Geodesic distances and intrinsic distances on some fractal sets

Masanori Hino (Kyoto Univ.)

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1. Introduction

Intrinsic distance in the framework of Dirichlet forms
(cf. Biloli–Mosco, Sturm etc.)

\((K, \lambda)\): a locally compact, separable metric measure space

\((\mathcal{E}, \mathcal{F})\): a strong local regular Dirichlet form on \(L^2(K; \lambda)\)

\(\mu_{\langle f \rangle}\): the energy measure of \(f \in \mathcal{F}\)

When \(f\) is bounded,

\[\int_K \varphi \, d\mu_{\langle f \rangle} = 2\mathcal{E}(f, f\varphi) - \mathcal{E}(f^2, \varphi) \quad \forall \varphi \in \mathcal{F} \cap \mathcal{C}_b(K).\]

If \(\mathcal{E}(f, g) = \frac{1}{2} \int_{\mathbb{R}^d} (a_{ij}(x) \nabla f(x), \nabla g(x))_{\mathbb{R}^d} \, dx,\)

then \(\mu_{\langle f \rangle}(dx) = (a_{ij}(x) \nabla f(x), \nabla f(x))_{\mathbb{R}^d} \, dx.\)
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d(x, y) := \sup \left\{ f(y) - f(x) \middle| f \in \mathcal{F}_{\text{loc}} \cap C(K), \text{ and } \mu(f) \leq \lambda \right\}.
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In this framework, various Gaussian estimates of the transition density have been obtained.
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In this framework, various Gaussian estimates of the transition density have been obtained.
Question:

Is $d$ identified with the **geodesic distance** (=shortest path metric)?

In particular, what if $K$ is a fractal set, which does not have a (usual) differential structure?

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2. Canonical Dirichlet forms on typical self-similar fractals

Case of the 2-dim. standard Sierpinski gasket

\[ K \supset V_2 \]

\[ V_n: \text{nth level graph approximation} \]

\[ \mathcal{E}^{(n)}(f,f) = \left( \frac{5}{3} \right)^n \sum_{x,y \in V_n, x \sim y} (f(x) - f(y))^2 \]
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\[ \text{scaling factor} \]
$E^{(n)}(f,f) / \exists E(f,f) \leq +\infty \ \forall f \in C(K)$.

$\mathcal{F} := \{ f \in C(K) \mid E(f,f) < +\infty \}$

Then, $(\mathcal{E}, \mathcal{F})$ is a strong local regular Dirichlet form on $L^2(K;\lambda)$. ($\lambda$: the Hausdorff measure on $K$)

$\{X_t\}$: “Brownian motion” on $K$

(invariant under scaling and isometric transformations)

Similar construction is valid for more general finitely ramified self-similar fractals.
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In many examples, $\mu(f) \perp \lambda$ (self-similar measure). Then, $d(x, y) = \sup\{f(y) - f(x) \mid f = \text{const.}\} = 0$.

(This is closely connected with the fact that the heat kernel density has a sub-Gaussian estimate.)

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By taking different measures as \( \lambda \), however, we have nontrivial quantities...
$K$: 2-dim. Sierpinski gasket

$(\mathcal{E}, \mathcal{F})$: the standard Dirichlet form on $L^2(K, \nu)$ with

$\nu := \mu_{\langle h_1 \rangle} + \mu_{\langle h_2 \rangle}$ (Kusuoka measure)

($h_i$: a harmonic function, $\mathcal{E}(h_i, h_j) = \delta_{i,j}$)

Theorem (Kigami '93, '08, Kajino '12)

- (Ki) $h: K \to h(K) \subset \mathbb{R}^2$ is homeomorphic;
- (Ka) The intrinsic distance $d$ coincides with the geodesic distance $\rho_h$ on $h(K)$ by the identifying $K$ and $h(K)$;
- (Ki, Ka) The transition density $p^\nu_t(x, y)$ has a Gaussian estimate w.r.t. $\rho_h (= d)$;
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- (Ki) The red line is the geodesic.
Observation: an alternative expression of $\rho_h$

For a continuous curve $\gamma: [0, 1] \to K$, the length $l_h(\gamma)$ of $\gamma$ based on $h$ is defined as

$$l_h(\gamma) := \sup \left\{ \sum_{i=1}^{n} |h(\gamma(t_i)) - h(\gamma(t_{i-1}))|_{\mathbb{R}^2}; 0 = t_0 < t_1 < \cdots < t_n = 1 \right\}.$$ 

Then, by identifying $K$ and $h(K)$,

$$\rho_h(x, y) = \inf \left\{ l_h(\gamma) \mid \gamma \text{ is a continuous curve connecting } x \text{ and } y \right\}.$$ 

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Brief formulation of the problem

