Existence of an Invariant Measure for the Kick-Forced Primitive Equations

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Outline

I will present some results for the primitive equations with physical boundary conditions. This is a joint result with Robert Gastler.

1. Explanation of the primitive equations.
2. Our initial goal and the obstacles we faced.
3. Statement of results.
4. Sketch of proof of results.

The Setup

The 3D Primitive Equations

Let $G = G_2 \times [-h, 0] \subset \mathbb{R}^3$ be a "lake". The velocity-components $u_k : G \to \mathbb{R}$, $k = 1, 2, 3$ and pressure $p : G \to \mathbb{R}$ satisfy the PDE

$$\begin{cases} \partial_t u_k - \nu \Delta u_k + \sum_{j=1}^{3} u_j \partial_j u_k + \partial_k p = \text{Forcing}, \quad k = 1, 2 \\ \text{div } u = \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 = 0 \end{cases} . \tag{1}$$

where *we also require that* $p = p(x, y)$.

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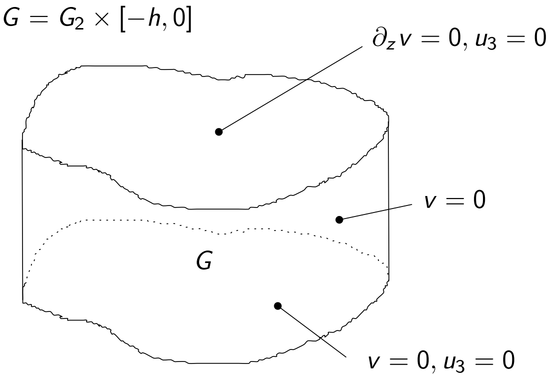
where *we also require that* $p = p(x, y)$.
Modification of the NS-equations where

- We drop the equation for u_3. (Relaxation)
- We demand that the pressure $p = p(x, y, z)$ be independent of z. (Restriction)

Let $u = (v, u_3)$ i.e. $v =$ horizontal velocity, $u_3 =$ vertical velocity. We consider the *physical boundary conditions*



Alternate Form of the Primitive Equations

From the divergence free condition we get

$$u_3(x,t) = -\int_{-h}^{z} \mathrm{div}_2 v(x,y,z',t)dz'$$

so we have the PDE for v,

$$\begin{cases} \partial_t v - \nu\Delta v + (v\cdot\nabla_2)v - \left(\int_{-h}^{z} \mathrm{div}_2 v(x,y,z',t)dz'\right)\partial_z v + \nabla_2 p = f \\ v(x,0) = v_0 \end{cases} . \tag{2}$$

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Note that the bilinear term is nastier than that of the NS-equations. This is the source of great trouble...

Functional Analytic Form of the Primitive Equations

Define the spaces

$$H = \text{“}L^2 \text{ \& Divergence Free \& Boundary conditions”}$$
$$V = \text{“}H^1 \text{ \& Divergence Free \& Boundary conditions”}$$

Then we can consider the evolution equation

$$\begin{cases} \partial_t v + \nu A v + B(v,v) = \Pi_H \left(\text{Forcing}\right) \\ v(x,0) = v_0 \end{cases}, \tag{3}$$

where Π_H denotes projection onto H, $A := -\Pi_H \Delta$, and $B(u,v) := \Pi_H \left[(u \cdot \nabla_2) v - \left(\int_{-h}^{z} \text{div}_2 u \; dz' \right) \partial_z v \right]$.

Types of Solution

v is a **weak solution** if

$$v \in L^\infty([0,T];H) \cap L^2([0,T];V),\ \partial_t v \in L^2([0,T];V^{-3})$$

(V^{-3} denotes the dual space to $V^3 := H^3 \cap V$) and the equalities in (2) hold in V^{-3}, i.e if $\forall w \in V^3$

$$\langle \partial_t v + (v\cdot\nabla_2)v - \left(\int_{-h}^{z} \mathrm{div}_2 v(x',z',t)dz'\right)\partial_z v, w\rangle + \langle \nu\nabla v, \nabla w\rangle = \langle f, w\rangle$$

$$\langle v(x,0), w\rangle = \langle v_0, w\rangle.$$

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$$\partial_t v + \nu \Delta v + (v \cdot \nabla_2)v - \left(\int_{-h}^{z} \mathrm{div}_2 v(x',z',t)dz'\right)\partial_z v + \nabla_2 p = f$$

$$v(x,0) = v_0.$$

Known Results

Non-Physical Boundary Conditions:

- (2005) Cao and Titi: Showed there exists a unique global strong solution for any $v_0 \in V$. (Not known for 3DNS!!!)
- (2007) Ning Ju: Showed the existence of a global attractor.
- (2009) Guo and Huang: Global well posedness for additive noise. Random attractors.

