REFLECTED BROWNIAN MOTION — DEFINITION AND APPROXIMATION

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Collaborators

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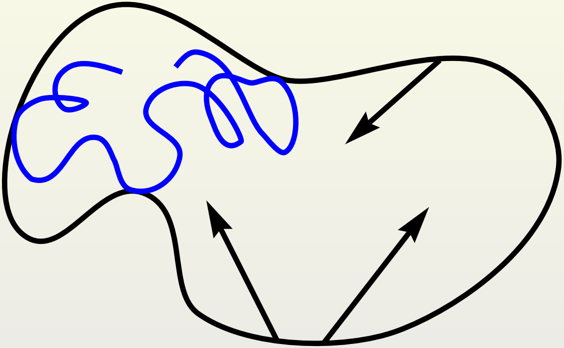
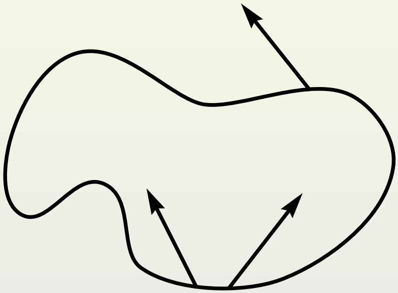
Obliquely reflected Brownian motion



Obliquely reflected Brownian motion in fractal domains

Technical challenges



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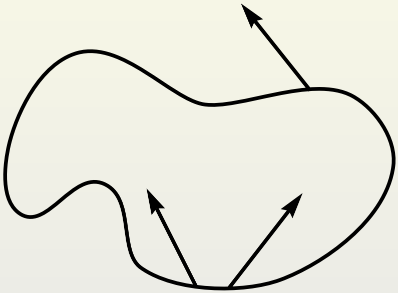

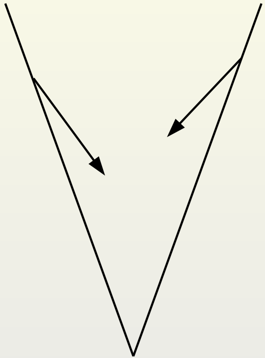

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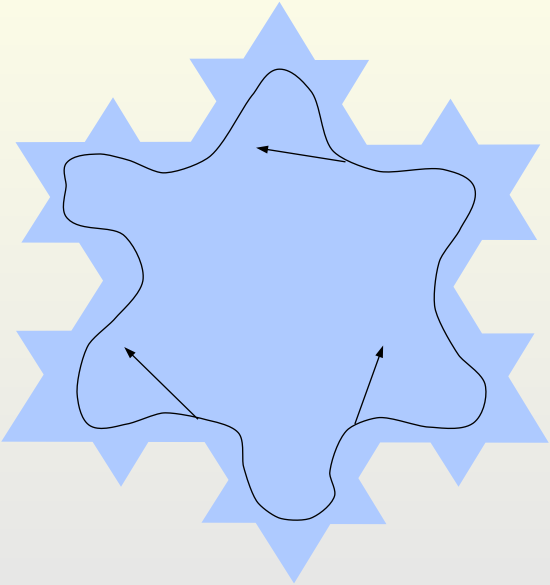

Smooth domain approximation

$D \subset \mathbb{R}^2$ – open bounded simply connected set
$D_k \subset D_{k+1}$, $\bigcup_k D_k = D$, D_k have smooth boundaries

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THEOREM (forthcoming; B, Chen, Marshall, Ramanan)

Suppose that θ_k converge as $k \to \infty$. Then obliquely reflected Brownian motions X^k converge, as $k \to \infty$, to a process in D.

Obliquely reflected Brownian motion – technical remarks

Classical Dirichlet form approach to Markov processes is limited to symmetric processes. Obliquely reflected Brownian motion is not symmetric.

