Harnack inequalities for subordinate Brownian motions

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joint work with Panki Kim
Introduction

\[ S = (S_t)_{t \geq 0} \]
subordinator
(i.e. an increasing
Lévy process in \( \mathbb{R} \))

\[ B = (B_t)_{t \geq 0} \]
Brownian motion in \( \mathbb{R}^d \)
independent of \( S \)

\[ X_t = B_{S_t} \]
subordinate
Brownian motion

Harmonic function

\[ u: \mathbb{R}^d \to [0, \infty) \] is harmonic in \( D \subset \mathbb{R}^d \) open and bounded (w.r.t \( X \)) if
for any open \( B \subset \overline{B} \subset D \)

\[ u(x) = \mathbb{E}_x[u(X_{\tau_B})] \quad \forall \ x \in B . \]

\[ \tau_B = \inf\{ t > 0 : X_t \notin B \} \]
Harnack inequality

**Harnack inequality** holds for $X$ if there is a constant $C > 0$ such that for any $r \in (0, 1)$ and any $u : \mathbb{R}^d \to [0, \infty)$ which is harmonic in $B(0, r)$

$$u(x) \leq C u(y) \quad \forall \ x, y \in B(0, \frac{r}{2}).$$

Some applications:

- boundary Harnack principle
- Green function estimates (→ talk of P. Kim)
- regularity estimates of harmonic functions
Known cases of SBM when HI holds:

- **rotationally invariant** $\alpha$-stable processes $(0 < \alpha < 2)$:
  \[
  \mathbb{E}[e^{i\xi \cdot X_t}] = e^{-t|\xi|^\alpha}
  \]
  or, more generally, when $0 < \alpha < 2$ and
  \[
  \mathbb{E}[e^{i\xi \cdot X_t}] = e^{-t \phi(|\xi|^2)}
  \]
  with
  \[
  \lim_{\lambda \to \infty} \frac{\phi(\lambda x)}{\phi(\lambda)} = x^{\alpha/2} \quad \forall \ x > 0
  \]

Our motivation:

- **geometric** $\beta$-stable process $(0 < \beta \leq 2)$:
  \[
  \mathbb{E}[e^{i\xi \cdot X_t}] = e^{-t \log(1+|\xi|^\beta)}
  \]
  
  \[
  \phi(\lambda) = \log(1 + \lambda^{\beta/2})
  \]
  
  \[
  \text{weaker form of HI was known before (Šikić, Song, Vondraček, PTRF '06)}:
  \]
  \[
  u(x) \leq C(r) u(y) \quad \forall \ x, y \in B(0, \frac{r}{2})
  \]
  with
  \[
  \lim_{r \to 0^+} C(r) = \infty
  \]
Subordinators and Subordinate Brownian Motions

\[ S = (S_t)_{t \geq 0} \text{ subordinator} \]

Laplace transform

\[ \mathbb{E}[e^{-\lambda S_t}] = e^{-t\phi(\lambda)}, \quad \lambda > 0 \]

Laplace exponent

\[ \phi(\lambda) = b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t})\mu(dt) \]

- \( b \geq 0 \)
- \( \mu \) Lévy measure

\[ \int_{(0,\infty)} (1 \wedge t)\mu(dt) < \infty \]

Potential measure

\[ U(A) = \mathbb{E} \left[ \int_0^\infty 1\{S_t \in A\} \, dt \right] \]

\( B = (B_t, \mathbb{P}_x) \) Brownian motion in \( \mathbb{R}^d \perp S \)

Subordinate Brownian motion \( X = (X_t, \mathbb{P}_x) \)

\[ X_t := B_{S_t}, \quad t \geq 0 \]
Properties of SBM

- $X$ is a Lévy process
- Characteristic exponent

\[ \mathbb{E}_x \left[ e^{i\xi \cdot (X_t - x)} \right] = e^{-t \Phi(\xi)} \quad \Phi(\xi) = \int_{\mathbb{R}^d} \left( 1 - e^{i\xi \cdot x} + i\xi \cdot x 1_{\{|x| < 1\}} \right) \Pi(dx) \]

Lévy measure $\Pi$ is of the form

\[ \Pi(dx) = j(|x|) \, dx \quad j(r) = (4\pi)^{-d/2} \int_0^\infty t^{-d/2} e^{-\frac{r^2}{4t}} \mu(dt) \]

- If $X$ is transient, the Green function is given by

\[ G(x, y) = (4\pi)^{-d/2} \int_0^\infty t^{-d/2} e^{-\frac{|x-y|^2}{4t}} U(dt), \quad x, y \in \mathbb{R}^d, x \neq y \]

- The Green function of an open set $D \subset \mathbb{R}^d$ is given by

\[ G_D(x, y) = G(x, y) - \mathbb{E}_x [G(X_{\tau_D}, y); \tau_D < \infty], \quad x, y \in D, x \neq y \]

Note: $G_D(x, y) \leq G(x, y)$
Assumptions

(A-1) $\mu(0, \infty) = \infty$ and

$$\mu(dt) = \mu(t) \, dt, \quad \mu: (0, \infty) \to (0, \infty) \text{ decreasing}$$

(A-2) $U(dt) = u(t) \, dt, \quad u: (0, \infty) \to (0, \infty) \text{ decreasing}$

(A-3) there exist $\sigma > 0$ and $\alpha \in [0, 2)$ such that

$$\frac{\phi'(\lambda x)}{\phi'(\lambda)} \leq \sigma x^{\frac{\alpha}{2} - 1} \quad \forall \ \lambda, x \geq 1$$

Remarks.

