On solutions of linear stochastic equations in the critical case.

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University of Wrocław

joint with Konrad Kolesko (Wrocław)

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where X and (A, B) are independent

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We assume that all the random variables are positive

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- If $\mathbb{E}\log A = \phi'(0) < 0$ and $\mathbb{E}\log^+|B| < \infty$, then equation I possesses a unique solution.
- If moreover $\phi(\alpha) = 1$ for some $\alpha > 0$ and .., then $\mathbb{P}[X > t] \sim t^{-\alpha}$

  (Kesten 73, Grincevicius 75, Goldie 91)

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Theorem [Brofferio, Damek, B.]

$$\lim_{x\to\infty} \nu(\alpha x, \beta x] = \log\frac{\beta}{\alpha}\ C_+, \qquad \nu(dx) \sim \frac{C_+ dx}{x}.$$

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$$\mathbb{P}[R > t] \sim \frac{C}{t^\beta}$$

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- one is the minimal one R and $\mathbb{P}[R > t] \sim t^{-\beta}$
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- there exists $\alpha \in (0,1)$ such that $\phi(\alpha) = 1$ and $\phi'(\alpha) = 0$;
- $\mathbb{E}[N^{1+\delta} + B^{\alpha+\delta} + \sum_{i=1}^{N}(A_i^{-\delta} + A_i^{\alpha+\delta})] < \infty$;
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Remark: existence of a solution was proved by Alsmeyer and Meiners for $\alpha < 1/5$.

Sketch of the proof

Let $\Lambda(s) = \mathbb{E}[e^{-sR}]$ be the Laplace transform of R. In view of the Tauberian theorem:

$$t^{\alpha} P[R > t] \sim L(t) \text{ as } t \to \infty \text{ iff } \frac{1 - \Lambda(s)}{s^{\alpha}} \sim L(1/s) \text{ as } s \to 0,$$

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$$\mathbb{E}[h(Y)] = \mathbb{E}\left[\sum_{i=1}^{N} h(-\log A_i) A_i^{\alpha}\right].$$

Then

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We consider the Poisson equation:

$$D(x) = \mathbb{E}[D(x+Y)] - G(x).$$

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We define $h_x(y) = D(x+y)/D(x)$, then the family h_x is relatively compact (in the topology of uniform convergence on compacts). We write the Poisson equation at $x+y$ and divide by $D(x)$:

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Since $h(0) = 1$ and h is positive: $h \equiv 1$.

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$\tau = \inf\{n : S_n > 0\}$, then for fixed n, $n \wedge \tau$ is a bounded stopping time.

By the optional stopping theorem

$$\mathbb{E}\left[D(x+S_{n\wedge\tau}) - \sum_{i=0}^{n\wedge\tau-1} G(x+S_i)\right] = \mathbb{E}[M_{n\wedge\tau}] = \mathbb{E}[M_0(x)] = D(x).$$

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We pass with n to infinity and use the duality lemma
($T_n = \inf\{n > T_{n-1} : S_n \le S_{T_{n-1}}\}$ - ladder times):

$$\mathbb{E}\big[D(x+S_\tau)\big] - D(x) = \mathbb{E}\left[\sum_{i=0}^{\infty} G(x+S_{T_i})\right].$$

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Integrating both sides of equation above we get

$$\int_0^x \mathbb{E}\Big[\big[D(z + S_\tau)\big] - D(z)\Big] dz = \int_0^x \mathbb{E}\Bigg[\sum_{i=0}^{\infty} G(z + S_{T_i})\Bigg] dz.$$

$$\mathbb{E}\Bigg[\int_0^{S_\tau} D(x+z)dz\Bigg] = \int_0^x \mathbb{E}\big[G * U_{T_1}(z)\big] dz + \mathbb{E}\Bigg[\int_0^{S_\tau} D(z)dz\Bigg].$$

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Finally, since $\lim_{x\to\infty} \frac{D(x+y)}{D(x)} = 1$

$$\lim_{x\to\infty} \frac{D(x)}{x} = \lim_{x\to\infty} \frac{D(x)}{x} \frac{1}{\mathbb{E}S_\tau} \mathbb{E}\Bigg[\int_0^{S_\tau} \frac{D(x+z)}{D(x)} dz\Bigg]$$
$$= \frac{1}{\mathbb{E}S_\tau} \lim_{x\to\infty} \frac{1}{x} \int_0^x G * U_{T_1}(z)dz = \frac{\int G}{\mathbb{E}S_\tau \mathbb{E}[-S_{T_1}]}$$

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If $\int G = 0$, then $D(x) \to \frac{2\int xG(x)dx}{\sigma^2} = C_2$.

Next we prove that $C_1 + C_2 > 0$, thus either $\mathbb{P}[R > t] \sim \frac{C_1 \log t}{t^\alpha}$ or $\mathbb{P}[R > t] \sim \frac{C_2}{t^\alpha}$

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Recall $R = \sum_{\gamma \in \mathcal{T}} \Pi_\gamma B_\gamma$. Assume $B_\gamma = 1$.

