On Exceptional times of Fleming-Viot processes with mutation

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The Feller diffusion

Let us start from the SDE in $\mathbb{R}$

$$Z_t = x + \int_0^t \sqrt{Z_s}\, dB_s, \qquad t \geq 0,$$

where $x \geq 0$ and B is a BM. By Yamada-Watanabe we have pathwise uniqueness and existence of strong solutions. In particular, 0 is a trap for Z.

This process is a CSBP: $Z_t^x + \hat{Z}_t^y \overset{(d)}{=} Z_t^{x+y}$, with Z and $\hat{Z}$ independent, since

$$d\left(Z_t^x + \hat{Z}_t^y\right)^2 = dm_t + \left(Z_t^x + \hat{Z}_t^y\right)\, dt$$

$$d\left(Z_t^x + \hat{Z}_t^y\right) = \sqrt{Z_t^x + \hat{Z}_t^y}\, dW$$

Z is the scaling limit of the dynamics of a population with critical (e.g. binary) branching.

Galton-Watson processes

Let $(\xi_i^n)_{i,n}$ i.i.d. in $\{0,1,\dots\}$ with $\mathbb{E}((\xi_j^n)^2) < +\infty$ and $\mathbb{E}(\xi_i^n) = 1$ $(\mathrm{Var}(\xi_i^n) = 1)$

$$GW_{n+1}^z = \sum_{i=1}^{GW_n^z} \xi_i^{n+1}, \qquad GW_0^z = z \in \mathbb{N}.$$

Note that $(GW_n^j - GW_n^{j-1}, n \geq 0)_{j \geq 1}$ is an i.i.d. sequence of processes, a discrete version of the CSBP property.

Then we have for $x \in \mathbb{R}_+$ and $N \to +\infty$

$$\left(\frac{1}{N} GW_{\lfloor Nt \rfloor}^{\lfloor Nx \rfloor}, t \geq 0 \right) \Longrightarrow (Z_t^x, t \geq 0).$$

We assign to each $j \in \mathbb{N}$ a genetic type u_j uniform in $[0,1]$ with $(u_j)_j$ i.i.d. and we define the measure-valued process $(X_n^N, n \geq 0)$

$$X_n^N := \sum_{j=1}^{N} (GW_n^j - GW_n^{j-1})\, \delta_{u_j}.$$

The measure-valued process

$$\left(\frac{1}{N}X^N_{[Nt]}, t \geq 0\right) \Longrightarrow (X_t, t \geq 0)$$

where $(X_t, t \geq 0)$ is a measure-valued Markov process s.t.

$$X_t(x) := X_t([0,x]) = x + \int_{[0,t]\times\mathbb{R}_+} \mathbb{1}_{\{u \leq X_s(x)\}} \, W(ds, du)$$

and W is a Brownian sheet. Then for all $x \geq 0$,

$$(X_t(x))_{t\geq 0} \stackrel{(d)}{=} (Z^x_t)_{t\geq 0}.$$

Pathwise uniqueness of the SDE has been proved by Dawson-Li [AOP 2012]. Then $(X_t(x))_{t\geq 0, x\in[0,1]} \stackrel{(d)}{=} (Y_t(x))_{t\geq 0, x\in[0,1]}$

$$Y_t = \sum_i e^i_t \, \delta_{u_i}, \qquad Y_t(x) = \sum_{u_i \leq x} e^i_t, \qquad t > 0,$$

where (u_i, e^i) is a PPP with intensity measure $du \otimes n(de)$,

Measure-valued branching process with immigration

We consider now a population with critical branching plus constant immigration-rate $\theta > 0$. The measure-valued process is

$$X_t(x,\theta) = x + \int_{[0,t]\times\mathbb{R}_+} \mathbb{1}_{\{u \le X_s(x,\theta)\}} \, W(ds,du) + x\,\theta t.$$

and $(X_t(x,\theta), t \ge 0)$ has the same law as the solution of

$$Z_t = x + \int_0^t \sqrt{Z_s}\, dB_s + x\,\theta\, t$$

i.e. of a squared-Bessel process.

Following Pitman-Yor (1982) one can prove the following representation: if (s_i, u_i, e^i) is a Poisson point process with intensity measure $(\delta_0(ds) + ds) \otimes du \otimes n(de)$ and we set

$$Y_t(x,\theta) := \sum_{s_i=0, u_i \le x} e^i_t + \sum_{0<s_i\le t, u_i\le\theta} e^i_{t-s_i}$$

then $(X_t(x,\theta))_{x,t\ge 0} \overset{(d)}{=} (Y_t(x,x\theta))_{x,t\ge 0}$.