1. (A-3) holds, in particular, when $\phi$ varies regularly at infinity with index $\alpha/2$ with $0 < \alpha < 2$ (by the Karamata monotone density theorem), i.e.

$$\lim_{\lambda \to \infty} \frac{\phi(\lambda x)}{\phi(\lambda)} = x^{\frac{\alpha}{2}} \quad \forall \ x > 0$$

2. (A-3) $\implies$

$$b = 0 \quad \text{and} \quad \frac{\phi(\lambda x)}{\phi(\lambda)} \leq \sigma' x^{\frac{\alpha}{2}} \quad \forall \ \lambda, x \geq 1$$
Main result

**Theorem (Harnack Inequality; Kim - M, EJP ’12)**

Let $S$ be a subordinator satisfying (A-1), (A-2) and (A-3). Assume that the Lévy density $J(x) = j(|x|)$ of the corresponding subordinate Brownian motion $X$ satisfies

$$j(r + 1) \leq j(r) \leq c j(r + 1) \quad \forall \ r > 1.$$  \hfill (*)

for some constant $c \geq 1$. Then the Harnack inequality holds for $X$, i.e. there exists a constant $C > 0$ such that

- for any $r \in (0, 1)$
- for any non-negative function $u$ which is harmonic in $B(0, r)$

$$u(x) \leq C u(y) \quad \forall \ x, y \in B(0, \frac{r}{2}).$$

**Remark.** New result for ’$\alpha = 0$’, e. g.

- geometric stable

$$\phi(\lambda) = \log(1 + \lambda^{\beta/2}), \ 0 < \beta \leq 2$$

- iterated geometric stable

$$\phi(\lambda) = \log(1 + \log(1 + \lambda^{\beta/2})^{\beta/2}), \ 0 < \beta \leq 2$$
Key Estimates 1

\[ \Pi(dx) = j(|x|) \, dx \quad j(r) = (4\pi)^{-d/2} \int_0^\infty t^{-d/2} e^{-\frac{r^2}{4t}} \mu(t) \, dt \]

\[ G(x, y) = g(|x - y|) \quad g(r) = (4\pi)^{-d/2} \int_0^\infty t^{-d/2} e^{-\frac{r^2}{4t}} u(t) \, dt \]

Proposition (Lévy and potential density, Green function; Kim-M, EJP ’12)

\[ \mu(t) \asymp t^{-2} \phi'(t^{-1}), \; t \to 0^+ \quad j(r) \asymp r^{-d-2} \phi'(r^{-2}), \; r \to 0^+ \]

\[ u(t) \asymp t^{-2} \frac{\phi'(t^{-1})}{\phi(t^{-1})^2}, \; t \to 0^+ \quad g(r) \asymp r^{-d-2} \frac{\phi'(r^{-2})}{\phi(r^{-2})^2}, \; r \to 0^+ \]

Example (Geometric stable process \( \phi(\lambda) = \log(1 + \lambda^{\beta/2}) \) (0 < \( \beta \) < 2))

\[ \mu(t) \asymp \frac{1}{t} \quad j(r) \asymp \frac{1}{r^d} \]

\[ u(t) \asymp \frac{1}{t(\log \frac{1}{t})^2} \quad G(x, y) \asymp \frac{1}{|x-y|^d} \left( \log \frac{1}{|x-y|} \right)^2 \]
Set $B_r := B(0, r)$. 

**Proposition (Green function of the ball; Kim - M, EJP ’12)**

There exist $0 < a < b < 1$ such that

$$G_{B_r}(x, y_1) \geq c |x - y_1|^{-d-2} \frac{\phi'(|x - y_1|^{-2})}{\phi(|x - y_1|^{-2})^2}$$

$$G_{B_r}(x, y_2) \asymp r^{-d-2} \frac{\phi'(r^{-2})}{\phi(r^{-2})} \mathbb{E}_y \tau_{B_r}$$
Idea of the Proof

Let $u$ be a nonnegative function that is harmonic in $B_{2r}$ and $x_1, x_2 \in B_{ar}$.

\[
\begin{align*}
    u(x) &= \mathbb{E}_x[u(X_{\tau_{B_r}})] \\
    &= \int_{B_r^c} \int_{B_r} G_{B_r}(x, y) j(|z - y|) u(z) \, dy \, dz \\
    &= \int_{B_r^c} \int_{B_r} + \int_{B_r^c} \int_{B_r \setminus B_{br}} =: u_1(x) + u_2(x)
\end{align*}
\]

- Green function estimate in $B_r \setminus B_{br} \implies$
  \[
  u_2(x_1) \leq c_1 \frac{r^{-d-2} \phi'(r^{-2})}{\phi(r^{-2})} \int_{B_r^c} \int_{B_r \setminus B_{br}} E_{y_2}[\tau_{B_r}] j(|z-y_2|) u(z) \, dy_2 \, dz \leq c_2 u_2(x_2)
  \]

- $(\star)$ + fact that $G_{B_r} \leq G +$ Green function lower estimate in $B_{ar} \implies$
  \[
  \begin{align*}
  u_1(x_1) &\leq c_3 \cdot \int_{B_{br}} G_{B_r}(x_1, y_1) \, dy_1 \cdot \int_{B_r^c} j(|z|) u(z) \, dz \leq c_6 u_1(x_2) \\
  &\leq \frac{c_4}{\phi(r^{-2})} \leq c_5 \int_{B_{ar}} G_{B_r}(x_2, y_1) \, dy_1
  \end{align*}
  \]
Thank you for your attention!