Second Quantised Representation of Mehler Semigroups Associated with Banach Space Valued Lévy processes

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Talk at *6th International Conference on Stochastic Analysis and its Applications*,
Bedlewo, Poland

13th September 2012

Talk based on joint work with Jan van Neerven (Delft)

Outline of Talk

- Mehler semigroups arise as transition semigroups of linear SPDEs with additive Lévy noise.
- Szymon Peszat has shown that these semigroups can be expressed functorially using second quantisation.
- Peszat's approach is based on chaotic decomposition formulae due to Last and Penrose.
- We pursue an alternative strategy using vectors related to exponential martingales.

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Reproducing Kernel Hilbert Space (RKHS)

E is a real separable Banach space, E^* is its dual,

$\langle \cdot, \cdot \rangle$ is pairing $E \times E^* \to \mathbb{R}$.

$T \in \mathcal{L}(E^*, E)$ is

- *symmetric* if for all $a, b \in E^*$, $\langle Ta, b \rangle = \langle Tb, a \rangle$,
- *positive* if for all $a \in E^*$, $\langle Ta, a \rangle \geq 0$.

If T is positive and symmetric, $[\cdot, \cdot]$ is an inner product on $\text{Im}(T)$, where

$$[Ta, Tb] = \langle Ta, b \rangle.$$

RKHS H_T is closure of $\text{Im}(T)$ in associated norm.

Inclusion $\iota_T : \text{Im}(T) \to E$ extends to a continuous injection $\iota_T : H_T \to E$.

$$T = \iota_T \circ \iota_T^*.$$

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μ a Borel measure on E. Reversed measure $\widetilde{\mu}(E) = \mu(-E)$. μ *symmetric* if $\widetilde{\mu} = \mu$.

μ a (Borel) probabililty measure on E Its *Fourier transform/ characteristic function* is the mapping $\widehat{\mu} : E^* \to \mathbb{C}$ defined for $a \in E^*$ by:

$$\widehat{\mu}(a) = \int_E e^{i\langle x, a\rangle} \mu(dx).$$

A measure $\nu \in \mathcal{M}(E)$ is a *symmetric Lévy measure* if it is symmetric and satisfies

(i) $\nu(\{0\}) = 0$,

(ii) The mapping from E^* to $\mathbb{R}$ given by

$$a \to \exp\left\{\int_E [\cos(\langle x, a\rangle) - 1]\nu(dx)\right\}$$

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We say that a probability measure μ on E is *infinitely divisible* if for all $n \in \mathbb{N}$, μ has a convolution nth root μ_n.

Equivalently for all $n \in \mathbb{N}$ there exists a probability measure μ_n on E such that $\widehat{\mu}(a) = (\widehat{\mu_n}(a))^n$ for all $a \in E^*$.

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Theorem (Lévy-Khintchine)

A probability measure $\mu \in \mathcal{M}_1(E)$ is infinitely divisible if and only if there exists $x_0 \in E$, a positive symmetric operator $R \in \mathcal{L}(E^*, E)$ and a Lévy measure ν on E such that for all $a \in E^*$,

$$\widehat{\mu}(a) = e^{\eta(a)},$$

where

$$\begin{aligned}\eta(a) &= i\langle x_0, a\rangle - \frac{1}{2}\langle Ra, a\rangle \\ &+ \int_E (e^{i\langle y,a\rangle} - 1 - i\langle y, a\rangle 1_{B_1}(y))\nu(dy).\end{aligned}$$

The triple (x_0, R, ν) is called the *characteristics* of the measure ν and η is known as the *characteristic exponent.*

See e.g. W.Linde, *Probability in Banach Spaces - Stable and Infinitely Divisible Distributions*, Wiley-Interscience (1986).

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Covariance Operators

A probability measure μ on E has *weak second order moments* if for all $a \in E^*$

$$\int_E |\langle x, a\rangle|^2 \mu(dx) < \infty.$$

In this case, there exists a *covariance operator* $Q \in \mathcal{L}(E^*, E)$ which is positive and symmetric:

$$\langle Qa, b\rangle = \int_E \langle x, a\rangle\langle x, b\rangle\mu(dx) - \left(\int_E \langle x, a\rangle\mu(dx)\right)\left(\int_E \langle x, b\rangle\mu(dx)\right).$$

Associated RKHS is H_Q.