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Define $\tilde{R} = \max_{\gamma \in \mathcal{T}} \Pi_\gamma$.
Define new random variable:

$$\mathbb{E}[h(Y)] = \mathbb{E}\left[\sum_{i=1}^{N} h(-\log A_i) A_i^\alpha\right].$$

Then Y is a centered random variable with second moment. Let Y_i be a sequence of iid copies of Y and S_n partial sums of Y_i's. Then by induction we prove

$$\mathbb{E}[e^{\alpha S_n} f(S_1, S_2, .., S_n)] = \mathbb{E}\left[\sum_{|\gamma|=n} f(-\log \Pi_{\gamma_1}, -\log \Pi_{\gamma_2}, .., -\log \Pi_{\gamma_n})\right].$$

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We have

$$\begin{aligned}
\mathbb{P}[\tilde{R} > t] &= \mathbb{P}[\max_{\gamma \in \mathcal{T}} \Pi_\gamma > t] \\
&\leq \sum_n \mathbb{E}\left[\sum_{|\gamma|=n} \mathbf{1}_{\{\Pi_{\gamma_1} < t, .., \Pi_{\gamma_{n-1}} < t, \Pi_{\gamma_n} > t\}}\right] \\
&= \sum_n \mathbb{E}\left[\sum_{|\gamma|=n} \mathbf{1}_{\{-\log \Pi_{\gamma_1} > -\log t, .., -\log \Pi_{\gamma_{n-1}} > -\log t, -\log \Pi_{\gamma_n} < -\log t\}}\right] \\
&= \sum_n \mathbb{E}\left[e^{\alpha S_n} \mathbf{1}_{\{S_1 > -\log t, .., S_{n-1} > -\log t, S_n < -\log t\}}\right] \leq t^{-\alpha}
\end{aligned}$$

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It is sufficient to prove that $\mathbb{P}[R > t, \tilde{R} < t] \leq \frac{C}{t^\alpha}$.

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\mathbb{P}[R > t, \tilde{R} < t] &= \mathbb{P}\left[\sum_{\gamma\in\mathcal{T}} \Pi_\gamma > t \text{ and } \max_{\gamma\in\mathcal{T}} \Pi_\gamma < t\right] \\
&\leq \mathbb{P}\left[\sum_{\gamma\in\mathcal{T}} \Pi_\gamma \mathbf{1}_{\{\Pi_{\gamma'} \leq t \text{ for } \gamma' \leq \gamma\}} > t\right] \\
&\leq \frac{1}{t}\mathbb{E}\left[\sum_{\gamma\in\mathcal{T}} \Pi_\gamma \mathbf{1}_{\{\Pi_{\gamma'} \leq t \text{ for } \gamma' \leq \gamma\}}\right] \\
&= \frac{1}{t}\sum_n \mathbb{E}\left[e^{\alpha S_n} e^{-S_n} \mathbf{1}_{\{-S_k \leq \log t \text{ for } k \leq n\}}\right] \\
&= \frac{1}{t} t^{1-\alpha} \sum_n \mathbb{E}\left[e^{(\alpha-1)(S_n + \log t)} \mathbf{1}_{\{S_k + \log t \leq 0 \text{ for } k \leq n\}}\right]
\end{aligned}$$

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Finally we have to prove that for a centered random walk S_n the function

$$R(x) = \mathbb{E}\Bigg[\sum_{n=0}^{\infty} e^{-(x+S_n)} \mathbf{1}_{\{S_j + x \geq 0 \text{ for } j \leq n\}}\Bigg] \text{ is bounded}$$

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Let $\tau = \inf\{n : S_n < 0\}$. Then

$$\begin{aligned}
R(x) &= \mathbb{E}\Big[\sum_{n=0}^{\infty} e^{-(x+S_n)} \mathbf{1}_{\{S_j+x\geq 0 \text{ for } j\leq n\}}\Big] \\
&= \mathbb{E}\Big[\sum_{n=0}^{\tau-1} e^{-(x+S_n)} \mathbf{1}_{\{S_j+x\geq 0 \text{ for } j\leq n\}}\Big] + \mathbb{E}\Big[\sum_{n=\tau}^{\infty} e^{-(x+S_n)} \mathbf{1}_{\{S_j+x\geq 0 \text{ for } j\leq n\}}\Big] \\
&= \mathbb{E}\Big[\sum_{n=0}^{\tau-1} e^{-(x+S_n)}\Big] \mathbf{1}_{\{x\geq 0\}} + \mathbb{E}\Big[\sum_{n=\tau}^{\infty} e^{-(x+S_n)} \mathbf{1}_{\{S_j+x\geq 0 \text{ for } j\leq n\}}\Big] \\
&= \mathbb{E}\Big[\sum_{n=0}^{\tau-1} e^{-S_n}\Big] e^{-x} \mathbf{1}_{\{x\geq 0\}} + \mathbb{E}\big[R(x+S_\tau)\big] \\
&= Ce^{-x}\mathbf{1}_{\{x\geq 0\}} + \mathbb{E}\big[R(x+S_\tau)\big]
\end{aligned}$$

Thus, $R = U * f$, where U is the potential and $f(x) = e^{-x}\mathbf{1}_{\{x\geq 0\}}$ and by the renewal theorem the function R is bounded.