Regularity of generalized Ornstein-Uhlenbeck processes

Jerzy Zabczyk

Institute of Mathematics, Polish Academy of Sciences, Warsaw

$$dX(t) = AX(t)\,dt + dZ(t), \quad X(0) = x \in H.$$

$$A : H \to H, \text{generator of } S(t) = e^{tA}, \quad Z \text{ — Lévy process on } U \supseteq H,$$

$$X(t) = S(t)x + \int_0^t S(t-s)dZ(s).$$

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P. Kotelenez *A submartingale type inequality with applications to stochastic evolution equations*, Stochastics, 8 (1982), 139–151.

If $Z(t) \in H$, $t \geqslant 0$, then X has a càdlàg modification.

Stochastic heat equation

$$dX(t,\xi) = \Delta X(t,\xi)dt + dZ(t,\xi)$$
$$X(0,\xi) = x(\xi), \quad \xi \in \mathcal{O} \subset \mathbb{R}^d, \; x_0 \in H$$
$$H = L^2(\mathcal{O}), \quad U \supseteq L^2(\mathcal{O}).$$

General message

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The càdlàg property for equations with Lévy perturbations is much less frequent than continuity for equations with Gaussian perturbations.

Content

1. Cylindrical Lévy processes
2. Càdlàg results for diagonal systems
3. Càdlàg results for general equations
4. Weaker concepts of time regularity
5. Spatial regularity

Lévy processes Z on U — Hilbert

$$\mathbb{E}\left(e^{i\langle \lambda, Z(t)\rangle}\right) = e^{-t\psi(\lambda)}, \qquad \lambda \in U,\ \ t \geqslant 0,$$

$$\psi(\lambda) = -\langle a, \lambda\rangle\, i + \frac{1}{2}\langle Q\lambda, \lambda\rangle + \int_U \left[1 - e^{i\langle \lambda, y\rangle} + \mathbb{1}_{[0,1]}(|y|) i \langle \lambda, y\rangle\right] \nu(dy)$$

$a \in U$; $Q \geqslant 0$, trace class, $\int (|y|^2 \wedge 1)\, \nu(dy) < +\infty$

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$a \in U$; $Q \geqslant 0$, trace class, $\int (|y|^2 \wedge 1)\,\nu(dy) < +\infty$
α – *stable processes*, $\alpha \in (0,2)$,

$$\psi(\lambda) = |y|^\alpha,\ \ \lambda \in R^d,\ \ \nu(dx) = \frac{c}{|x|^{d+\alpha}}dx,\ \ x \in R^d.$$

Cylindrical Lévy processes

$$Z(t) = \sum_{n=1}^{+\infty} \beta_n Z^n(t) e_n, \quad (e_n) \text{ o.c.b. in } U$$

Z^n, Lévy processes.

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Identity covariance.

$$E \langle Z(t), \varphi \rangle \langle Z(s), \psi \rangle = ct \wedge s \langle \varphi, \psi \rangle$$

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$$Z(t) \in \widetilde{U} \supset U.$$

Cylindrical noise ⇒ good distributional properties of X ≈ ellipticity

Diagonal systems

$$dX = AX\,dt + dZ$$

$$Ae_n = -\gamma_n e_n, \quad \gamma_n \uparrow +\infty, \quad (e_n) \text{ o.c.b. in } H$$

$$X^n(t) = \langle X(t), e_n \rangle_H, \quad H \sim l^2,$$

$$\beta_n Z^n(t) = \langle Z(t), e_n \rangle_H.$$

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$$dX^n(t) = -\gamma_n X^n(t)\,dt + \beta_n\,dZ^n(t)$$
$$X(t) = (X_n(t)) \in H = l^2.$$

Z^n Lévy processes, $Z(t) = (\beta_n Z^n(t))$ in $U = l^2_w$.

Paths continuity of X in l^2 if Z^n Wiener processes, see

[Iscoe, Marcus, McDonald, Talagrand and Zinn, *Continuity of l^2–valued Ornstein-Uhlenbeck processes*, Annals of Probability, 18(1990), 68–84]

α- stable noise. Collaboration with E.Priola.

$$dX^n(t) = -\gamma_n X^n(t)\,dt + \beta_n\,dZ^n(t), \quad H = l^2,$$

- If Z^n α-stable, i.i.d., $\alpha \in (0,2]$, then $X(t) \in H$, $\mathbb{P}$-a.s. iff

(1)
$$\sum_{n \geqslant 1} \frac{\beta_n^\alpha}{\gamma_n} < +\infty$$

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- If $\beta_n \equiv 1$, $X(t) \in H \iff \sum_n \frac{1}{\gamma_n} < +\infty$.

