Geodesic distances and intrinsic distances on some fractal sets

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1. Introduction

Intrinsic distance in the framework of Dirichlet forms

(cf. Biloli–Mosco, Sturm etc.)

(K,λ): a locally compact, separable metric measure space

$(\mathcal{E},\mathcal{F})$: a strong local regular Dirichlet form on $L^2(K;\lambda)$

$\mu_{\langle f\rangle}$: the energy measure of $f\in\mathcal{F}$

When f is bounded,

$\int_K \varphi\,d\mu_{\langle f\rangle} = 2\mathcal{E}(f,f\varphi)-\mathcal{E}(f^2,\varphi)\ \ {}^\forall\varphi\in\mathcal{F}\cap C_b(K)$.

If $\mathcal{E}(f,g)=\frac{1}{2}\int_{\mathbb{R}^d}(a_{ij}(x)\nabla f(x),\nabla g(x))_{\mathbb{R}^d}\,dx$,

then $\mu_{\langle f\rangle}(dx)=(a_{ij}(x)\nabla f(x),\nabla f(x))_{\mathbb{R}^d}\,dx$.

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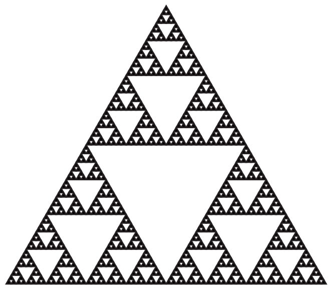
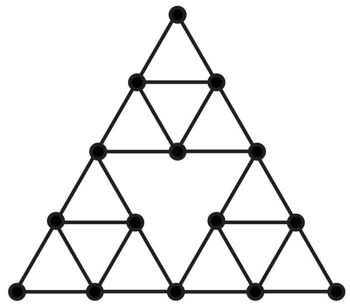
2. Canonical Dirichlet forms on typical self-similar fractals

Case of the 2-dim. standard Sierpinski gasket





$$K \supset V_2$$

V_n: nth level graph approximation

$$\mathcal{E}^{(n)}(f,f) = \left(\frac{5}{3}\right)^n \sum_{x,y \in V_n,\ x \sim y} (f(x) - f(y))^2$$

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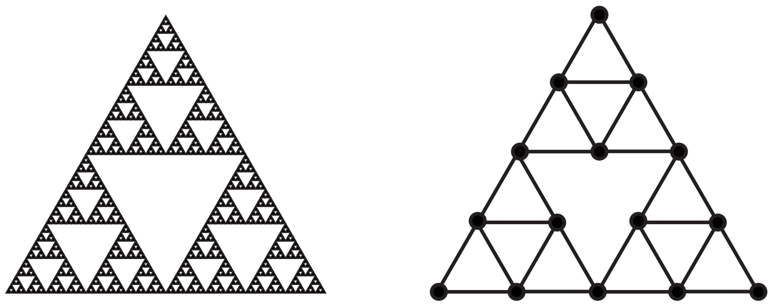

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↑ scaling factor

$$\mathcal{E}^{(n)}(f,f) \nearrow {}^{\exists}\mathcal{E}(f,f) \le +\infty \quad {}^{\forall}f \in C(K).$$

$$\mathcal{F} := \{f \in C(K) \mid \mathcal{E}(f,f) < +\infty\}$$

Then, $(\mathcal{E},\mathcal{F})$ is a strong local regular Dirichlet form on $L^2(K;\lambda)$. (λ: the Hausdorff measure on K)

$\rightsquigarrow \{X_t\}$: "Brownian motion" on K
(invariant under scaling and isometric transformations)