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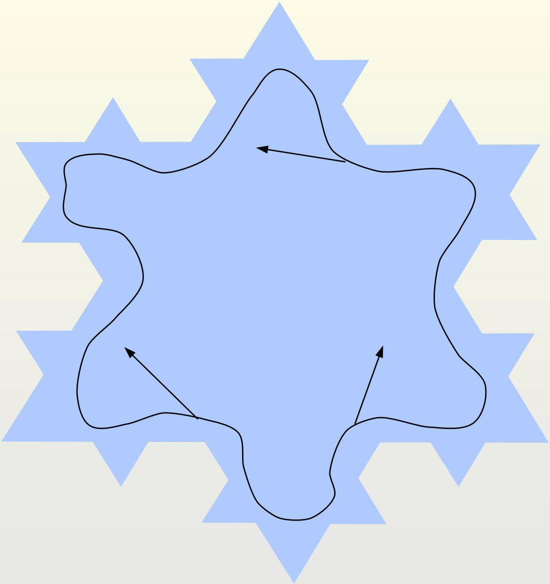
We apply conformal invariance of Brownian motion.

Obliquely reflected Brownian motion in higher dimensions



Parametrization of obliquely reflected Brownian motions

D – unit disc in $\mathbb{R}^2$
$\theta(x)$ – angle of reflection at $x \in \partial D$

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For an arbitrary measurable θ, obliquely reflected Brownian motion X in D with the oblique angle of reflection θ exists.

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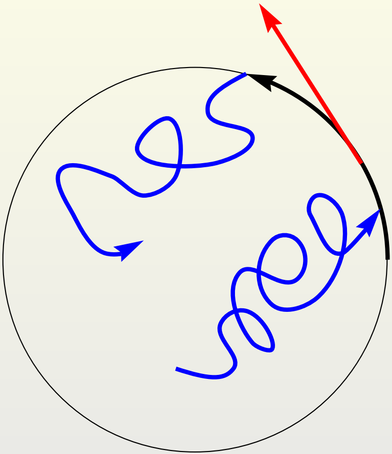
Lions and Sznitman (1984), Harrison, Landau and Shepp (1985), Varadhan and Williams (1985)

Jumps on the boundary



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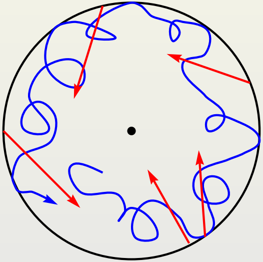
μ – rate of rotation



Parametrization of obliquely reflected Brownian motions

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THEOREM (forthcoming; B, Chen, Marshall, Ramanan)

$$h(z) = \frac{\Re \exp(\widetilde{\theta}(z) - i\theta(z))}{\pi \Re(e^{-i\theta(0)})} = \frac{\Re \exp(\widetilde{\theta}(z) - i\theta(z))}{\pi \cos\theta(0)}, \qquad z \in D,$$

$$\mu = \tan\theta(0) = \int_D \tan\theta(z) h(z) dz,$$

$$\theta(z) = -\arg\left(h(z) + i\widetilde{h}(z) - i\mu/\pi\right), \qquad z \in D.$$

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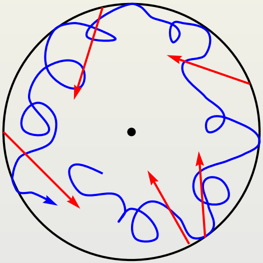
Harrison, Landau and Shepp (1985): smooth θ

Rate of rotation of obliquely reflected Brownian motion

THEOREM (forthcoming; B, Chen, Marshall, Ramanan)

1. Assume that θ is C^2. Then, with probability 1, X is continuous and, therefore, $\arg X_t$ is well defined for $t > 0$. The distributions of $\frac{1}{t} \arg X_t - \mu$ converge to the Cauchy distribution when $t \to \infty$.
2. Let $\arg^* X_t$ be $\arg X_t$ "without large excursions from ∂D." Then, a.s.,

$$\lim_{t \to \infty} \arg^* X_t / t = \mu.$$



Obliquely reflected Brownian motion in fractal domains

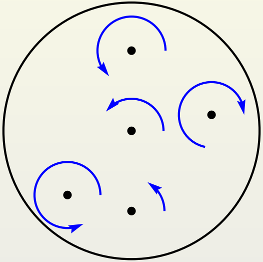
D – simply connected bounded open set in $\mathbb{R}^2$

THEOREM (forthcoming; B, Chen, Marshall, Ramanan)

For every positive harmonic function h in D with L^1 norm equal to 1 and every real number μ, there exists a (unique in distribution) obliquely reflected Brownian motion in D with the stationary distribution $h(x)dx$ and rate of rotation μ.