\((\mathcal{E}, \mathcal{F}), h = (h_1, \ldots, h_N) \in \mathcal{F}^N \cap C(K \to \mathbb{R}^N)\): given \(d_h\): the intrinsic distance based on \((\mathcal{E}, \mathcal{F})\) and

\[ \nu := \mu\langle h_1 \rangle + \cdots + \mu\langle h_N \rangle \]

\(\rho_h\): the geodesic distance based on \(h\)

The relation between \(d_h\) and \(\rho_h\), in particular when the underlying space has a fractal structure?
3. General framework

\((K, d_K)\): a separable and compact metric space

\(\lambda\): a finite Borel measure on \(K\)

\((\mathcal{E}, \mathcal{F})\): a strong local regular Dirichlet form on \(L^2(K, \lambda)\)

\(N \in \mathbb{N}, \ h = (h_1, \ldots, h_N) \in \mathcal{F}^N \cap C(K \to \mathbb{R}^N)\)

\(\nu := \mu(h) := \sum_{j=1}^{N} \mu(h_j)\)

The intrinsic distance \(d_h(x, y)\) based on \((\mathcal{E}, \mathcal{F})\) and \(h\) is defined as

\[d_h(x, y) := \sup \left\{ f(y) - f(x) \ \bigg| \ f \in \mathcal{F} \cap C(K) \text{ and } \mu(f) \leq \mu(h) \right\}.\]
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For a continuous curve \( \gamma \in C([0, 1] \rightarrow K) \), its length based on \( h \) is defined as

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    l_h(\gamma) := \sup \left\{ \sum_{i=1}^{n} |h(\gamma(t_i)) - h(\gamma(t_{i-1}))| \right\}_{\mathbb{R}^N},
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where \( n \in \mathbb{N}, 0 = t_0 < t_1 < \cdots < t_n = 1 \).

The geodesic distance \( \rho_h(x, y) \) based on \( h \) is defined as

\[
    \rho_h(x, y) = \inf \left\{ l_h(\gamma) \mid \gamma \text{ is a continuous curve connecting } x \text{ and } y \right\}.
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**Problem:** The relation between \( d_h \) and \( \rho_h \)?
4. Results

Theorem 1 \( \rho_h(x, y) \leq d_h(x, y) \) if the following hold:

(A1) (Finitely ramified cell structure) There exists an increasing sequence of finite subsets \( \{V_m\}_{m=0}^{\infty} \) of \( K \) such that

(i) \( \bigcup_{m=0}^{\infty} V_m \) is dense in \( K \);

(ii) For each \( m \), \( K \setminus V_m \) is decomposed as a finite number of connected components \( \{U_\lambda\}_{\lambda \in \Lambda_m} \);

(iii) \( \lim_{m \to \infty} \max_{\lambda \in \Lambda_m} \text{diam } U_\lambda = 0 \).

(A2) \( \mathcal{F} \subset C(K) \).

(A3) \( \mathcal{E}(f, f) = 0 \) if and only if \( f \) is a constant function.
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\[ V_2 \] 

\[ \begin{array}{cccc}
V_0 & V_1 & V_2 & V_3 \\
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Theorem 2 \( \rho_h(x, y) \geq d_h(x, y) \) if

- \( K \): a 2-dimensional (generalized) Sierpinski gasket that is also a nested fractal;
- \( (\lambda: \text{the normalized Hausdorff measure;}) \)
- \( (\mathcal{E}, \mathcal{F}) \): the self-similar Dirichlet form associated with the Brownian motion on \( K \);
- \( h = (h_1, \ldots, h_d) \); each \( h_i \) is a harmonic function;
- The harmonic structure associated with \( (\mathcal{E}, \mathcal{F}) \) is nondegenerate. (That is, for any nonconstant harmonic function \( g \), \( g \) is not constant on any nonempty open sets.)
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The nondegeneracy assumption holds for 2-dim. level $l$ S. G. with $l \leq 50$ (by the numerical computation).

Remark Theorem 2 is valid under more general situations. Essential assumptions (for the current proof) are:

- #the vertex set $= 3$;
- The harmonic structure is near to symmetric.
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5. Idea of the proof of Theorem 1 \((\rho_h \leq d_h)\)

In the case of Riemannian manifolds \(M\), the proof is given as follows:

For \(x \in M\), define \(\rho(y)\) as the geodesic distance between \(x\) and \(y \in M\). Then, \(\rho\) is Lipschitz and \(|\nabla \rho| \leq 1\).

Therefore, \(\rho(x, y) = \rho(y) - \rho(x) \leq d(x, y)\).

In Theorem 1, the main part of the proof is to prove \(\rho \in \mathcal{F}\) and \(\mu(\rho) \leq \mu(h)\). This is done by the discrete approximation.
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In Theorem 1, the main part of the proof is to prove \(\rho \in \mathcal{F}\) and \(\mu_{\langle \rho \rangle} \leq \mu_{\langle h \rangle}\). This is done by the discrete approximation.
More precisely,

- \( \rho(y) := \rho_h(x, y) \); the distance function from \( x \in K \)
  It is sufficient to prove that \( \rho \in \mathcal{F} \) and \( \mu(\rho) \leq \mu(\nu_h) \).