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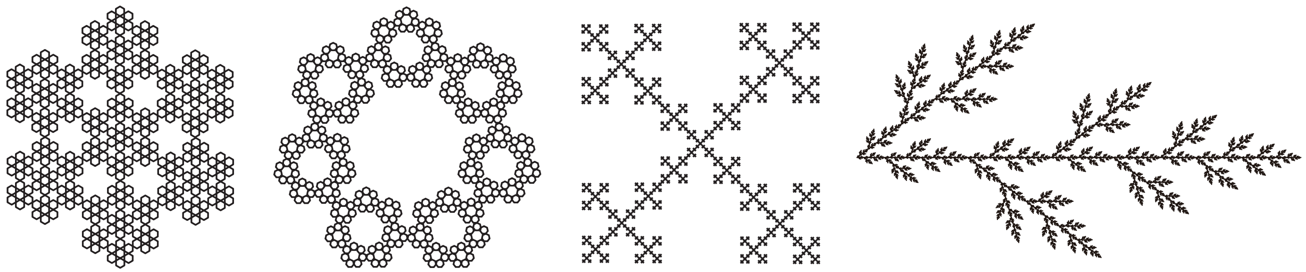

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Then, $\mathbf{d}(x, y) = \sup\{f(y) - f(x) \mid f = \text{const.}\} = 0$.

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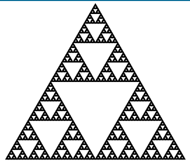

$(\mathcal{E}, \mathcal{F})$: the standard Dirichlet form on $L^2(K, \nu)$ with

$\nu := \mu_{\langle h_1 \rangle} + \mu_{\langle h_2 \rangle}$ (Kusuoka measure)

(h_i: a harmonic function, $\mathcal{E}(h_i, h_j) = \delta_{i,j}$)

Theorem (Kigami '93, '08, Kajino '12)

- (Ki) $h : K \to h(K) \subset \mathbb{R}^2$ is homeomorphic;
- (Ka) The intrinsic distance $\mathbf{d}$ coincides with the geodesic distance ρ_h on $h(K)$ by the identifying K and $h(K)$;
- (Ki, Ka) The transition density $p_t^\nu(x, y)$ has a Gaussian estimate w. r. t. $\rho_h (= \mathbf{d})$;
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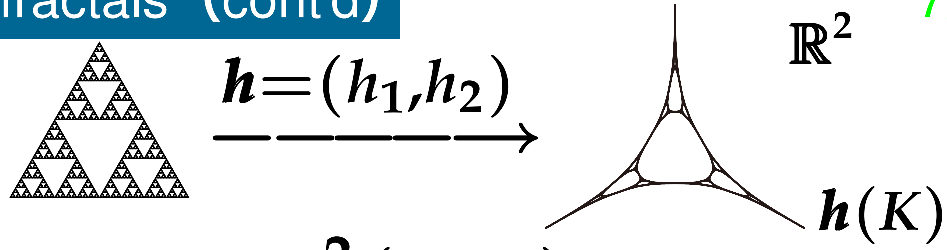

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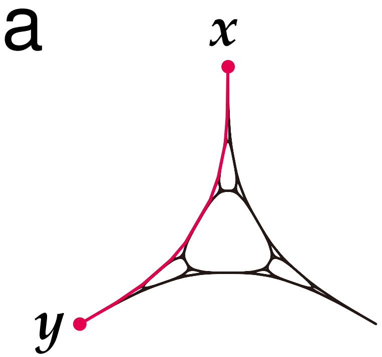
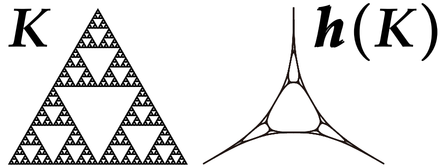




Observation: an alternative expression of $\rho_{\boldsymbol{h}}$

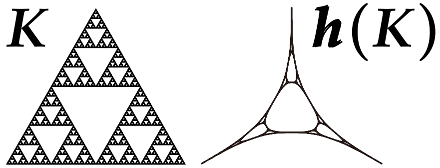
For a continuous curve $\gamma\colon [0,1] \longrightarrow K$, the length $l_{\boldsymbol{h}}(\gamma)$ of γ based on $\boldsymbol{h}$ is defined as

$$l_{\boldsymbol{h}}(\gamma) := \sup\left\{ \sum_{i=1}^{n} |\boldsymbol{h}(\gamma(t_i)) - \boldsymbol{h}(\gamma(t_{i-1}))|_{\mathbb{R}^2};\ 0 = t_0 < t_1 < \cdots < t_n = 1 \right\}.$$

Then, by identifying K and $\boldsymbol{h}(K)$,

$$\rho_{\boldsymbol{h}}(x,y) = \inf\left\{ l_{\boldsymbol{h}}(\gamma) \,\middle|\, \gamma \text{ is a continuous curve connecting } x \text{ and } y \right\}.$$

In this expression, $\boldsymbol{h}$ does not need to be homeomorphic.