Physical Boundary Conditions:

- (2007) Kukavica and Ziane: Showed there exists a unique global strong solution for any $v_0 \in V$.
- (2008) Kukavica and Ziane: Uniform gradient bounds.

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Show the existence (and then uniqueness?) of an invariant measure for the 3D primitive equations (with physical boundary conditions) under random forcing.

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  - Can only define a solution operator $S(t) : V \to V$.
  - So can only define a Markov process on V (but it is Feller at least!)
- But how to get compactness in V? Hard to get estimates on $||Av||$...

Ning Ju (2007) proves the following result (for non-physical BCs):

$$\text{The solution operator } S(t) : V \to V \text{ is compact.}$$

We take advantage of this to show the existence of an invariant measure for *kick forcing*!

Kick Forcing

Idea: Run primitive equations with no forcing and give a random kick every T seconds.
Defn: Given $v_0 \in V$, let $X_n : \Omega \to V$ be the random variables

$$X_0 \equiv v_0 \text{ and } X_n(\omega) = S(T)\left[X_{n-1}(\omega)\right] + \xi_n(\omega), \text{ for } n = 1, 2, \dots, \quad (4)$$

where the $\{\xi_n\}_{n=1}^{\infty}$ are i.i.d. V-valued random variables (the "kicks").

Our Results

Let $v(t)$ be the strong solution with $v_0 \in V$ and forcing f.

Theorem 1 (Bounded absorbing set in V under constant forcing). Suppose $||v_0||_V^2 \leq R$ and $f \in H$ is constant. Then there exists $K_V > 0$, $T_V > 0$ depending only on R, $||f||_H$ such that

$$||v(t)||_V^2 < K_V, \quad \forall t > T_V$$

(Note: This result is also proven in Kukavica-Ziane (2008))

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Theorem 2 (Decay in V-norm under no forcing). Suppose $||v_0||_V^2 \leq R$ and $f \equiv 0$. Then $\forall \varepsilon > 0, \exists T_V = T_V(R, \varepsilon)$ such that

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Theorem 3 (Invariant measure for kick-forcing). Suppose the kicks are bounded in H^2-norm, i.e.

$$\exists R > 0 \text{ s.t } ||A\xi_n||_H^2 \leq R \; \forall n$$

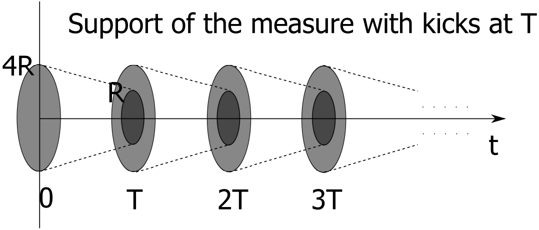
Then there exists a time $T = T(R)$ for which there exists an invariant measure for kick-forcing at interval T.

Proof of Theorem 3

By Theorem 2, if we take $T = T_V(4R, R)$, then $||X_n(\omega)||_V^2 \leq 4R$ implies

$$\begin{aligned} ||X_{n+1}(\omega)||_V^2 =& ||S(T)[X_n(\omega)] + \eta_{n+1}||_V^2 \\ \leq& 2||S(T)[X_n(\omega)]||_V^2 + 2||\eta_{n+1}||_V^2 \leq 2R + 2R = 4R \end{aligned}$$

as well. Hence $||X_n(\omega)||_V^2 \leq 4R$ for all n.



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$\Longrightarrow$ The measures μ_n are tight.

$\Longrightarrow$ By Krylov-Bugolybov there exists an invariant measure. Q.E.D.

