Some relationships between potential theories of classical and hyperbolic Brownian motion

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Hyperbolic space $\mathbb{H}^n$

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$$\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n > 0\}.$$

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- Hyperbolic distance formula

$$\cosh(d_{\mathbb{H}^n}(x, y)) = 1 + \frac{|x - y|^2}{2x_n y_n}.$$

Hyperbolic space $\mathbb{H}^n$

- Volume element

$$dV_n = \frac{1}{x_n^n} dx_1 \dots dx_n$$

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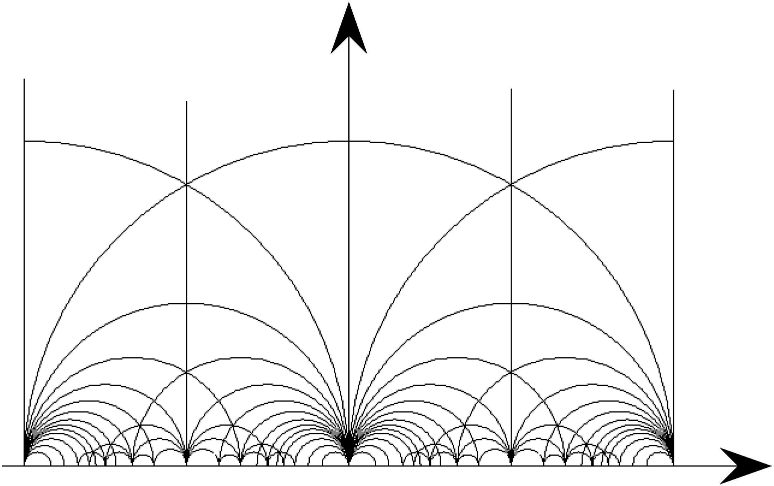
- Laplace - Beltrami operator

$$\Delta = div(grad) = x_n^2 \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} - (n-2) x_n \frac{\partial}{\partial x_n}$$

Hyperbolic space $\mathbb{H}^n$



Hyperbolic Brownian motion

- Stochastic differential equations.

$$\begin{cases} dX_1(t) & = & X_n(t)dB_1 \\ & \vdots & \\ dX_{n-1}(t) & = & X_n(t)dB_{n-1} \\ dX_n(t) & = & X_n(t)dB_n(t) - \frac{n-2}{2}X_n(t)dt, \end{cases}$$

$X(0) = x \in \mathbb{H}^n$, $\mathbb{E}B_i^2(t) = t, i = 1, ..., n.$

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- System of SDE has unique strong solution and it is called hyperbolic Brownian motion.

Hyperbolic Brownian motion

- Solution

$$\begin{cases} X_1(t) & = & x_1 + \int_0^t x_n \exp\left(B_n(u) - (n-1)/2u\right) dB_1(u) \\ & \vdots & \\ X_{n-1}(t) & = & x_{n-1} + \int_0^t x_n \exp\left(B_n(u) - (n-1)/2u\right) dB_{n-1}(u) \\ X_n(t) & = & x_n \exp\left(B_n(t) - (n-1)/2t\right). \end{cases}$$

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- Generator $= \frac{1}{2}\Delta$

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  transition density function - formula

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- (2012) J.Małecki, GS
  Green function and Poisson kernel of the set
  $\{x \in \mathbb{H}^n : x_1 > 0\}$ for HBM with drift - formulas and estimates

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Relationships

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Notation:

- $G_U^E(x,y)$, $P_U^E(x,y)$ - Green function and Poisson kernel of a set U for standard (euclidean) Brownian motion.
- $G_U^H(x,y)$, $P_U^H(x,y)$ - Green function and Poisson kernel of a set U for hyperbolic Brownian motion (with respect to the Lebesgue measure).

Relationships

Let consider sets of the form

$$\begin{aligned} \{x \in \mathbb{H}^n : (x_1, ..., x_{n-1}) \in D\} &= D \times (0, \infty), \\ \{x \in \mathbb{R}^{2n} : (x_1, ..., x_{n-1}) \in D\} &= D \times \mathbb{R}^{n+1}, \end{aligned}$$

where $D \subseteq \mathbb{R}^{n-1}$.