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Brief formulation of the problem

$(\mathcal{E},\mathcal{F}), \boldsymbol{h}=(h_1,\dots,h_N)\in\mathcal{F}^N\cap C(K\to\mathbb{R}^N)$: given

$\mathbf{d}_{\boldsymbol{h}}$: the intrinsic distance based on $(\mathcal{E},\mathcal{F})$ and

$$\nu:=\mu_{\langle h_1\rangle}+\cdots+\mu_{\langle h_N\rangle}$$

$\rho_{\boldsymbol{h}}$: the geodesic distance based on $\boldsymbol{h}$

The relation between $\mathbf{d}_{\boldsymbol{h}}$ and $\rho_{\boldsymbol{h}}$, in particular when the underlying space has a fractal structure?

3. General framework

(K, d_K): a separable and compact metric space

λ: a finite Borel measure on K

$(\mathcal{E}, \mathcal{F})$: a strong local regular Dirichlet form on $L^2(K, \lambda)$

$N \in \mathbb{N}$, $\boldsymbol{h} = (h_1, \ldots, h_N) \in \mathcal{F}^N \cap C(K \to \mathbb{R}^N)$

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Problem: The relation between $\mathbf{d}_{\boldsymbol{h}}$ and $\rho_{\boldsymbol{h}}$?

4. Results

Theorem 1 $\rho_h(x,y) \leq \mathbf{d}_h(x,y)$ if the following hold:

(A1) (Finitely ramified cell structure) There exists an increasing sequence of finite subsets $\{V_m\}_{m=0}^{\infty}$ of K such that

(i) $\bigcup_{m=0}^{\infty} V_m$ is dense in K;

(ii) For each m, $K \setminus V_m$ is decomposed as a finite number of connected components $\{U_\lambda\}_{\lambda \in \Lambda_m}$;

(iii) $\lim_{m\to\infty} \max_{\lambda\in\Lambda_m} \operatorname{diam} U_\lambda = 0$.

(A2) $\mathcal{F} \subset C(K)$.

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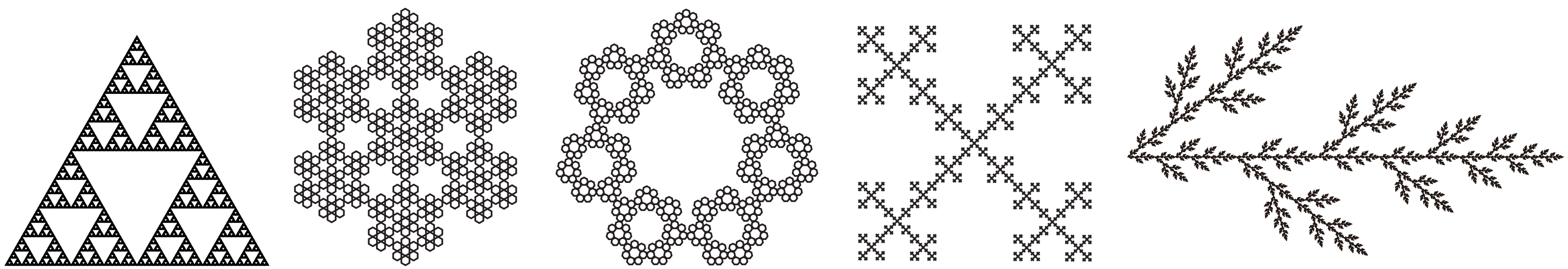