If μ is infinitely divisible with characteristics (x_0, R, ν) and has weak second order moments:

$$Qa = Ra + \int_E \langle x, a\rangle x\, \nu(dx).$$

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Mehler Semigroups

Let $(\mu_t, t \geq 0)$ be a family of probability measures on E with $\mu_0 = \delta_0$ and $(S(t), t \geq 0)$ be a C_0-semigroup on E. Define $T_t : B_b(E) \to B_b(E)$ by

$$T_t f(x) = \int_E f(S(t)x + y)\mu_t(dy).$$

$(T_t, t \geq 0)$ is a semigroup, i.e. $T_{t+s} = T_t T_s$ if and only if $(\mu_t, t \geq 0)$ is a *skew-convolution semigroup*, i.e.

$$\mu_{t+u} = \mu_u * S(u)\mu_t$$

(where $S(u)\mu_t := \mu_t \circ S(u)^{-1}$.)

Note that $T_t : C_b(E) \to C_b(E)$ but it is not (in general) strongly continuous.

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From now on, we assume that the skew-convolution semigroup $(\mu_t, t \geq 0)$ is *F-differentiable*, i.e. $a \in E^*, t \to \widehat{\mu_t}(a)$ is differentiable.

$$\text{Define } \xi(a) := \frac{d}{dt}\widehat{\mu_t}(a)\bigg|_{t=0}.$$

Then

$$\widehat{\mu_t}(a) = e^{\eta_t(a)} := \exp\left\{\int_0^t \xi(S(u)^* a)du\right\}.$$

From this it follows that μ_t is infinitely divisible for all $t \geq 0$.

Furthermore ξ is the characteristic exponent of an infinitely divisible probability measure ρ with characteristics (b, R, ν) (say) and the characteristics of μ_t are (b_t, R_t, ν_t) where:

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$$x_t = \int_0^t S(r)bdr + \int_0^t \int_E S(r)y(1_B(S(r)y) - 1_B(y))\nu(dy)dr,$$

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see Bogachev, Röckner, Schmuland, PTRF **105**, 193 (1996); Furhman,Röckner, Pot. Anal. **12**, 1 (2000)

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Let H_t be RKHS of Q_t. Then $H_t \subseteq H_{t'}$ if $t \leq t'$. From the above $S(r)Q(t)S(r)^* = Q_{t+r} - Q_r$ and so $S(r)$ maps $\mathrm{Im}(Q_t S(r)^*) \subseteq H_t$ to H_{r+t}. In fact

$$S(r)H_t \subseteq H_{t+r} \text{ and } ||S(r)||_{\mathcal{L}(H_t, H_{t+r})} \leq 1.$$

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$$\begin{aligned} Q_t &= \int_0^t S(r)QS(r)^* dr \\ &= R_t + \int_0^t \int_E \langle S(r)y, a\rangle S(r)y\nu(dy) \end{aligned}$$

from which it follows that

$$Q_{t+s} = Q_t + S(t)Q_sS(t)^*.$$

Let H_t be RKHS of Q_t. Then $H_t \subseteq H_{t'}$ if $t \leq t'$. From the above $S(r)Q(t)S(r)^* = Q_{t+r} - Q_r$ and so $S(r)$ maps $\text{Im}(Q_tS(r)^*) \subseteq H_t$ to H_{r+t}. In fact

$$S(r)H_t \subseteq H_{t+r} \text{ and } ||S(r)||_{\mathcal{L}(H_t,H_{t+r})} \leq 1.$$

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Contraction Properties

Theorem

If $(T_t, t \geq 0)$ is a Mehler semigroup then T_t is a contraction from $L^p(E, \mu_{t+u})$ to $L^p(E, \mu_u)$ for all $u \geq 0, 1 \leq p < \infty$.

Proof. For each $f \in L^p(E, \mu_{t+u})$,

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\begin{aligned}
||T_t f||^p_{L^p(\mu_u)} &= \int_E |T_t f(x)|^p \mu_u(dx) \\
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Let A be the infinitesimal generator of the semigroup $(S(t), t \geq 0)$.