Theorem

If $0 < p < \alpha \leqslant 2$, $0 < \delta < 1 \wedge \alpha$, and

$$\sum_n \frac{\beta_n^\alpha}{\gamma_n^{1-\delta}} < +\infty. \tag{2}$$

Then

$$\exists c > 0,\ \mathbb{E}|X(t) - X(s)|^p \leqslant c|t-s|^{\delta p/\alpha} \Big(\sum_{n \geqslant 1} \frac{\beta_n^\alpha}{\gamma_n^{1-\delta}} \Big)^{p/\alpha} \tag{3}$$

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(2) $$\sum_n \frac{\beta_n^\alpha}{\gamma_n^{1-\delta}} < +\infty.$$

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If $\alpha = 2$ then (3) holds for arbitrary $\delta \in (0,1)$ and arbitrary $p > 0$.

- If $\alpha = 2$, (2) $\Rightarrow$ Hölder continuity of paths with arbitrary exponent smaller than $\delta/2$.

Theorem

1. $\alpha < 2$, $\sum \frac{\beta_n^\alpha}{\gamma_n} < +\infty$, $\beta_n \not\to 0$, then X is not càdlàg in H
2. $\alpha < 2$ and $\sum \beta_n^\alpha < +\infty$, then X is càdlàg in H.

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3. If $\alpha < 2$ and X is càdlàg in H then $\sum \beta_n^\alpha < +\infty$.

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First and second part special case of a result formulated below, from [Brzeźniak, Gołdys, Imkeller, Peszat, Priola, JZ. (CRASP 2011)]. Third part from [Liu and Zhai (CRASP, 2012)].

Theorem

Assume

1. *A generates a C_0-semigroup on H*
2. $(e_n) \in D(A^*)$, (e_n) *o.c.b. in H*
3. $Z(t) = \sum_{n\geqslant 1} \beta_n Z^n(t) e_n$
4. Z^n *i.i.d. Lévy,* $\nu \neq 0$.

Then

i) *If* $(\beta_n) \not\to 0$, *solution X of* $dX = AX\,dt + dZ$ *has no càdlàg modification.*

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2. $(e_n) \in D(A^*)$, (e_n) *o.c.b. in* H
3. $Z(t) = \sum_{n \geq 1} \beta_n Z^n(t) e_n$
4. Z^n *i.i.d. Lévy,* $\nu \neq 0$.

Then

i) *If* $(\beta_n) \nrightarrow 0$, *solution* X *of* $dX = AX\,dt + dZ$ *has no càdlàg modification.*

ii) *Moreover, with probability* 1, *trajectories of* X *do not have left and right limits in* H *at any point* $t \geq 0$.

Liu and Zhai; A. Jakubowski (Ann. Inst. H.Poincare, 1986); Peszat and JZ.

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Theorem

Assume that X_n are càdlàg. (i) Process X has a càdlàg modification if and only if

$$\mathbb{P}\left(\lim_{N\to\infty} \sup_{t\in[0,T]} \sum_{n=N}^{\infty} X_n^2(t) = 0 \right) = 1. \tag{4}$$

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(ii) Process X has a weakly càdlàg modification if and only if

$$\mathbb{P}\left(\sup_{t\in[0,T]} \sum_{n=1}^{\infty} X_n^2(t) < \infty \right) = 1. \tag{5}$$

Càdlàg modification

Joint work with S. Peszat

$$dX(t) = AX(t)dt + dZ(t)$$

A generates an analytic semigroup S on $H \subset U$. If $\rho > 0$, $H_\rho = D(-A)^\rho, H_{-\rho}$ dual to H_ρ, with the identification $H = H^*$

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Applications to stochastic parabolic equations.
H_ρ, $H_{-\rho}$ Sobolev spaces.

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Applications to stochastic parabolic equations.
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Case of hyperbolic equations

Theorem

Z a Lévy process on $U = H_{-\rho}$, without the Gaussian part, $\nu(U \setminus H) = 0$,$\rho < 1/2$, $\epsilon > 0$ and

$$\int_H (|z|^2_{-\rho} + |z|^4_{\epsilon})\nu(dz) < \infty. \tag{6}$$

Then X has a càdlàg modification in H.