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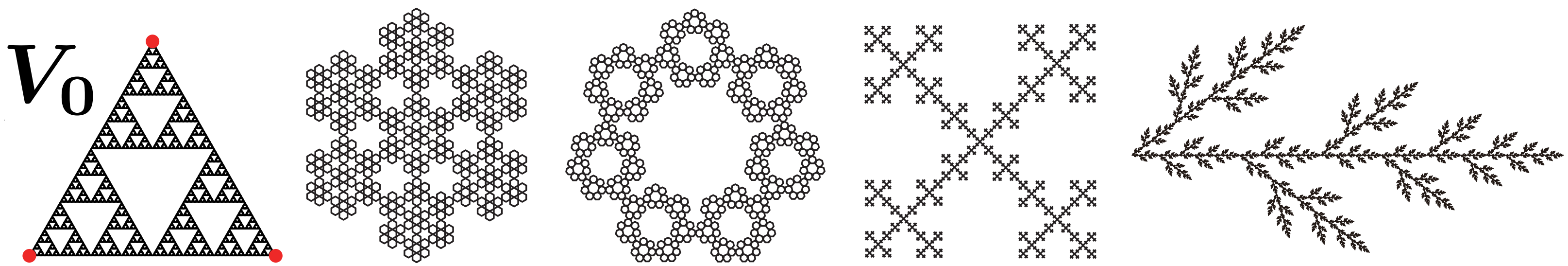

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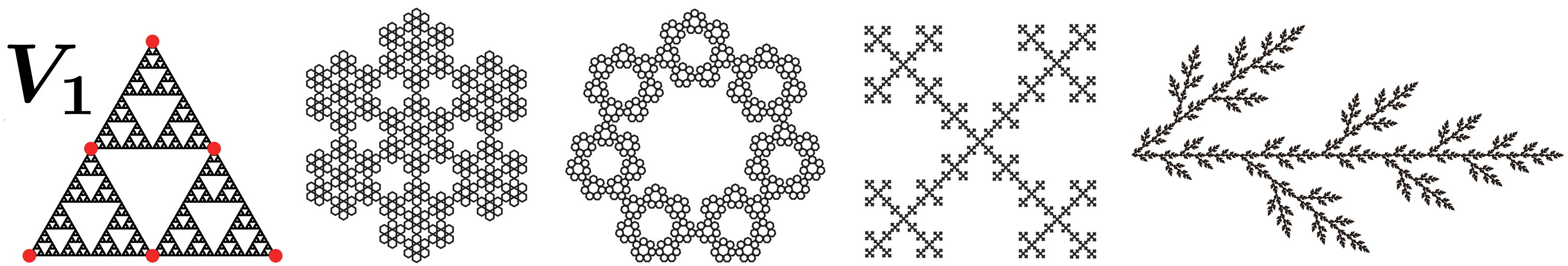

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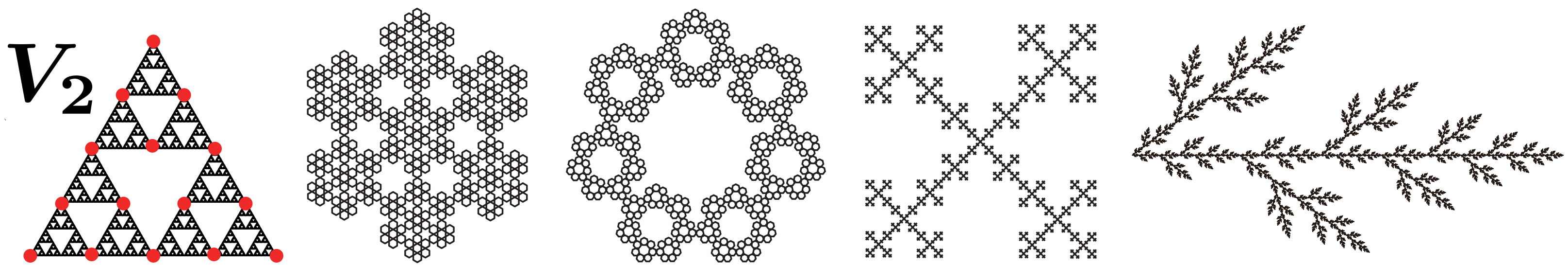

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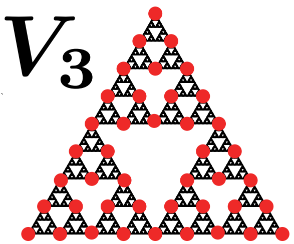