Proof of Theorem 1

From careful (and messy) analysis of the argument in Kukavica and Ziane (2007) we deduce

Lemma

(Growth Control Lemma) There is an $\eta > 0$ *s.t. if* $0 \leq \tau_1 \leq \tau_3$ *are close in that*

$$|\tau_3 - \tau_1| \leq 1 \text{ and } \int_{\tau_1}^{\tau_3} ||v(\tau)||_V^2 \, d\tau \leq \eta, \tag{5}$$

then

$$||v(\tau_2)||_V^2 \leq e^{C(1+||v(\tau_1)||_V^2)^4} \left[||v(\tau_1)||_V^2 + ||f||_H^2 \right] =: \Gamma\left(||v(\tau_1)||_V^2 \right)$$

for any $\tau_2 \in [\tau_1, \tau_3]$*, where* $C = C(\nu, \eta, ||f||_H)$*.*

So provided τ_1 and τ_3 are close enough, V-norm only grows so much.

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where λ_1 is the first eigenvalue of A (Poincaré inequality).

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By basic ODE theory,

$$\implies \exists K_H > 0,\ T_H = T_H(R, ||f||_H) \text{ such that } ||v(t)||_H^2 \leq K_H \text{ for all } t \geq T_H.$$

Also, by integrating (6), we get

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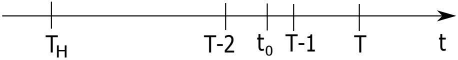

Then

$$\int_{T-2}^{T-1} ||v(\tau)||_V^2 \, d\tau \leq K_H + C||f||_H^2.$$

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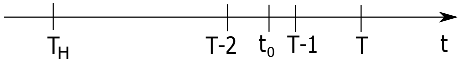

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$\Longrightarrow \exists\ t_0 \in [T-2, T-1]$ such that $||v(t_0)||_V^2 \le K_H + C||f||_H^2$.

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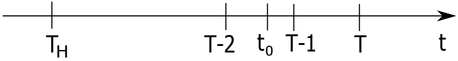

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And on the interval $[t_0, T]$,

$$\int_{t_0}^{T} ||v(\tau)||_V^2 \, d\tau \le K_H + 2C||f||_H^2 < \infty.$$

So we can divide the interval $[t_0, T]$ into L intervals $[t_k, t_{k+1}]$ satisfying

$$\int_{t_k}^{t_{k+1}} ||v(\tau)||_V^2 \, d\tau < \eta, \ |t_{k+1} - t_k| < 1$$

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$\Longrightarrow$ by the Growth Control Lemma,

$$||v(T)||_V^2 \leq \Gamma^{(L)} \left(||v(t_0)||_V^2 \right) \leq \Gamma^{(L)} \left(K_H + ||f||_H^2 \right),$$

where $\Gamma^{(L)}(\cdot)$ denotes L-fold composition.

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where $\Gamma^{(L)}(\cdot)$ denotes L-fold composition.
So Theorem 1 is proven with $T_V = T_H + 2$ and $K_V = \Gamma^{(L)}\left(K_H + ||f||_H^2\right)$.

Proof of Theorem 2

As before, take the PDE $\partial_t v + Av + B(v,v) = \Pi_H f$, multiply by v and integrate to get

$$\partial_t ||v||_H^2 \leq -\lambda_1 ||v||_H^2$$

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Arguing as before (with $f \equiv 0$ now), to get

$$T_H \leq T - 2 \leq t_0 \leq T - 1 < T$$

and then

$$||v(T)||_V^2 \leq \Gamma^{(L)} \left(||v(t_0)||_V^2 \right) \leq \Gamma^{(L)} (\varepsilon) \, .$$

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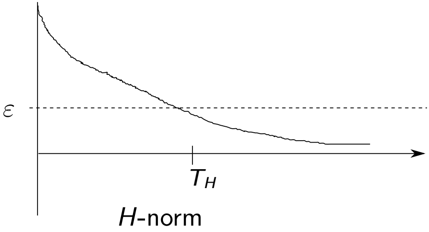
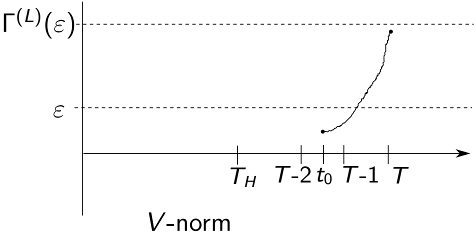
$$||v(T)||_V^2 \leq \Gamma^{(L)}\left(||v(t_0)||_V^2\right) \leq \Gamma^{(L)}(\varepsilon).$$

So Theorem 2 holds with $T_V\left(R, \Gamma^{(L)}(\varepsilon)\right) = T_H(R,\varepsilon) + 2$. (Sufficient as $\Gamma^{(L)}(\varepsilon) \to 0$ as $\varepsilon \to 0$).

Rough Picture





The End