Semilinear obstacle problem with measure data and generalized reflected BSDE

Andrzej Rozkosz
(joint work with T. Klimsiak)

Nicolaus Copernicus University
(Toruń, Poland)

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Parabolic obstacle problem

Let $D \subset \mathbb{R}^d$ $(d \geq 2)$ be a bounded domain, $D_T = [0, T] \times D$. We consider the problem

$$\begin{cases} \frac{\partial u}{\partial t} + Au = -f(\cdot, u) - \mu \text{ on } \{h_1 < u < h_2\}, \\ h_1 \leq u \leq h_2\,, \\ u(T, \cdot) = \varphi, \quad u(t, \cdot)_{|\partial D} = 0, \quad t \in (0, T), \end{cases} \qquad (*)$$

where A is a uniformly elliptic operator of the form

$$A = \sum_{i,j=1}^{d} \left(\frac{\partial}{\partial x_j}(a_{ij}(t, x)\frac{\partial}{\partial x_i}\right)$$

and

- $f : D_T \times \mathbb{R} \to \mathbb{R}$, $\varphi : D \to \mathbb{R}$ - measurable functions satisfying some conditions (to be specified later on), μ is a bounded smooth measure on D_T,
- $h_1, h_2 : D_T \to \bar{\mathbb{R}}$ are measurable functions such that $h_1 \leq h_2$ a.e..

Parabolic obstacle problem

Problem: solve $(*)$ for

- measurable barriers h_1, h_2,
- $\varphi \in L^1(D)$, f satisfying the monotonicity condition in u (for instance $f(x,u) = -|u|^{q-1}u$ for some $q > 1$) and mild integrability conditions,
- $\mu \in \mathcal{M}_{0,b}$ ($\mathcal{M}_{0,b}$ - space of all smooth signed measures on D_T with bounded total variation (for instance $\mu(dt\,dx) = g\,dt\,dx$ for some $g \in L^1(D_T)$).

Known results for irregular barriers:

- M. Pierre (1979, 1980) - linear case ($f = f(x)$) with L^2 data (i.e. $\varphi, f, g \in L^2$).
- T. Klimsiak (2012) - $y \mapsto f(x,y)$ satisfies the Lipschitz condition and the linear growth condition, L^2 data (i.e. $\varphi, f(\cdot,0), g \in L^2$).

Cauchy-Dirichlet problem

Let

$$f_u = f(\cdot, u).$$

We first consider the problem

$$\begin{cases} \frac{\partial u}{\partial t} + Au = -f_u - \mu \text{ on } (0,T) \times D, \\ u(T, \cdot) = \varphi, \quad u(t, \cdot)_{|\partial D} = 0, \quad t \in (0,T). \end{cases} \qquad (**)$$

Cauchy-Dirichlet problem

$\mathbb{X} = \{(X, P_{s,x})\}$ - Markov family associated with A.

Proposition

Any $\mu \in \mathcal{M}^+_{0,b}$ admits a unique positive AF A^μ of $\mathbb{X}$ such that for q.e. $(s,x) \in D_T$,

$$E_{s,x} \int_s^T f(t, X_t)\, dA^\mu_{s,t} = \int_s^T \int_D f(t,y) p(s,x,t,y)\, d\mu(t,y)$$

for every positive Borel function $f : D_T \to \mathbb{R}$.

Here p is the transition density for $\mathbb{X}$ (weak fundamental solution for A).
Ex. If $\mu(dt, dx) = g(t,x)\, dt\, dx$ for some $g \in L^1(D_T)$ then

$$A^\mu_{s,t} = \int_s^t g(\theta, X_\theta)\, d\theta.$$

Assumptions.

(H1) $\mu \in \mathcal{M}_{0,b}(D_T)$,

(H2) $\varphi \in L^1(D)$,

(H3) $f : [0, T] \times D \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that

- (a) $y \mapsto f(t, x, y)$ is continuous for q.e. $(t, x) \in D_T$.
- (b) There exists $\mu \in \mathbb{R}$ such that $(f(t, x, y) - f(t, x, y'))(y - y') \le \mu|y - y'|^2$ for every $y, y' \in \mathbb{R}$ and a.e. $(t, x) \in D_T$.
- (c) $f(\cdot, \cdot, 0) \in L^1(D_T)$,
- (d) $\forall_{c>0}\, (t, x) \mapsto \sup_{|y|\le c} |f(t, x, y)| \in L^1(D_T)$.

Let

$$\xi_s = \inf\{t \geq s : X_t \notin D\}.$$

Theorem (T. Klimsiak, A.R.)

Assume (H1)–(H3)*. Then there exists a unique renormalized solution* u *of* $(**)$*. Moreover,*

$$u(s,x) = E_{s,x}\left\{\mathbf{1}_{\{\xi_s > T\}}\varphi(X_T) + \int_s^{\xi_s \wedge T} f_u(t, X_t)\,dt + \int_s^{\xi_s \wedge T} dA^{\mu}_{s,t}\right\}$$

for q.e. $(s,x) \in D_T$*.*

Remark

u *is an entropy solution of* $(**)$*.*

Renormalized solutions

Theorem (Droniou, Porretta & Prignet)

Each measure $\mu \in \mathcal{M}_{0,b}(D_T)$ admits a decomposition of the form

$$\mu = g_t + div(G) + f,$$

where $g \in L^2(0,T;H_0^1(D))$, $G = (G^1,\dots,G^d) \in L^2(D_T)^d$, $f \in L^1(D_T)$.