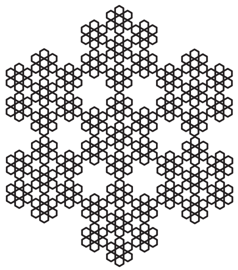

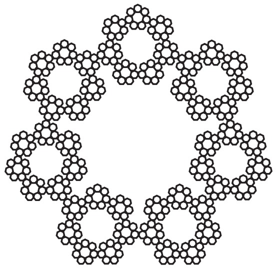

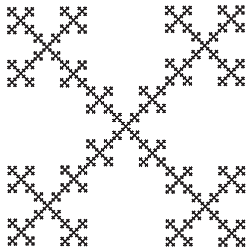

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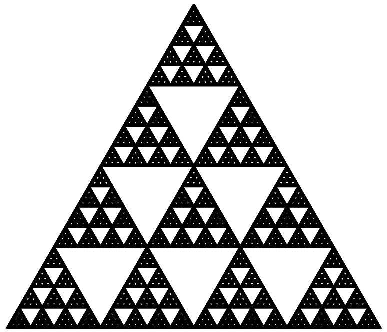

- K: a 2-dimensional (generalized) Sierpinski gasket that is also a nested fractal;
- (λ: the normalized Hausdorff measure;)
- $(\mathcal{E}, \mathcal{F})$: the self-similar Dirichlet form associated with the Brownian motion on K;
- $\boldsymbol{h} = (h_1, \ldots, h_d)$; each h_i is a harmonic function;
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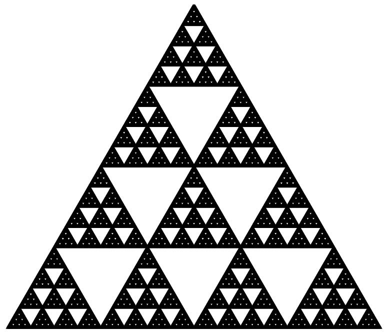

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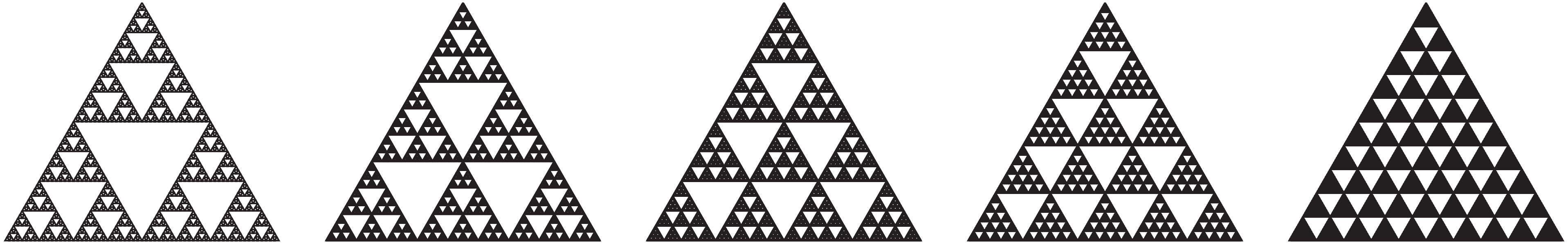

(level l S. G. with $l = 2, 3, 4, 5, 10$)

Remark Theorem 2 is valid under more general situations. Essential assumptions (for the current proof) are:

- #the vertex set $= 3$;
- The harmonic structure is near to symmetric.

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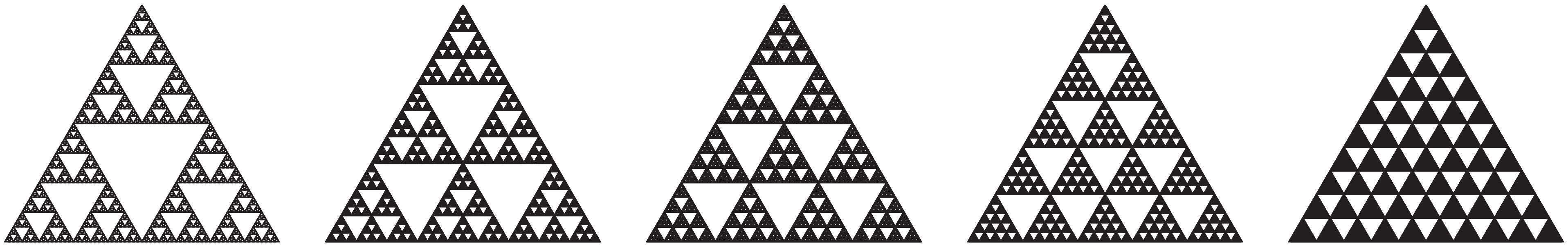

(level l S. G. with $l = 2, 3, 4, 5, 10$)

Remark Theorem 2 is valid under more general situations. Essential assumptions (for the current proof) are:

- #the vertex set $= 3$;
- The harmonic structure is near to symmetric.