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where $D \subseteq \mathbb{R}^{n-1}$.
Let $x \in \mathbb{R}^m$, m>n. We denote

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Then for $x, y \in D \times \mathbb{R}^{n+1}$ we have

$$\begin{aligned} G^E_{D\times\mathbb{R}^{n+1}}(x, y) &= G^E_{D\times\mathbb{R}^{n+1}}(x^-, y^-, |x^+ - y^+|^2), \\ P^E_{D\times\mathbb{R}^{n+1}}(x, y) &= P^E_{D\times\mathbb{R}^{n+1}}(x^-, y^-, |x^+ - y^+|^2). \end{aligned}$$

Relationships

Theorem (GS 2012)

For $x, y \in D \times (0,\infty)$ we have

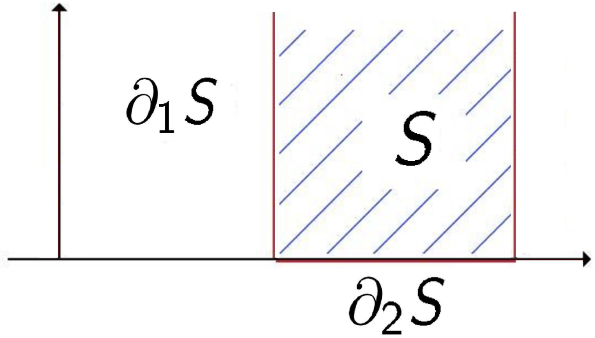
$$G^H_{D\times(0,\infty)}(x,y) = \frac{2\pi^{n/2} x_n y_n^{n-1}}{\Gamma(n/2)} \int_{-1}^{1} (1-u^2)^{n/2-1} \times$$

$$G^E_{D\times\mathbb{R}^{n+1}}\left(x^-, y^-, x_n^2 + y_n^2 + 2x_n y_n u\right) du.$$

Relationships

We divide the boundary ∂S of the set $S = D \times (0, \infty) \subseteq \mathbb{H}^n$, $D \subseteq \mathbb{R}^{n-1}$ into two parts

- $\partial_1 S = \partial D \times (0, \infty)$
  It is a boundary S as a subset of $\mathbb{H}^n$.
- $\partial_2 S = D \times \{0\}$
  It is not a subset of $\mathbb{H}^n$, but hyperbolic Brownian motion can hit it in infinite time.



Relationships

Theorem (GS 2012)

For $x \in S, y \in \partial_1 S$ we have

$$P_S^H(x,y) = \frac{2\pi^{n/2} y_n x_n^{n-1}}{\Gamma(n/2)} \int_{-1}^{1} (1-u^2)^{n/2-1} \times$$

$$P^E_{D\times\mathbb{R}^{n+1}}\left(x^-, y^-, x_n^2 + y_n^2 + 2x_n y_n u\right) du.$$

For $x \in S, y \in \partial_2 S$ we have similar formula. However, it is more complicated and less useful.

Hyperbolic strip

For $a > 0$ we define

$$S_a = \{x \in \mathbb{H}^n : x_1 \in (0, a)\}.$$

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We divide the boundary as before:

$$\begin{aligned} \partial_1 S_a &= \{0, a\} \times \mathbb{R}^{n-2} \times (0, \infty), \\ \partial_2 S_a &= (0, a) \times \mathbb{R}^{n-2} \times \{0\}. \end{aligned}$$

Poisson kernel of strip - formulae

Theorem (GS 2012)

