Schrödinger perturbations of transition densities

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Motivation

Let $x, y \in \mathbb{R}^d$. For $s < t$,

$$g(s, x, t, y) = [4\pi(t - s)]^{-d/2} \exp\left(-|y - x|^2/(t - s)\right),$$

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Fundamental solution:

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} g(s,x,u,z)\left[\partial_u + \Delta_z\right]\phi(u,z)dzdu = -\phi(s,x),$$

for $s \in \mathbb{R}$, $x \in \mathbb{R}^d$ and $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$.

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GOAL: Given function $q \geq 0$ on *time-space*, find a transition density $\tilde{g}$ such that for all $s \in \mathbb{R}$, $x \in \mathbb{R}^d$, $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$,

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \tilde{g}(s, x, u, z) \big[\partial_u + \Delta_z + q(u, z)\big] \phi(u, z) dz du = -\phi(s, x).$$

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Here $\quad g_n = (gq)^n g = (gq)g_{n-1} \quad$ and $\quad g_0 = g\,.$

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Definition

Function $p\colon \mathbb{R} \times X \times \mathbb{R} \times X \to [0, \infty]$ *is called a transition density if*

$$\int_X p(s,x,u,z)p(u,z,t,y)\,dz = p(s,x,t,y)\,, \quad s < u < t.$$

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Definition (Schrödinger perturbation of p by q)

$$\tilde{p}(s,x,t,y) = \sum_{n=0}^{\infty} p_n(s,x,t,y)\,.$$

Khasminski's lemma

Assume that for $\varepsilon \geq 0$ and $s < t$, $x, y \in X$,

$$p_1(s,x,t,y) = \int_s^t \int_X p(s,x,u,z)q(u,z)p(u,z,t,y)dzdu \leq \varepsilon p(s,x,t,y).$$

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Finally,

$$\tilde{p} \leq p \sum_{n=0}^{\infty} \varepsilon^n = \left(\frac{1}{1-\varepsilon}\right) p\,,$$

for $\varepsilon \in (0, 1)$.

Local smallness, global growth control

Assume that for all $s < t$, $x, y \in X$,

$$p_1(s, x, t, y) \leq \left[\eta + Q(s, t)\right] p(s, x, t, y), \qquad (\bigstar)$$

where $\eta \geq 0$ and $0 \leq Q(s, u) + Q(u, t) \leq Q(s, t)$, if $s < u < t$.

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Theorem (T. Jakubowski, 2009)

If $(\bigstar)$ *holds, then for all* $s < t$, $x, y \in X$,

$$\tilde{p}(s,x,t,y) \le \left(\frac{1}{1-\eta}\right)^{1+\frac{Q(s,t)}{\eta}} p(s,x,t,y)\,, \qquad \text{if} \quad 0 < \eta < 1\,,$$

and

$$\tilde{p}(s,x,t,y) \le e^{Q(s,t)} p(s,x,t,y)\,, \qquad \text{if} \quad \eta = 0\,.$$

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  Estimates of gradient perturbation series, 2012

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3G theorems

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Assume that 3G inequality holds for p,

$$p(s,x,u,z) \wedge p(u,z,t,y) \leq c\, p(s,x,t,y)\,, \quad c>0.$$

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We then verify $(\bigstar)$ by considering

$$\frac{p_1(s,x,t,y)}{p(s,x,t,y)} \leq c \int_s^t \int_X \Big[p(s,x,u,z) + p(u,z,t,y)\Big] q(u,z) dz du.$$

3G theorems

For instance, let p be α-stable transition density, $\alpha \in (0,2)$. Then

$$p(s,x,t,y) \approx (t-s)^{-d/\alpha} \wedge \frac{t-s}{|y-x|^{d+\alpha}}\,.$$

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- Szczypkowski- 2012
  Gradient perturbations of the sum of two fractional Laplacians

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3G does not hold for g:
$z-x = y-z = \xi$, $u-s = t-u = \tau$

$$g(s,x,u,z) \wedge g(u,z,t,y) = (4\pi\tau)^{-d/2} \exp\left(-|\xi|^2/4\tau\right),$$
$$g(s,x,t,y) = (8\pi\tau)^{-d/2} \exp\left(-|\xi|^2/2\tau\right).$$

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Q. S. Zhang- 2003

A sharp comparison result concerning schrödinger heat kernels.

A majorant

Consider auxiliary transition density p^* and $C \geq 1$ such that

$$p(s,x,t,y) \leq C\, p^*(s,x,t,y)\,, \qquad s < t,\ x,y \in X\,. \qquad (\bigstar\bigstar)$$

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Definition

We say that $q \in \mathcal{N}(p,p^*,C,\eta,Q)$, if $(\bigstar\bigstar)$ holds and

$$\int_s^t \int_X p(s,z,u,z)q(u,z)p^*(u,z,t,y)\,dzdu \qquad (\bigstar\bigstar\bigstar)$$
$$\leq \big[\eta + Q(s,t)\big]\, p^*(s,x,t,y),$$

where $\eta \geq 0$, Q is continuous and superadditive.

Q is called *superadditive* if

$$0 \leq Q(s,u) + Q(u,t) \leq Q(s,t)\,, \qquad s<u<t$$

The main result

Theorem

Let $q \in \mathcal{N}(p, p^*, C, \eta, Q)$. *Then for all* $s < t$, $x, y \in X$,

$$\tilde{p}(s,x,t,y) \leq \left(\frac{C}{1-2\eta}\right)^{1+\frac{Q(s,t)}{\eta}} p^*(s,x,t,y)\,, \quad \text{if} \quad 0 < \eta < 1/2\,,$$

and

$$\tilde{p}(s,x,t,y) \leq (2C)^{1+2Q(s,t)} p^*(s,x,t,y)\,, \quad \text{if} \quad \eta = 0\,.$$

An application

For $c > 0$ consider

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$$\tilde{g}_b(s,x,t,y) \leq g_a(s,x,t,y) \left(\frac{C}{1-2\eta}\right)^{1+\frac{Q(s,t)}{\eta}}.$$

Gaussian transition density

Define $L(\alpha) = \max_{\tau \geq \alpha \vee 1/\alpha} \left[\ln(1+\tau) - \frac{\tau - \alpha}{1+\tau} \ln(\alpha\tau) \right]$.

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Theorem (sharp 4G)

Let $M = [b/(b-a)]^{d/2} \exp\left[\frac{d}{2} L(\frac{a}{b-a})\right]$, *then*

$$g_b(s,x,u,z) g_a(u,z,t,y) \\ \leq M g_a(s,x,t,y) \left[g_{b-a}(s,x,u,z) \vee g_a(u,z,t,y) \right].$$

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We verify (★★★) by

$$\int_s^t \int_{\mathbb{R}^d} g_b(s,u,z)q(u,z)g_a(u,z,t,y)dzdu \\ \leq M\, g_a(s,x,t,y) \int_s^t \int_{\mathbb{R}^d} \left[g_{b-a}(s,x,u,z) + g_a(u,z,t,y)\right] q(u,z)dzdu.$$

Thank you!