Remark. The above decomposition means that

$$\int_{D_T} \eta\, d\mu = -\int_0^T \langle g, \frac{\partial \eta}{\partial t}\rangle\, dt - (G,\nabla\eta)_{L^2} + \int_{D_T} f\eta\, dm_1$$

for every $\eta \in \mathcal{W}(D_T)$ (m_1 denotes the Lebesgue measure on D_T) , where

$$\mathcal{W} = \{\eta \in L^2(0,T;H_0^1(D)) : \frac{\partial \eta}{\partial t} \in L^2(0,T;H^{-1}(D)).$$

Renormalized solutions

Definition (Droniou, Porretta & Prignet)

A measurable $u : D_T \to \mathbb{R}$ is a renormalized solution of $(**)$ if

(a) $f_u \in L^1(D_T)$,

(b) *For some decomposition (g, G, f) of μ,*
$u - g \in L^\infty(0, T; L^2(D))$, $T_k(u - g) \in L^2(0, T; H^1(D))$ *and*

$$\lim_{n\to+\infty} \int_{\{n\le|u-g|\le n+1\}} |\nabla u| \, dm_1 = 0,$$

(c) *For any $S \in W^{2,\infty}(\mathbb{R})$ with compact support,*

$$\frac{\partial}{\partial t}(S(u-g)) + div(a\nabla u S'(u-g)) - S''(u-g)a\nabla u \cdot \nabla(u-g)$$
$$= -S'(u-g)f - div(GS'(u-g)) + GS''(u-g) \cdot \nabla(u-g)$$

in the sense of distributions,

(d) $T_k(u-g)(T) = T_k(\varphi)$ *in $L^2(D)$ for all $k \ge 0$.*

Definition

We say that a pair (u,ν) consisting of a measurable function $u : D_T \to \mathbb{R}$ and a measure ν on D_T is a solution of $(*)$ if

(a) $f_u \in L^1(D_T)$, $\nu \in \mathcal{M}_{0,b}(D_T)$, $h_1 \le u \le h_2$, m_1-a.e.,

(b) *For q.e.* $(s,x) \in D_T$,

$$u(s,x) = E_{s,x}\left\{\mathbf{1}_{\{\xi_s > T\}}\varphi(X_T) + \int_s^{\xi_s \wedge T} f_u(t,X_t)\,dt + \int_s^{\xi_s \wedge T} d(A^{\mu}_{s,t} + A^{\nu}_{s,t})\right\},$$

i.e. u is a renormalized solution of the problem

$$\begin{cases} \frac{\partial u}{\partial t} + Au = -f_u - \mu - \nu \text{ on } (0,T)\times D, \\ u(T,\cdot) = \varphi, \quad u(t,\cdot)_{|\partial D} = 0, \quad t \in (0,T). \end{cases}$$

We say that $u \in \mathcal{FD}$ if the process $[s, T] \mapsto u(t, X_t)$ is càdlàg under $P_{s,x}$ for q.e. $(s, x) \in D_T$.

Definition (continued)

(c) *For every $h_1^*, h_2^* \in \mathcal{FD}$ such that $h_1 \le h_1^* \le u \le h_2^* \le h_2$, m_1-a.e. we have*

$$\int_s^{\xi_s \wedge T} (u(t, X_t) - h_{1-}^*(t, X_t))\, dA_{s,t}^{\nu^+} = 0, \quad P_{s,x}\text{-a.s.}$$

and

$$\int_s^{\xi_s \wedge T} (h_{2-}^*(t, X_t) - u(t, X_t))\, dA_{s,t}^{\nu^-} = 0, \quad P_{s,x}\text{-a.s.}$$

for q.e. $(s, x) \in D_T$ (Here $h_{i-}^*(t, X_t) = \lim_{s<t, s\to t} h_i^*(s, X_s)$).

Comments.

1. If h_1, h_2 are quasi-continuous and $h_1(T,\cdot) \le \varphi \le h_2(T,\cdot)$, m-a.e. then condition (c) says that

$$\int_{D_T} (u - h_1)\, d\nu^+ = \int_{D_T} (h_2 - u)\, d\nu^- = 0.$$

2. In the linear case with L^2 data condition (c) coincides with the condition introduced by M. Pierre (1979).

Additional assumption.

(H4) There exists a renormalized solution v of the problem

$$\begin{cases} \frac{\partial v}{\partial t} + Av = -\lambda, \\ v(T,\cdot) = \psi, \quad v(t,\cdot)_{|\partial D} = 0, \quad t \in (0,T) \end{cases}$$

with some $\lambda \in \mathcal{M}_{0,b}(D_T)$ and measurable ψ satisfying $\psi \geq \varphi$ such that $f_v \in L^1(D_T)$ and $h_1 \leq v \leq h_2$, m_1-a.e. on D_T.

