Some inequalities concerning Dirichlet forms of stable processes

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If $\lambda I \leq A \leq \Lambda I$, then

$$\int_B \langle \nabla u, A\nabla u \rangle \, dx \asymp \int_B |\nabla u|^2 \, dx.$$

If $\lambda|x-y|^{-d-\alpha} \leq k(x,y) \leq \Lambda|x-y|^{-d-\alpha}$, then

$$\int_B \int_B (u(x)-u(y))^2 k(x,y) dy \, dx \asymp \int_B \int_B (u(x)-u(y))^2 |x-y|^{-d-\alpha} dy \, dx.$$

Question: under what assumption the latter inequalities hold?

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Motivation

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If $\lambda |x-y|^{-d-\alpha} \leq k(x,y) \leq \Lambda |x-y|^{-d-\alpha}$, then

$$\int_B \int_B (u(x) - u(y))^2 k(x,y) dy \, dx \asymp \int_B \int_B (u(x) - u(y))^2 |x-y|^{-d-\alpha} dy \, dx.$$

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Notation etc.

Notation:

$$\mathcal{E}_D^k(u,u) = \int_D \int_D (u(y) - u(x))^2 k(x,y)\, dx\, dy,$$

$$\mathcal{E}_D^\alpha(u,u) = \alpha(2-\alpha) \int_D \int_D (u(y) - u(x))^2 |x-y|^{-d-\alpha}\, dx\, dy.$$

We will always assume that

$$0 \le L(x-y) \le k(x,y) \le U(x-y)$$

for some *symmetric* functions L and U: $L(x) = L(-x)$, $U(x) = U(-x)$, and then we will impose some conditions on them.

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Local comparability

Theorem

If

(K) $L(x-y) \le k(x,y) \le U(x-y)$,

(U) *there exists* $C_1 > 0$ *such that for every* $r \in (0,1]$

$$\int_{\mathbb{R}^d} \left(r^2 \wedge |z|^2\right) U(z)\, dz \le C_1 r^{2-\alpha},$$

(L) *there exist* $a > 1$ *and* C_2, C_3 *such that every annulus* $B_{a^{-n+1}} \setminus B_{a^{-n}}$ $(n = 0, 1, \ldots)$ *contains a ball* B_n *with radius* $C_2 a^{-n}$, *such that*

$$L(z) \ge C_3(2-\alpha)|z|^{-d-\alpha}, \quad z \in B_n,$$

then for all balls B *of radius* ≤ 1

$$c\mathcal{E}_B^{\alpha}(u,u) \le \mathcal{E}_B^{k}(u,u) \le c'\mathcal{E}_B^{\alpha}(u,u).$$

Cut-off function

For the Harnack inequality we will need the following assumption.

For some constant $c > 0$, and every $R, \rho \in (0,1)$ there is a nonnegative function $\tau \in C^\infty(\mathbb{R}^d)$ with $\mathrm{supp}(\tau) = \overline{B_{R+\rho}}$, $\tau(x) \equiv 1$ on B_R

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \big(\tau(y) - \tau(x)\big)^2 k(x,y)\, dy \leq c\rho^{-\alpha}\,. \tag{B}$$

(B) follows from (K) and (U) from the previous theorem.

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Harnack inequality – reformulation

Remark for the $\Delta^{\alpha/2}$ case $(0 < \alpha \le 2)$. Let
$H(f|B_r) = \int_{B_r^c} f(y) \frac{C_\alpha^d}{(|y|^2 - r^2)^{\alpha/2}} \left(r^2 - |x|^2\right)^{\alpha/2} |y-x|^{-d} \, dy$, $\alpha < 2$.