For $x \in S_a$, $y \in \partial_1 S_a$ we have

$$P^H_{S_a}(x,y) = \frac{(-2)^{1-n}\pi^{\frac{n}{2}}}{a\Gamma(\frac{n}{2})} \frac{x_n}{y_n^{n-1}} \int_{-1}^{1} (1-u^2)^{\frac{n-2}{2}} \frac{\partial^{n-1}}{\partial u^{n-1}} \frac{\sin\left(\frac{\pi(x_1-y_1)}{a}\right) du}{\cosh\left(\frac{\pi}{a}\sqrt{2x_n y_n(\cosh\tilde{\rho}+u)}\right) - \cos\left(\frac{\pi}{a}(x_1-y_1)\right)},$$

Poisson kernel of strip - formulae

Theorem (GS 2012)

and for $y \in \partial_2 S_a$ *we have*

$$P^H_{S_a}(x,y) = \frac{2x_n^{n-1}}{\Gamma\left(\frac{n+1}{2}\right)} \frac{(-1)^n}{\pi^{\frac{n-1}{2}}} \times \frac{\partial^{n-1}}{\partial \xi^{n-1}} \ln \left[1 - \frac{2 \sin x_1 \sin y_1}{\cosh\left(\frac{\pi}{a}\sqrt{\xi}\right) + \cos\left(\frac{\pi(x_1 - y_1)}{a}\right)} \right]_{\xi = |\tilde{x} - \tilde{y}|^2},$$

where $\tilde{x} = (x_2, x_3, \ldots, x_n)$.

Estimates in euclidean case

$$P^{E}_{(0,a)\times\mathbb{R}^{2n-1}}(x,y) \overset{c}{\asymp} \frac{x_1(a-x_1)\exp\left(-\frac{\pi}{a}|x-y|\right)}{a^{n+2}} \frac{a^{n+1}+|x-y|^{n+1}}{|x-y|^{2n}}.$$

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Theorem (GS 2012)

Let $x \in (0,a)\times\mathbb{H}^{n-1}$. For $y \in \partial_1 S_a$ we have

$$P^H_{S_a}(x,y) \overset{c}{\asymp} \frac{x_1(a-x_1)e^{-\frac{a}{\pi}|x-y|}}{a^2|x-y|^n} \left(\frac{x_n}{y_n}\right)^{n/2} \frac{a^{n+1}+|x-y|^{n+1}}{(a^2\cosh\rho+|x-y|)^{n/2}},$$

Poisson kernel of strip - estimates

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and for $y \in \partial_2 S_a$ we have

$$P^H_{S_a}(x,y) \overset{c}{\asymp} \frac{x_n^{n-1}}{a^{n+3}} \exp\left(-\frac{\pi}{a}|x-y|\right) \frac{x_1 y_1 (a-x_1)(a-y_1)}{|\tilde{x}-\tilde{y}|^2 + (a-|x_1+y_1-a|)^2} \frac{a^{n+1}+|x-y|^{n+1}}{|x-y|^{2n-2}}.$$

The constant c depends only on dimension n.

Idea of theorems

We define the functional

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$$X(t) \overset{(d)}{=} Y\left(A_{x_n}(t)\right),$$

where

$$Y(t) = \left(\beta_1\left(t\right), ..., \beta_{n-1}\left(t\right), R^{\left(-\frac{n-1}{2}\right)}\left(t\right)\right).$$

Here process $R^{(-\frac{n-1}{2})}(t)$ is the Bessel process starting from $x_n > 0$ and independent of the process $\beta(t)$. The integral functional $A_{x_n}(t)$ is independent of $\beta(t)$ too.

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RemarkPoisson kernels for processes $X(t)$ and $Y(t)$ are the same.

Idea of theorems

Process $Y(t)$ without last coordinate is just standard $n-1$ - dimensional Brownian motion. For the last coordinate we have

$$\mathbb{P}^x\left(R_t^{(-\frac{n-1}{2})} \in dy\right) = x^{n-1}\mathbb{P}^x\left(\left(R_t^{(\frac{n-1}{2})}\right)^{-n-1}, R_t^{(\frac{n-1}{2})} \in dy\right).$$

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$R_t^{(\frac{n-1}{2})}$ is the absolute value of $n+1$ - dimensional Brownian motion.