Parabolic obstacle problem - main result

Theorem (T. Klimsiak, A.R.)

Assume (H1)–(H4).

(i) *There exists a unique solution* (u,ν) *of* $(*)$. *Moreover,*

(a) $u\in\mathcal{FD}$ *and*

$$E_{s,x}\sup_{s\le t\le T}|u(t,X_t)|^q<\infty,\quad E_{s,x}\Big(\int_s^T|u(t,X_t)|^2\,dt\Big)^{q/2}<\infty$$

for $q\in(0,1)$,

(b) $T_ku\in L^2(0,T;H^1_0(D))$ *for* $k>0$, *where* $T_ku=(-k)\vee u\wedge k$,

(c) $u\in L^q(0,T;W^{1,q}_0(D))$ *for* $q<\frac{d+2}{d+1}$.

(ii) *If* h_1,h_2 *are quasi-continuous and* $h_1(T,\cdot)\le\varphi\le h_2(T,\cdot)$, *m-a.e. then* u *is quasi-continuous.*

Parabolic obstacle problem - main result

Theorem (continued)

(iii) *Let u_n be a renormalized solution of the problem*

$$\begin{cases} \frac{\partial u_n}{\partial t} + A_t u_n = -f_{u_n} - \mu - n(u_n - h_1)^- + n(u_n - h_2)^-, \\ u_n(T,\cdot) = \varphi, \quad u_n(t,\cdot)_{|\partial D} = 0, \quad t \in (0,T). \end{cases}$$

Then $u_n \to u$ q.e. on D_T and $\nabla u_n \to \nabla u$ in measure m_1,

(iv) *If $h_2 = +\infty$ then $\nu_n \to \nu$ weakly, where $\nu_n = n(u_n - h_1)^-$. Similar statement in case $h_1 = -\infty$. In the general case more complicated formulation.*

Sketch of proof (for $\varphi = 0$)

(i) We prove existence of a unique solution (Y, Z, L) of RBSDE

$$Y_t = \int_{t\wedge t}^{\tau} f(X_t, Y_t)\,dt + \int_{\tau\wedge t}^{\tau} dA_t^{\mu} + \int_{\tau\wedge t}^{\tau} dL_t\,dt + \int_{\tau\wedge t}^{\tau} dM_s, \quad P_{s,x}\text{-a.s.}$$

for q.e. $(s, x) \in D_T$ and show that $L = A^{\nu}$ for some $\nu \in \mathcal{M}_{0,b}$.

(ii) Taking $t = s$ and integrating with respect to $P_{s,x}$ we conclude that u defined by $u(s, x) = Y_s$, $P_{s,x}$-a.s. satisfies the nonlinear Fenynman-Kac formula

$$u(s, x) = E_{s,x}\left\{\int_s^{\xi_s\wedge T} f_u(t, X_t)\,dt + \int_s^{\xi_s\wedge T} d(A^{\mu}_{s,t} + A^{\nu}_{s,t})\right\}.$$

(iii) We show that $f_u \in L^1(D_T)$.

Sketch of proof (for $\varphi = 0$)

(iv) We know that

$$u(s,x) = E_{s,x} \int_s^{\xi_s \wedge T} dA^{\gamma}_{s,t}, \quad \gamma = f_u \cdot m + \mu + \nu.$$

We choose a generalized nest $\{F_n\}$ (i.e. $F_n \subset F_{n+1}$, $\mathrm{cap}(K \setminus F_n) \to 0$, $|\gamma|(D_T \setminus \bigcup_{n=1}^{\infty} F_n) = 0$) such that $\mathbf{1}_{F_n} \cdot \gamma \in \mathcal{M}_{0,b} \cap \mathcal{W}'$, where $\mathcal{W}'$ is the space dual to the space

$$\mathcal{W} = \{\eta \in L^2(0,T; H^1_0(D)) : \frac{\partial \eta}{\partial t} \in L^2(0,T; H^{-1}(D)).$$

and we define

$$u_n(s,x) = E_{s,x} \int_s^{\xi_s \wedge T} A^{\mathbf{1}_{F_n} \cdot \gamma}_{s,t}.$$

Sketch of proof (for $\varphi = 0$)

(iv) Then (L^2 theory of linear equations) u_n is a weak solution of the problem

$$\begin{cases} \frac{\partial u_n}{\partial t} + Au_n = -\mathbf{1}_{F_n} \cdot \gamma \text{ on } (0,T) \times D, \\ u_n(T,\cdot) = 0, \quad u_n(t,\cdot)_{|\partial D} = 0, \quad t \in (0,T), \end{cases}$$

and hence u_n is a renormalized solution.

(v) Since $\|\mathbf{1}_{F_n} \cdot \gamma - (f_u \cdot m + \mu + \nu)\|_{TV}$, it follows that $u_n \to u$ q.e. in D_T and u is a renormalized solution of $(**)$.

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