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Then X has a càdlàg modification in H.Moreover for arbitrary $T > 0$ and $p \in [1, 4)$

$$\mathbb{E} \sup_{t \in [0,T]} |X(t)|^p_H < \infty.$$

Diagonal case

$$Ae_n = -\gamma_n e_n, \quad \gamma_n \uparrow +\infty, \quad (e_n) \text{ o.c.b. in } H = L^2(\mathcal{O}).$$

$$Z(t) = (\beta_n Z_n(t)),$$

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$$|z|_{-\rho} = \sum_{1}^{+\infty} z_n^2 \gamma_n^{-2\rho}, \; |z|_\epsilon = \sum_{1}^{+\infty} z_n^2 \gamma_n^{2\epsilon}, \; \rho > 0, \epsilon > 0.$$

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Then condition (6) is of the form

$$\sum_{n=1}^{+\infty} [(\frac{\beta_n}{\gamma_n^\rho})^2 + (\beta_n^4 \gamma_n^{4\epsilon})] < +\infty \quad \int_R z^4 \mu(dz) < +\infty.$$

Theorem

The diagonal case. Measure μ has bounded support, $\gamma_n = n^\alpha$, $\beta_n = \frac{1}{n^\beta}$, $\alpha, \beta > 0$.

1. Z takes values in $H = H_0$ iff $\beta > 1/2$.

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3. If $\alpha > 1 - 2\beta$ and $\beta > 1/4$ then X has càdlàg modification.
4. If $1/4 < \beta \leq 1/2$ and $\alpha > 1 - 2\beta$, then X has càdlàg modification and Z is cylindrical

Chentsov and Kinney's theorems

Theorem

$X(t),\ t \in [0, T]$ a process with values in a Banach space E, K, p, r constants such that for $0 \le t - h < t < t + h \le T$:

$$\mathbb{E}|X(t) - X(t-h)|_E^p |X(t+h) - X(t)|_E^p \le K h^{1+r}.$$

Then X has a càdlàg modification.

Theorem

$X(t)$, $t \in [0, T]$ a Markov process with values in a Banach space E, such that for each $r > 0$,

$$lim_{t\to 0} \sup_{x\in E} P_t(x, B^c(x, r)) = 0.$$

Then X has a càdlàg modification.

$$P_t(x, B^c(x,r)) = \mathbb{P}(|S(t)x + \int_0^t S(t-s)dZ(s) - x| > r) = 0$$

Weaker concepts of time regularity

- Càdlàg modification in the weak topology
- Cylindrical càdlàg :For arbitrary $z \in H$, $< X(t), z >$ has càdlàg modification

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Theorem

If $X(t) = \sum_n^\infty X_n(t) e_n$, $\mathbb{P}$ - a.s. convergent, X_n, càdlàg.

Then X has a càdlàg modification in the weak topology iff

$$\mathbb{P}\left(\sup_{t \in [0,T]} \sum_n^\infty (X_n(t))^2 < \infty \right) = 1.$$

Theorem

If Z_n real,independent, with Lévy measures μ_n and for each $r > 0$

$$\sum_n^{+\infty} \mu_n(y; |y| \geq r) = +\infty,$$

Then X does not have a weak càdlàg modification.

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Then X does not have a weak càdlàg modification.

Problem:

Construct equation with the solution X having weak càdlàg modification but not strong càdlàg modification.

Subordinated Wiener process

$$Z(t) = W(Y(t)), \qquad t \geqslant 0,$$

$$E \subset H \subset U, \qquad W(0) \in U$$

Y subordinator, ρ Lévy measure

Subordinators (increasing Lévy processes):

$$\mathbb{E}(e^{-\gamma Z(t)}) = e^{-t\phi(\gamma)}, \;\; \phi(\gamma) = \gamma a \; + \int_0^{+\infty} (1 - e^{\gamma z})\nu(dz),$$

$$\phi(\gamma) = \gamma^{\alpha}, \;\; \alpha \in (0, 1), \;\; \nu(dz) = \frac{c}{z^{1+\alpha}} dz, \;\; z \in (0, +\infty),$$

$\gamma > 0, \;\; a \geq 0.$

Spatial regularity (with Z. Brzeźniak)

Theorem

Assume $p \in (0,2)$, $\rho \neq 0$. *If*

$$\int_0^1 \xi^{p/2}\,\rho(d\xi) < +\infty, \qquad \int_0^1 \|e^{At}\|^p_{L(U,E)}\,dt < +\infty$$

then

i) $X(t) \in E$, $t \geqslant 0$

ii) X *has no* E*-càdlàg version.*

$$dX(t,\xi) = -(-\Delta)^{\gamma} X(t,\xi)\, dt + dZ^{\alpha}(t,\xi)$$
$$H = L^2(\mathcal{O}), \quad \mathcal{O} \subset \mathbb{R}^d$$
$$Z^{\alpha} = W(Y^{\alpha}(t)), \quad W \text{ cylindr. } L^2(\mathcal{O})$$
$$Y^{\alpha} \ \alpha\text{-stable, increasing, } 0 < \alpha < 1.$$

Theorem

If $0 < \alpha < 1$, $0 < \delta < \gamma - \frac{d}{2}$, $\gamma > 0$, then $X(t) \in C_0^{\delta}(\mathcal{O})$, but X is not càdlàg in $L^2(\mathcal{O})$.

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