Let $(X(t), t \geq 0)$ be an E-valued Lévy process. Consider the linear SPDE with additive noise:

$$dY(t) = AY(t) + dX(t) \quad ; \quad Y(0) = Y_0$$

Unique solution is *generalised Ornstein-Uhlenbeck process*:

$$Y(t) = S(t)Y_0 + \int_0^t S(t-u)dX(u).$$

Transition semigroup $T_t f(x) = \mathbb{E}(f(Y(t))|Y_0 = x)$ is a Mehler semigroup. Skew convolution semigroup μ_t is law of $\int_0^t S(t-u)dX(u) \stackrel{d}{=} \int_0^t S(u)dX(u)$ and is F-differentiable with ξ the characteristic exponent of $X(t)$, i.e. $\mathbb{E}(e^{i\langle X(t),a\rangle}) = e^{t\xi(a)}$ for all $a \in E^*, t \geq 0$.

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Second Quantisation

"First quantisation is a mystery, second quantisation is a functor."
Ed. Nelson

H a complex Hilbert space. $\Gamma(H)$ is symmetric Fock space over H.

$$\Gamma(H) := \bigoplus_{n=0}^{\infty} H_s^{(n)}$$

$H^{(0)} = \mathbb{C}, H^{(1)} = H, H^{(n)}$ is n fold symmetric tensor product

Exponential vectors $\{e(f), f \in H\}$ are linearly independent and total where

$$e(f) = \left(1, f, \frac{f \otimes f}{\sqrt{2!}}, \ldots, \frac{f^{\otimes^n}}{\sqrt{n!}}, \ldots\right), \quad \langle e(f), e(g) \rangle = e^{\langle f, g \rangle}.$$

$$n\text{-particle vector } f^{\otimes^n} = \frac{1}{\sqrt{n!}} \frac{d}{da} e(af)\Big|_{a=0}.$$

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Gaussian Spaces μ a Gaussian measure on E (i.e. infinitely divisible with characteristics $(0, R, 0)$).

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Since $S(t)$ is a contraction from H_u to $H_{u+t} \Rightarrow S(t)^*$ is a contraction from H_{u+t} to H_u.

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J van Neerven, JFA **155**, 495 (1998)

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Non-Gaussian Mehler Semigroups

Assume μ infinitely divisible with $\widehat{\mu}(a) = e^{\eta(a)}$ for $a \in E^*$. We need analogues of exponential vectors. For each $a \in E^*$ define $K_a \in L^2_{\mathbb{C}}(E, \mu)$ by

$$K_a(x) = e^{i\langle x, a\rangle - \eta(a)}.$$

Theorem

The set $\{K_a, a \in E^*\}$ is total in $L^2_{\mathbb{C}}(E, \mu)$.

Proof. Let $\psi \in L^2_{\mathbb{C}}(E, \mu)$ be such that for all $a \in E^*$, $\int_E K_a(x)\overline{\psi(x)}\mu(dx) = 0$. Then $\int_E e^{i\langle x, a\rangle}\mu_\psi(dx) = 0$, where $\mu_\psi(dx) := \overline{\psi(x)}\mu(dx)$ is a complex measure. It follows by injectivity of the Fourier transform that $\mu_\psi = 0$ and hence $\psi = 0$ (a.e.) as was required. $\square$

Non-Gaussian Mehler Semigroups

Assume μ infinitely divisible with $\widehat{\mu}(a) = e^{\eta(a)}$ for $a \in E^*$. We need analogues of exponential vectors. For each $a \in E^*$ define $K_a \in L^2_{\mathbb{C}}(E, \mu)$ by

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The set $\{K_a, a \in E^*\}$ is total in $L^2_{\mathbb{C}}(E, \mu)$.

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Theorem

The set $\{K_a, a \in E^*\}$ is linearly independent in $L^2_{\mathbb{C}}(E, \mu)$.

Proof. Let $a_1, \ldots a_n \in E^*$ be distinct and $c_1, \ldots, c_n \in \mathbb{C}$ for some $n \in \mathbb{N}$ and assume that $\sum_{i=1}^n c_i K_{a_i} = 0$.

Define $\tilde{c}_i := e^{-\eta(a_i)} c_i$ for $1 \leq i \leq n$ and replace x by tx where $t \in \mathbb{R}$.

Then we have $\sum_{i=1}^n \tilde{c}_i e^{it\langle x, a \rangle} = 0$ for all $t \in \mathbb{R}$. Let $t = 0$ to see that $\sum_{i=1}^n \tilde{c}_i = 0$.

Now differentiate r times with respect to t (where $1 \leq r \leq n-1$) and then put $t = 0$. This yields $\sum_{i=1}^n \tilde{c}_i \langle x, a_i \rangle^r = 0$.