Fleming-Viot processes

Setting

$$T(t) := \int_0^t X_s(1)^{-1}\, ds, \qquad FV_t := \frac{X_{T^{-1}(t)}}{X_{T^{-1}(t)}(1)}, \qquad t \geq 0,$$

then $(FV_t, t \geq 0)$ is a Markov process in the space of probability measures on $[0,1]$, describing the evolution of a population under the effect of genetic drift and constant-rate mutation. The invariant measure is the Poisson-Dirichlet(θ) distribution. Since a.s. X_t is atomic for all $t > 0$, so is FV_t.

- for $\theta = 0$ a.s. FV_t has finitely many atoms for all $t > 0$
- for $\theta > 0$ and $t > 0$ a.s. FV_t has infinitely many atoms.

Schmuland ['91] proved using capacities and Dirichlet forms that

- for $0 < \theta < 1$ a.s. there exist exceptional times when FV_t has finitely many atoms.
- for $\theta \geq 1$ a.s. FV_t has infinitely many atoms for all $t > 0$.

Continuous-state branching processes

The Feller diffusion is only an example of a class of Markov processes with the branching property $Z_t^x + \hat{Z}_t^y \overset{(d)}{=} Z_t^{x+y}$, see Lamperti ['60]. An important subclass is given by CSBPs with stable branching mechanism

$$Z_t = x + \int_0^t Z_{s-}^{1/\alpha} \, dL_s$$

where $\alpha \in (1,2)$ and L is a spectrally positive α-stable Lévy process with Laplace exponent for $\lambda \geq 0$

$$\log \mathbb{E}[e^{-\lambda L_1}] = \lambda^\alpha = \int_0^\infty \left(e^{-\lambda x} - 1 + \lambda x \right) c_\alpha \, x^{-1-\alpha} \, dx.$$

Z is the scaling limit of a Galton-Watson process with critical branching in the domain of attraction of a α-stable law.

Measure-valued processes

For $x \in [0,1]$ and Lipschitz $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$

$$
\begin{aligned}
X_t(x) =& x + \int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}_+} \mathbb{1}_{\{u \le X_{s-}(x)\}} \, y \, N(dy, ds, du) \\
&+ x \int_0^t g(X_s(1)) \, ds
\end{aligned}
\tag{1}
$$

where $N := M - \tilde{M}$, $\tilde{M}(dy, ds, du) := c_\alpha y^{-1-\alpha} dy \, ds \, du$ and M is a Poisson point process with intensity $\tilde{M}$.

Then the measure defined by $X_t(]a,b]) := X_t(b) - X_t(a)$, $b \ge a \ge 0$ is a measure-valued branching process with α-stable branching mechanism and immigration rate $g(X_s(1))$.

Pitman-Yor representation

Now the immigration rate is not constant anymore. We consider again a PPP (s_i, u_i, e^i) with intensity measure $(\delta_0(ds) + ds) \otimes du \otimes n(de)$ and we look for a process such that

$$Z_t(x) := \sum_{s_i=0, u_i \le x} e^i_t + \sum_{0<s_i\le t} e^i_{t-s_i} \, \mathbb{1}_{(u_i \le x g(Z_{s_i-}(1)))} \qquad (2)$$

Theorem

The equations (1) *and* (2) *have unique solutions, and*

$$(X_t(x), t \ge 0, x \in [0,1]) \overset{(d)}{=} (Z_t(x), t \ge 0, x \in [0,1]).$$

More general results can be found in a recent book by Zenghu Li.

Fleming-Viot processes

Let $g(z) := \theta z^{2-\alpha}$. We define the Fleming-Viot process

$$T_t := \int_0^t (X_s(1))^{1-\alpha}\, ds, \qquad Y_t := \frac{X_{T_t^{-1}}}{X_{T_t^{-1}}(1)},$$

then Y is Markovian with values in the space of probability measures on $[0,1]$ (this would not be true for different immigration-rates). Y solves an infinite-dimensional SDE.

One can see that Y_t is a.s. an atomic measure for all $t > 0$ and if $\theta = 0$ then a.s. there are only finitely many atoms.

Using the Pitman-Yor representation, we obtain

Theorem

For $\alpha < 2$, the Schmuland dichotomy does not arise, i.e. for all $\theta > 0$ a.s. Y_t has infinitely many atoms for all $t > 0$.