Rotation rate field

Let D be the unit disc and $\mu(z)$ be the rate of rotation around $z \in D$. In other words, the distributions of $\frac{1}{t}\arg(X_t - z) - \mu(z)$ converge to the Cauchy distribution when $t \to \infty$.



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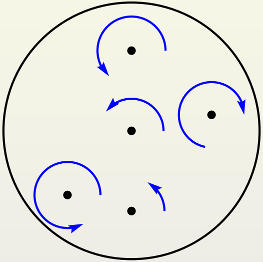

THEOREM (forthcoming; B, Chen, Marshall, Ramanan)

The function $\mu(z)$ is harmonic in D.

Rotation rate field

D – unit disc in $\mathbb{R}^2$
$\theta(x)$ – angle of reflection at $x \in \partial D$
$h(x)dx$ – stationary distribution
$\mu(z)$ – rate of rotation around z

$$\boxed{\theta \leftrightarrow (h, \mu(0)) \leftrightarrow \{\mu(z)\}_{z \in D}}$$

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Arrows indicate one to one mappings. Are the mappings surjective?

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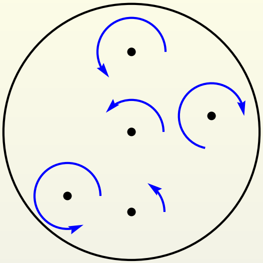
Arrows indicate one to one mappings. Are the mappings surjective?
In the first case, yes.

Rotation rate field – limitations



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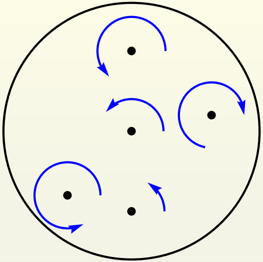

Suppose that $\phi(z)$ is harmonic in the unit disc D. $\phi(z)$ does not have to be positive.

For each ϕ there exists $b_0 \geq 0$ such that for all $a \in \mathbb{R}$ and $b \in [0, b_0]$, the function $\mu(z) = a + b\phi(z)$ is the rotation field of an obliquely reflected Brownian motion in D.

For $b > b_0$, the function $\mu(z) = a + b\phi(z)$ does not represent the rotation field of an obliquely reflected Brownian motion in D.

Integrability of harmonic functions

Let $\delta_D(x) = \text{dist}(x, \partial D)$ and $x_0 \in D$. We say that D is a John domain with John constant $c_J > 0$ if each $x \in D$ can be joined to x_0 by a rectifiable curve γ such that $\delta_D(y) \geq c_J \ell(\gamma(x, y))$ for all $y \in \gamma$, where $\gamma(x, y)$ is the subarc of γ from x to y and $\ell(\gamma(x, y))$ is the length of $\gamma(x, y)$.

THEOREM (Aikawa, 2000)

(i) If $D \subset \mathbb{R}^2$ is a bounded John domain with John constant $c_J \geq 7/8$ then all positive harmonic functions in D are in $L^1(D)$.

(ii) If $D \subset \mathbb{R}^2$ is a bounded Lipschitz domain with constant $\lambda < 1$ then all positive harmonic functions in D are in $L^1(D)$.

(iii) There exists a bounded Lipschitz domain D with constant $\lambda = 1$ and a positive harmonic function h in D which is not in $L^1(D)$.

Bounded harmonic functions

The modulus of continuity of f is $\omega_f(a) = \sup_{|x-y|<a} |f(x) - f(y)|$ and f is Dini continuous if $\int_0^b (\omega_f(a)/a) da < \infty$ for some $b > 0$.

THEOREM

(i) (Garnett, 2007) If θ is Dini continuous then h is bounded.

(ii) Suppose that ω is an increasing continuous concave function on $[0, \pi/2]$ such that $\omega(0) = 0$, $\omega(\pi/2) = \pi/4$, and $\int_0^{\pi/2} (\omega(a)/a) da = \infty$. Then there exists θ such that $\omega_\theta(a) = \omega(a)$ for $a \leq \pi/2$ and h is unbounded.