There exists c such that if $u : \mathbb{R}^d \to \mathbb{R}$ is harmonic in B_1, then

$$u(x) \le cu(y) + c^2 H(u^-|B_{3/4})(0), \quad x, y \in B_{1/2}.$$

If $u \ge 0$ in B_1, then

- when $\alpha = 2$,

$$u(x) \le cu(y), \quad x, y \in B_{1/2},$$

- when $\alpha < 2$,

$$u(x) \le cu(y) + c'\alpha(2-\alpha) \int_{\mathbb{R}^d \setminus B_1} \frac{u^-(z)}{|z|^{d+\alpha}} \, dz, \quad x, y \in B_{1/2},$$

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There exists c such that if $u : \mathbb{R}^d \to \mathbb{R}$ is harmonic in B_1, then

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Theorem (Weak Harnack inequality)

Assume $\mathcal{E}^{\alpha}_{B}(u,u) \asymp \mathcal{E}^{k}_{B}(u,u)$ *and (B) holds. There are* $p_0, c > 0$ *such that for every* $u \in L^{\infty}(\mathbb{R}^d) \cap H^{\alpha/2}_{loc}(B_1)$ *with* $u \geq 0$ *in* B_1 *satisfying* $\mathcal{E}^{k}_{\mathbb{R}^d}(u,\phi) \geq 0$ *for every nonnegative* $\phi \in C^{\infty}_{c}(B_1)$ *the following inequality holds:*

$$\Big(\int\limits_{B_{1/2}} u(x)^{p_0}\, dx \Big)^{1/p_0} \leq c \inf_{B_{1/4}} u + c \sup_{x \in B_{1/2}} \int\limits_{\mathbb{R}^d \setminus B_1} u^{-}(z) k(x,z)\, dz\,.$$

Similarly, (K), (U), (L) and $\limsup\limits_{R\to\infty} R^{\gamma} \int\limits_{|z|>R} U(z)\, dz \leq 1$ for some $\gamma \in (0,\alpha)$ imply Hölder estimates for weak *solutions*.

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Harnack inequality or Hölder regularity

- Komatsu 1995
- Bass, Levin 2002
- Chen, Kumagai 2003
- Kassmann 2007
- ...
- Bogdan, Stós, Sztonyk 2003, ...
- Silvestre 2006
- Cafarelli, Silvestre 2009
- Barles, Chasseigne, Imbert 2011
- ...

Poincaré inequality – notation

For a function u we denote

$$u_{B_r} = \frac{1}{|B_r|} \int_{B_r} u(x)\, dx,$$

$$u_{B_r}^{\phi} = \frac{\int_{B_r} u(x)\phi(x)\, dx}{\int_{B_r} \phi(x)\, dx},$$

where ϕ is a 'nice' weight function (i.e., radial, strictly positive on B_1, with a nonincreasing càdlàg profile Φ).

Known:

$$\int_{B_r} |u(x) - u_{B_r}|^p \, dx \le C_{p,d} r^p \int_{B_r} |\nabla u(x)|^p \, dx, \qquad (1)$$

for every $u \in W^1_p(B_r)$

Corollary

Let $p \ge 1$. Let $X = \mathbb{R}^d$ equipped with the Lebesgue measure and the Euclidean metric. There exists a constant $c_{p,d,\phi}$ such that

$$\int_{B_1} |u(x) - u^{\phi}_{B_1}|^p \phi(x) \, dx \le c_{p,d,\phi} \int_{B_1} |\nabla u(x)|^p \phi(x) \, dx, \qquad (2)$$

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Saloff-Coste, *Aspects of Sobolev-Type-Inequalities*, 2002

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Theorem

If for any $u \in L^2(B_1)$ *and* $\frac{1}{2} < r \leq 1$ *it holds*

$$\int_{B_r} |u(x) - u_{B_r}|^2 \, dx \leq C \int_{B_r} \int_{B_r} |u(x) - u(y)|^2 k(x,y) \, dy \, dx,$$

then also

$$\int_{B_1} |u(x) - u^{\phi}_{B_1}|^2 \phi(x) \, dx$$

$$\leq C \frac{2^{3p}|B_1|}{|B_{1/2}|} \frac{\Phi(0)}{\Phi(\frac{1}{2})} \int_{B_1} \int_{B_1} |u(x) - u(y)|^2 k(x,y)(\phi(y) \wedge \phi(x))(*) dy \, dx,$$

where ϕ *is a 'nice' weight function.*

Chen, Kim, Kumagai, 2008: additional factor $R^{2-\alpha} 1_{|x-y| \leq \frac{1}{R}}$ in $(*)$.

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