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We have a system of n linear equations in $\tilde{c}_1, \ldots \tilde{c}_n$ and it has a non-zero solution if and only if

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \langle x, a_1 \rangle & \langle x, a_2 \rangle & \cdots & \langle x, a_n \rangle \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \langle x, a_1 \rangle^{n-1} & \langle x, a_2 \rangle^{n-1} & \cdots & \langle x, a_n \rangle^{n-1} \end{vmatrix} = 0.$$

This ia a Vandermonde determinant and so the equation simplifies to

$$\prod_{1 \leq i,j \leq n} (\langle x, a_i \rangle - \langle x, a_j \rangle) = 0.$$

Hence there exists k, l with $1 \leq k, l \leq n$ such that $\langle x, a_k - a_l \rangle = 0$ for all $x \in E$. The choice of k and l here depends on x. We now prove that in fact they are independent of the choice of vector in E.

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To this end for each $1 \leq i, j \leq n$ define $F_{ij} := \{x \in E; \langle x, a_i - a_j \rangle = 0\}$. Then F_{ij} is closed and $\bigcup_{i,j=1}^{n} F_{ij} = E$.

By the Baire category theorem, at least one pair (i, j) must be such that F_{ij} has non-empty interior O_{ij}. Let (k, l) be such a pair and fix $x_0 \in O_{kl}$.

Then by linearity $\langle x - x_0, a_k - a_l \rangle = 0$ for all $x \in O_{kl}$. In other words $\langle y, a_k - a_l \rangle = 0$ for all $y \in O_{kl} - \{x_0\}$.

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non-Gaussian Second Quantisation

Let $T \in \mathcal{L}(E^*)$. We define its *second quantisation* $\Gamma(T)$ to be the densely defined linear operator with domain $\mathcal{E} = \text{lin span}\{K_a, a \in E^*\}$ defined by linear extension of the prescription

$$\Gamma(T)K_a = K_{Ta}.$$

It is straightforward to verify the functorial property:

If $T_1, T_2 \in \mathcal{L}(E^*)$ then

$$\Gamma(T_1 T_2) = \Gamma(T_1)\Gamma(T_2).$$

In the case where $\mu = \mu_t$, we write $K_{t,a}$ instead of K_a.

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Let $T \in \mathcal{L}(E^*)$. We define its *second quantisation* $\Gamma(T)$ to be the densely defined linear operator with domain $\mathcal{E} = \text{lin span}\{K_a, a \in E^*\}$ defined by linear extension of the prescription

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The Main Result

Theorem

Let $(\mu_t, t \geq 0)$ be an F-differentiable skew convolution semigroup. For all $t, u > 0$

$$T_t = \Gamma(S(t)^*_{t+u \to u}).$$

Proof. For all $a \in E^*, x \in E$

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$$\begin{aligned}\eta_t(a) - \eta_{t+u}(a) &= -\int_t^{t+u} \xi(S(r)^* a) dr \\ &= -\int_0^u \xi(S(r)^* S(t)^* a) dr \\ &= -\eta_u(S(t)^* a).\end{aligned}$$

From this we see that

$$\begin{aligned}T_t K_{t+u,a}(x) &= e^{i\langle x, S(t)^* a\rangle - \eta_u(S(t)^* a)} \\ &= K_{u, S(t)^* a}(x),\end{aligned}$$

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Some comments:

We did not assume that μ_t has second moments and made no use of a RKHS. So our second quantisation

$$\Gamma : \mathcal{L}(E^*) \to \text{closeable lin.ops on } L^2_{\mathbb{C}}(E, \mu) \text{ preserving } \mathcal{E}.$$

If we assume that μ_t has second moments for all t then $S(t)^*$ is a contraction from H_{t+u} to H_u.

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Invariant Measures

λ is an invariant measure for the Mehler semigroup $(T_t, t \geq 0)$ if and only if for all $t \geq 0$

$$\rho = \mu_t * S(t)\rho.$$

If ρ exists it is infinitely divisible (operator self-decomposable.)

e.g. if $\mu_\infty = \text{weak-}\lim_{n \to \infty} \mu_t$ exists it is an invariant measure.