5. Idea of the proof of Theorem 1 $(\rho_h \le \mathbf{d}_h)$

In the case of Riemannian manifolds M, the proof is given as follows:

For $x \in M$, define $\rho(y)$ as the geodesic distance between x and $y \in M$. Then, ρ is Lipschitz and $|\nabla\rho| \le 1$.

Therefore, $\rho(x,y) = \rho(y) - \rho(x) \le \mathbf{d}(x,y)$.

In Theorem 1, the main part of the proof is to prove $\rho \in \mathcal{F}$ and $\mu_{\langle\rho\rangle} \le \mu_{\langle h\rangle}$. This is done by the discrete approximation.

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More precisely,

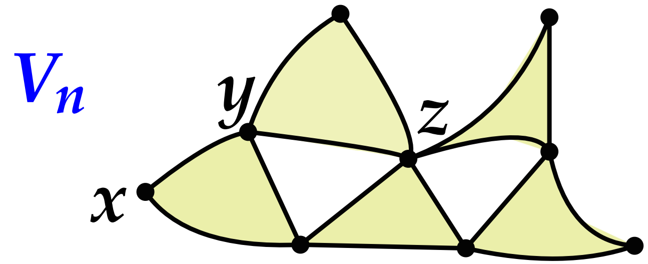
- $\rho(y) := \rho_{\boldsymbol{h}}(x, y)$; the distance function from $x \in K$

  It is sufficient to prove that $\rho \in \mathcal{F}$ and $\mu_{\langle\rho\rangle} \le \mu_{\langle \boldsymbol{h}\rangle}$.

- Discrete approximation. Assume $x \in V_m$.

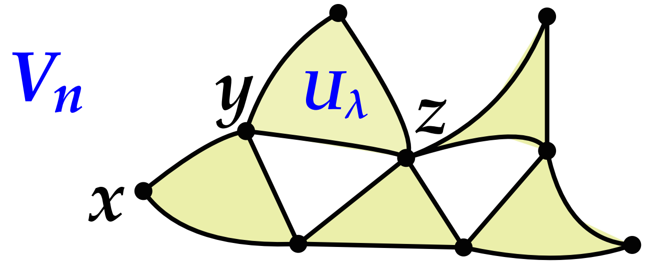
  $f^{(n)}(y) := \rho_{\boldsymbol{h}}^{(n)}(x, y), \quad y \in V_n, n \ge m;$

  the discrete version of the geodesic distance



$$|f^{(n)}(y) - f^{(n)}(z)| \le |\boldsymbol{h}(y) - \boldsymbol{h}(z)|_{\mathbb{R}^N}, \ y, z \in V_n.$$

(cont'd)



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- ▶ $g^{(n)}$: the harmonic extension of $f^{(n)}$

  $\mu_{\langle g^{(n)}\rangle}(U_\lambda) \le \mu_{\langle \boldsymbol{h}\rangle}(U_\lambda)$ for any U_λ
- ▶ $g^{(n(k))} \longrightarrow \rho$ in $\mathcal{F}$ and $\mu_{\langle\rho\rangle} \le \mu_{\langle \boldsymbol{h}\rangle}$
- ▶ For general $x \in K$, an argument of approximation is available.