Myopic conditioning

$D \subset \mathbb{R}^d$ – open bounded connected set
$\varepsilon > 0$
X_t^ε – a continuous process in D

DEFINITION (Myopic Brownian motion)

Given $\{X_t^\varepsilon, 0 \le t \le k\varepsilon\}$, the process $\{X_t^\varepsilon, k\varepsilon \le t \le (k+1)\varepsilon\}$ is Brownian motion conditioned not to hit D^c (during the time interval $[k\varepsilon, (k+1)\varepsilon]$).

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THEOREM (B, Chen)

Processes X^ε converge weakly, as $\varepsilon \to 0$, to reflected Brownian motion in D.

Myopic conditioning (2)

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B – Brownian motion in $\mathbb{R}^d$, $\tau_D = \inf\{t \ge 0 : B_t \notin D\}$
$Y_k^\varepsilon = X_{k\varepsilon}^\varepsilon, \quad k \ge 1$
$m_\varepsilon(dx) = P^x(\tau_D > \varepsilon)dx$

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LEMMA (B, Chen)

(i) $m_\varepsilon \to$ Lebesgue measure on D as $\varepsilon \to 0$.
(ii) $m_\varepsilon(dx)$ is a reversible (stationary) measure for Y_k^ε.

Increasing families of domains

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THEOREM (B, Chen; 1998)

Reflected Brownian motions X^k converge, as $k \to \infty$, to reflected Brownian motion in D.

Invariance principle for reflected random walks

D – open connected bounded set
X^k – reflected random walk on $D \cap (2^{-k}\mathbb{Z}^2)$
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Assume that D is an extension domain. Then reflected random walks X^k, with sped-up clocks, converge weakly to reflected Brownian motion in D, as $k \to \infty$.

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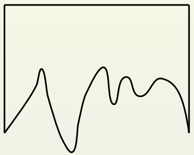
Examples of extension domains.

1. Smooth domains
2. Lipschitz domains
3. Uniform domains
4. NTA domains
5. Von Koch snowflake

Invariance principle in domains above graphs of continuous functions

D – bounded domain
∂D is locally the graph of a continuous function



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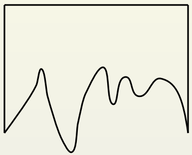

Fact: D is an extension domain.

COROLLARY (B, Chen; 2008)

Assume that D lies locally above the graph of a continuous function. Then reflected random walks X^k, with sped-up clocks, converge weakly to reflected Brownian motion in D, as $k \to \infty$.

Invariance principle – a counterexample

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THEOREM (B, Chen; 2008)

There exists a bounded domain $D \subset \mathbb{R}^2$ such that reflected random walks X^k, with sped-up clocks, do not converge weakly to reflected Brownian motion in D, when $k \to \infty$.

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Example: Remove suitable dust from a square.

Invariance principle (improved)

D – open connected bounded set
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D – open connected bounded set
D_k – subset of $D \cap (2^{-k}\mathbb{Z}^2)$; contains all vertices of the union of adjacent cubes in D
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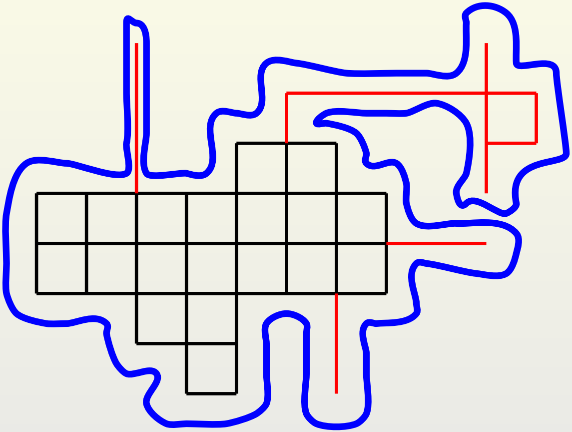
Reflected random walks X^k on D_k, with sped-up clocks, converge weakly to reflected Brownian motion in D, as $k \to \infty$.