If E is a Hilbert space and we are in the Ornstein-Uhlenbeck case:

A.Chojnowska-Michalik Stochastics, **21** 251 (1987)

e.g. Assume $(S_t, t \geq 0)$ is *exponentially stable* i.e. $||S(t)|| \leq Me^{-\lambda t}$ for $M \geq 1, \lambda > 0$. Then necessary and sufficient conditions for unique invariant measure are

$$\int_0^\infty \int_E (||S(t)y||^2 \wedge 1)\nu(dy)dt < \infty,$$

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Further if $\int_0^\infty \int_E ||S(t)y||^2 \nu(dy)dt < \infty$, then μ_∞ has covariance operator

$$\begin{aligned} Q_\infty &= \int_0^\infty S(r)QS(r)^* dr \\ &= R_\infty + \int_0^\infty \int_E \langle S(r)y, a \rangle S(r)y\nu(dy) \end{aligned}$$

We get RKHS H_∞ with for all $t \geq 0$

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The Chaos Approach in the non-Gaussian case.

Based on work by

S.Peszat, JFA **260**, 3457 (2011)

$(\Omega, \mathcal{F}, P)$ is a probability space. Let Π be a Poisson random measure defined on a measurable space $(E, \mathcal{B})$ with intensity measure λ. Let $\mathbb{Z}_+(E)$ be the non-negative integer valued measures on $(E, \mathcal{B})$. Regard Π as a random variable on Ω taking values in $\mathbb{Z}_+(E)$ by

$$\Pi(\omega)(E) = \Pi(E, \omega)$$

Let P_π be the law of Π and for $F \in L^2(P_\pi), \xi \in \mathbb{Z}_+(E)$ define the "Malliavin derivative":

$$D_y F(\xi) = F(\xi + \delta_y) - F(\xi)$$

Define $T^n : L^2(P_\pi) \to L^2_{\text{symm}}(E^n, \lambda^n)$ by

$$(T^n F)(y_1, \ldots, y_n) = \mathbb{E}(D^n_{y_1,\ldots,y_n} F(\Pi)).$$

Chaos expansion

$$\mathbb{E}(F(\Pi)G(\Pi)) = \mathbb{E}(F(\Pi))\mathbb{E}(G(\Pi)) + \sum_{n=1}^{\infty}\frac{1}{n!}\langle T^n F, T^n G\rangle_{L^2(E^n,\lambda^n)}$$

from which it follows that

$$F(\Pi) = \sum_{n=0}^{\infty}\frac{1}{n!}I_n(T^n F),$$

where I_n is usual multiple Itô integral w.r.t. compensator $\tilde{\Pi} := \Pi - \lambda$.

So here $L^2(P_\pi) = \Gamma(L^2(E,\lambda))$.

see G.Last, M.Penrose, PTRF **150**, 663 (2011)

Peszat: If E is a Hilbert space, $R \in \mathcal{L}(E)$, define $\rho_R^{(n)} \in \mathcal{L}(L^2(E^n,\lambda^n))$ by

$$\rho_R^{(n)} f(y_1,\ldots,y_n) = f(Ry_1,\ldots,Ry_n).$$

Second quantisation: $\Gamma_0(R) : L^2(P_\pi) \to L^2(P_\pi)$,

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Connecting The Two Approaches in the non-Gaussian Case

For all $t \geq 0$ let $S_t := [0, t) \times E$.

Let Π be a Poisson random measure defined on $[0, \infty) \times E$ so that Π_t has intensity measure λ_t.

The natural filtration of $\Pi_t(\cdot) := \Pi(t, \cdot)$ is denoted $(\mathcal{F}_t, t \geq 0)$.

For $t \geq 0, f \in L^2(S_t, \lambda_t)$ define the process $(X_f(t), t \geq 0)$ by

$$X_f(t) = \int_0^t \int_E f(s, x)\tilde{\Pi}(ds, dx).$$

$\mathbb{E}(|X_f(t)|^2) = ||f||^2_{L^2(S_t, \lambda_t)} < \infty$.