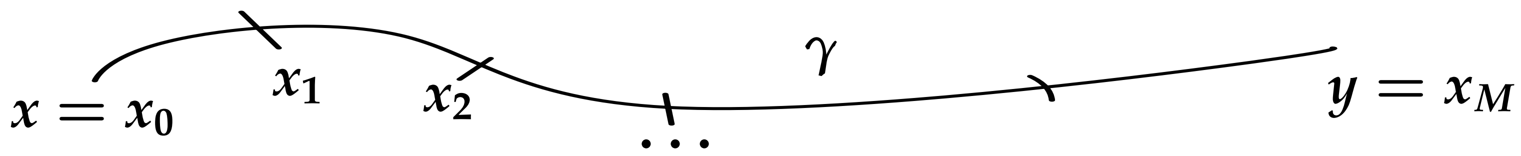
6. Idea of the proof of Theorem 2 ($\rho_h \geq \mathrm{d}_h$)

- Let $f \in \mathcal{F}$ with $\mu_{\langle f \rangle} \leq \mu_{\langle h \rangle}$, and take a continuous curve γ connecting x and y.
- It is sufficient to prove that $f(y) - f(x) \leq l_h(\gamma)$.
- $^{\exists}M > 0$, $^{\forall}\epsilon > 0$, we can take finitely many points $x_1, x_2, \dots, x_M \in \bigcup_{m=0}^{\infty} V_m$ on the curve γ such that

$$f(x_{i+1}) - f(x_i) \leq (1+\epsilon)|h(x_{i+1}) - h(x_i)|_{\mathbb{R}^N}$$

for most of i (when x_i is a good point),

$$f(x_{i+1}) - f(x_i) \leq M|h(x_{i+1}) - h(x_i)|_{\mathbb{R}^N} \text{ for all } i.$$



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By summing up, $f(y) - f(x) \leq (1 + o(1))l_h(\gamma)$.

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- A analog of "$|\nabla f| \leq 1$ a.e. on domain $D \subset \mathbb{R}^d$ implies that f has local Lipschitz constant 1"
- The obstacle is that the "Riemaniann metric" on K is degenerate on many points; on "nondegenerate" points for $\boldsymbol{h}$ we have the above inequalty.
- The assumption that #the vertex set $= 3$ assures that, on each small cell, either of the vertices the curve passes is nondegenerate w.r.t. $\boldsymbol{h}$.

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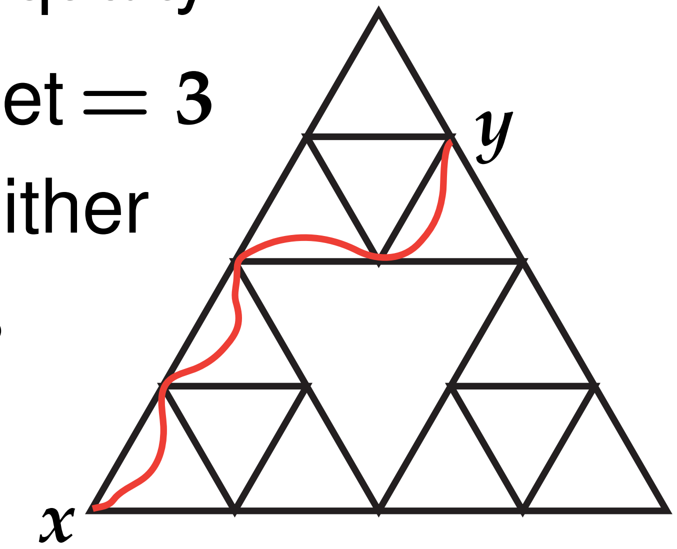

7. On the nondegeneracy condition

The nondegeneracy condition is summarized as follows:

u: function on V_0

$^{\exists 1}v$: the harmonic extesion of u to V_1, that is,

$v = u$ on V_0, and for all $x \in V_1 \setminus V_0$,

$\sum_{y \sim x}(v(y) - v(x)) = 0$.

Condition: If u is not constant, then

v is not constant on every .

