^{n-j} \frac{(-1)^k \, (2n-k)!}{k! \, (n-k-j)! \, (n-k+j+1)!} = 0$$

for $j = 0, 1, 2, \ldots, n-1$

$$\sum_{k=0}^{n-j} \frac{(-1)^k \, (2n+1-k)!}{k! \, (n-k-j)! \, (n-k+j+2)!} = 0$$

for $j = 0, 1, \ldots, n-1$.

Proofs follow from the Chu-Vandermonde identity for ${}_2F_1$ hypergeometric functions.