Two approximations



THEOREM (B, Chen; 2012)

Suppose that $D \subset \mathbb{R}^d$ is a domain with finite volume. There exists a countable sequence of bounded functions $\{\varphi_j\}_{j\geq 1} \subset W^{1,2}(D) \cap C^\infty(D)$ such that

1. $\{\varphi_j\}_{j\geq 1}$ is dense in $W^{1,2}(D)$,
2. $\{\varphi_j\}_{j\geq 1}$ separates points in D,
3. for each $j \geq 1$,

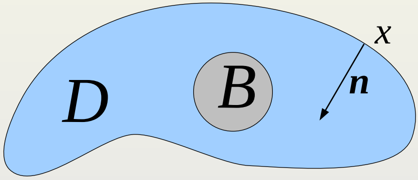
$$\limsup_{k\to\infty} 2^{k(2-d)} \sum_{\overline{xy}\in D_k} (\varphi_j(x) - \varphi_j(y))^2 \leq 2 \int_D |\nabla \varphi_j(x)|^2 \, dx.$$

Robin problem

$$\Delta u(x) = 0, \qquad x \in D \setminus B,$$

$$\frac{\partial u}{\partial \mathbf{n}}(x) = cu(x), \qquad x \in \partial D,$$

$$u(x) = 1, \qquad x \in \partial B.$$



Robin problem in fractal domains

Example: von Koch snowflake.

The normal vector does not exist at almost all boundary points.

Naive approach

Approximate the snowflake domain D with an increasing sequence of smooth domains D_k, such that $\bigcup_k D_k = D$.

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Let u_k be the solution to the Robin boundary problem in D_k, with the same c (adsorption rate) for all k, and let

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$$u(x) = \lim_{k\to\infty} u_k(x).$$

Then u satisfies the Dirichlet boundary conditions $u(x) = 0$ on ∂D.

Reformulation of Robin problem

Assuming that D is smooth, the Green-Gauss formula implies that for $u, v \in C^2(\overline{D})$,

$$\int_D \nabla u(x) \cdot \nabla v(x) dx = -\int_D v(x) \Delta u(x) dx - \int_{\partial D} v(x) \frac{\partial u}{\partial \mathbf{n}}(x) \sigma(dx),$$

where σ is the surface measure on ∂D.

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where σ is the surface measure on ∂D.

A weak solution u to the Robin problem is characterized by

$$\int_D \nabla u(x) \cdot \nabla v(x) dx = -\int_{\partial D} cu(x) v(x) \sigma(dx),$$

for every $v \in C^2(\overline{D})$ that vanishes on B.

Solution to Robin problem in von Koch snowflake

$d = \log 4 / \log 3$
Let μ be d-dimensional Hausdorff measure.

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Let μ be d-dimensional Hausdorff measure.

DEFINITION

We will say that a function u is a weak solution to the Robin problem in the snowflake domain if for all smooth v,

$$\int_D \nabla u(x) \cdot \nabla v(x) dx = - \int_{\partial D} c u(x) v(x) \mu(dx).$$

Alternative representation

D – von Koch snowflake domain
X – reflected Brownian motion in D
σ_B – hitting time of B

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D – von Koch snowflake domain
X – reflected Brownian motion in D
σ_B – hitting time of B
L – "local time" on ∂D, i.e., a continuous additive functional of X with Revuz measure μ

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X – reflected Brownian motion in D
σ_B – hitting time of B
L – "local time" on ∂D, i.e., a continuous additive functional of X with Revuz measure μ

THEOREM (forthcoming; B, Chen)

- The continuous additive functional L with Revuz measure μ exists.

Alternative representation

D – von Koch snowflake domain
X – reflected Brownian motion in D
σ_B – hitting time of B
L – "local time" on ∂D, i.e., a continuous additive functional of X with Revuz measure μ

THEOREM (forthcoming; B, Chen)

- The continuous additive functional L with Revuz measure μ exists.
- The function

$$u(x) = \mathbb{E}_x \left[\exp \left(-\frac{c}{2} \int_0^{\sigma_B} dL_s \right) \right], \qquad x \in \overline{D} \setminus B,$$

is the unique weak solution to the Robin problem.