It is conjectured that every level l gasket

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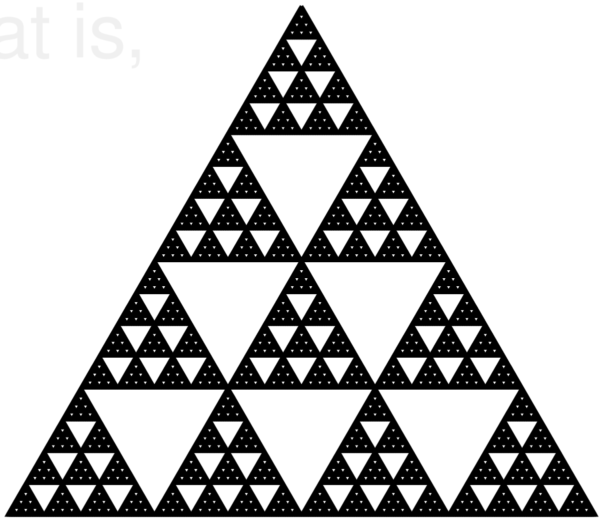
(This is a problem of linear algebras.)



level 4 gasket

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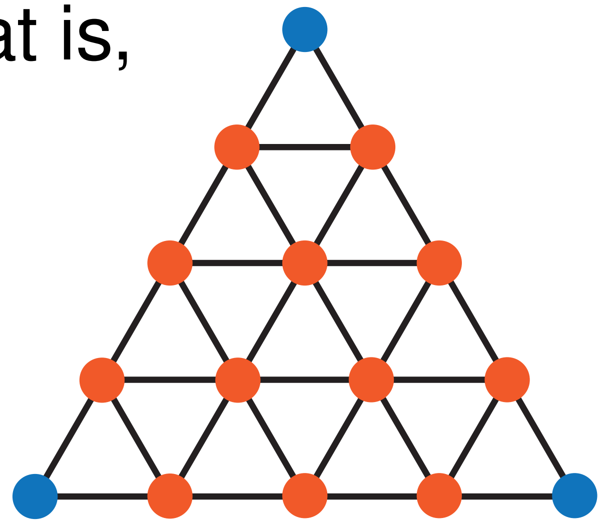

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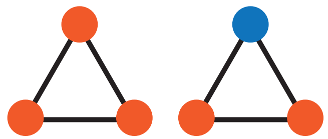
v is not constant on every .

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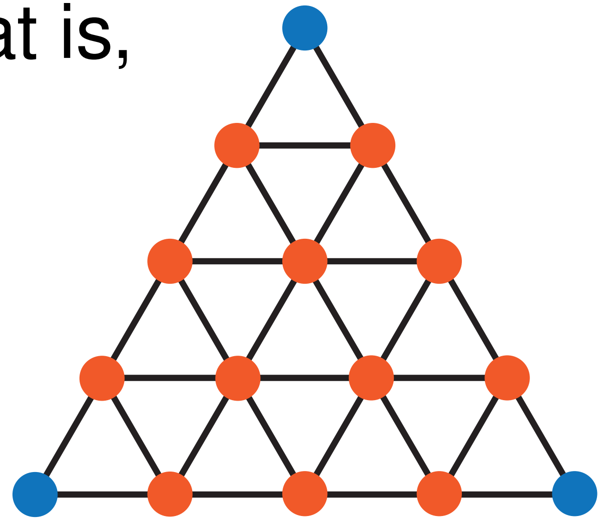

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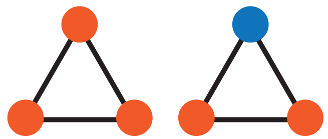
v is not constant on every  .

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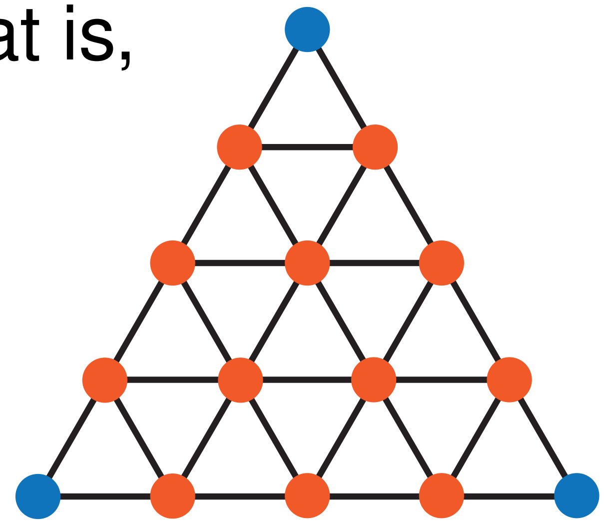

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