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$$\eta_f(t) = \int_{S_t} (e^{if(s,x)} - 1 - if(s, x))\lambda(ds, dx).$$

Connecting The Two Approaches in the non-Gaussian Case

For all $t \geq 0$ let $S_t := [0, t) \times E$.
Let Π be a Poisson random measure defined on $[0, \infty) \times E$ so that Π_t has intensity measure λ_t.
The natural filtration of $\Pi_t(\cdot) := \Pi(t, \cdot)$ is denoted $(\mathcal{F}_t, t \geq 0)$.
For $t \geq 0, f \in L^2(S_t, \lambda_t)$ define the process $(X_f(t), t \geq 0)$ by

$$X_f(t) = \int_0^t \int_E f(s, x)\tilde{\Pi}(ds, dx).$$

$\mathbb{E}(|X_f(t)|^2) = ||f||^2_{L^2(S_t, \lambda_t)} < \infty$.

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$$\eta_f(t) = \int_{S_t} (e^{if(s,x)} - 1 - if(s, x))\lambda(ds, dx).$$

Define the process $(M_f(t), t \geq 0)$ by

$$M_f(t) = \exp\{iX_f(t) - \eta_f(t)\}.$$

Then $(M_f(t), t \geq 0)$ is a square-integrable martingale with

$$dM_f(t) = \int_{S_t} (e^{if(s,x)} - 1)M_f(s-)\tilde{\Pi}(ds, dx),$$

and for all $t \geq 0$,

$$\mathbb{E}(|M_f(t)|^2) = \exp\left\{\int_{S_t} |e^{if(s,x)} - 1|^2 \lambda(ds, dx)\right\} \tag{1.1}$$

Lemma

For all $t \geq 0$,

$$\mathbb{E}(|M_f(t)|^2) \leq e^{||f||^2_{L^2(S_t,\lambda_t)}}.$$

Proof. Using the well known inequality $1 - \cos(y) \leq \frac{y^2}{2}$ for $y \in \mathbb{R}$

$$\begin{aligned}
\mathbb{E}(|M_f(t)|^2) &= \exp\left\{2\int_{S_t}(1-\cos(f(s,x)))\lambda(ds,dx)\right\} \\
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Now let $(Y_f(t), t \geq 0)$ be the Doléans-Dade exponential which is the unique solution of the stochastic differential equation

$$dY_f(t) = Y_f(t-)dX_f(t),$$

with initial condition $Y_f(0) = 1$ (a.s.)

$$Y_f(t) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(f^{\otimes^n}) \text{ and } \mathbb{E}(|Y_f(t)|)^2 = e^{||f||^2_{L^2(S_t,\lambda_t)}}.$$

Let $\mathcal{K}(t)$ be the linear span of $\{M_f(t), f \in L^2(S_t, \lambda_t)$.

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Both sets are total in $L^2(\Omega, \mathcal{F}_t, P)$.
The map $C : \mathcal{K}(t) \to \mathcal{L}(t)$ which takes each $M_f(t)$ to $Y_f(t)$ extends to an invertible linear operator on $L^2(\Omega, \mathcal{F}_t, P)$ which we continue to denote by C.
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Now assume that μ_t has weak second order moments and for each $a \in E^*, t \geq 0$ define

$$f_a \in L^2(S_t, \lambda) \text{ by } f_a(s, x) = \langle x, a \rangle 1_{[0,t)}(s) \text{ for each } 0 \leq s \leq t, x \in E.$$

Then we have $M_f(t) = M_{t,a}$ where

$$M_{t,a}(x) = \exp\left\{ i \int_E \langle x, a \rangle \tilde{\Pi}(t, dx) - \eta_t(x) \right\},$$

$$\eta_t(x) = \int_E (e^{i\langle x, a \rangle} - 1 - i\langle x, a \rangle)\lambda_t(dx).$$

Then $M_{t,a}$ is precisely the image of $K_{t,a}$ in $L^2(\Omega, \mathcal{F}_t, P)$ under the natural embedding of $L^2(E, \mu_t)$ into that space. From now on we will identify these vectors.

For each $t \geq 0$, we write the Doléans-Dade exponential $Y_a(t)$ when $f = f_a$ as above.

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Theorem

For each $S \in \mathcal{L}(E^*)$

$$\Gamma(S) = C^{-1}\Gamma_0(S)C,$$

Proof. For each $a \in E^*, t \geq 0$,

$$\begin{aligned}
\Gamma(S)C^{-1}Y_a(t) &= \Gamma(S)K_{t,a} \\
&= K_{t,Sa} \\
&= C^{-1}Y_{Sa}(t) \\
&= C^{-1}\Gamma_0(S)Y_a(t),
\end{aligned}$$

and the result follows. □

Thank you for listening.
Dziekujemy za nasluchiwanie.