- Discrete approximation. Assume \( x \in V_m \).
  \[
f^{(n)}(y) := \rho_h^{(n)}(x, y), \quad y \in V_n, n \geq m;
  \]
  the discrete version of the geodesic distance
  \[
  |f^{(n)}(y) - f^{(n)}(z)| \leq |h(y) - h(z)|_{\mathbb{R}^N}, \quad y, z \in V_n.
  \]
5. Idea of the proof of Theorem 1 \((\rho_h \leq d_h)\) (cont’d)

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- \(\rho(y) := \rho_h(x, y)\)
- \(f^{(n)}(y) := \rho^{(n)}_h(x, y), \ y \in V_n, n \geq m\)

\[
|f^{(n)}(y) - f^{(n)}(z)| \leq |h(y) - h(z)|_{\mathbb{R}^N}, \ y, z \in V_n
\]

- \(g^{(n)}\): the harmonic extension of \(f^{(n)}\)

\[
\mu\langle g^{(n)} \rangle(U_\lambda) \leq \mu\langle h \rangle(U_\lambda) \text{ for any } U_\lambda
\]

- \(g^{(n(k))} \to \rho \text{ in } \mathcal{F} \) and \(\mu\langle \rho \rangle \leq \mu\langle h \rangle\)

- For general \(x \in K\), an argument of approximation is available.
6. Idea of the proof of Theorem 2 \((\rho_h \geq d_h)\)

- Let \(f \in \mathcal{F}\) with \(\mu_{\langle f \rangle} \leq \mu_{\langle h \rangle}\), and take a continuous curve \(\gamma\) connecting \(x\) and \(y\).

- It is sufficient to prove that \(f(y) - f(x) \leq l_h(\gamma)\).

- \(\exists M > 0, \forall \epsilon > 0\), we can take finitely many points \(x_1, x_2, \ldots, x_M \in \bigcup_{m=0}^{\infty} V_m\) on the curve \(\gamma\) such that
  
  \[f(x_{i+1}) - f(x_i) \leq (1 + \epsilon) |h(x_{i+1}) - h(x_i)|_{\mathbb{R}^N}\]

  for most of \(i\) (when \(x_i\) is a good point),

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- Let $f \in \mathcal{F}$ with $\mu\langle f \rangle \leq \mu\langle h \rangle$, and take a continuous curve $\gamma$ connecting $x$ and $y$.

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- $\exists M > 0, \forall \epsilon > 0$, we can take finitely many points $x_1, x_2, \ldots, x_M \in \bigcup_{m=0}^{\infty} V_m$ on the curve $\gamma$ such that

  $$f(x_{i+1}) - f(x_i) \leq (1 + \epsilon)|h(x_{i+1}) - h(x_i)|_{\mathbb{R}^N}$$

  for most of $i$ (when $x_i$ is a good point),

  $$f(x_{i+1}) - f(x_i) \leq M|h(x_{i+1}) - h(x_i)|_{\mathbb{R}^N}$$

  for all $i$.

By summing up, $f(y) - f(x) \leq (1 + o(1))l_h(\gamma)$. 
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\[ f(x_{i+1}) - f(x_i) \leq (1 + \epsilon) |h(x_{i+1}) - h(x_i)|_{\mathbb{R}^N} \]
for most of \( i \):

- A analog of "\( |\nabla f| \leq 1 \) a.e. on domain \( D \subset \mathbb{R}^d \) implies that \( f \) has local Lipschitz constant 1"
- The obstacle is that the "Riemaniann metric" on \( K \) is degenerate on many points; on "nondegenerate" points for \( h \) we have the above inequality.
- The assumption that \#the vertex set = 3 assures that, on each small cell, either of the vertices the curve passes is nondegenerate w.r.t. \( h \).
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- The assumption that #the vertex set = 3 assures that, on each small cell, either of the vertices the curve passes is nondegenerate w.r.t. \( h \).
On the proof of
\[ f(x_{i+1}) - f(x_i) \leq (1 + \epsilon) |h(x_{i+1}) - h(x_i)|_{\mathbb{R}^N} \]
for most of \( i \):

- A analog of “|\nabla f| \leq 1 \text{ a.e. on domain } D \subset \mathbb{R}^d \implies \text{that } f \text{ has local Lipschitz constant } 1”

- The obstacle is that the “Riemaniann metric” on \( K \) is degenerate on many points; on “nondegenerate” points for \( h \) we have the above inequality.

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7. On the nondegeneracy condition

The nondegeneracy condition is summarized as follows:

\( u \): function on \( V_0 \)

\( \exists \upsilon : \) the harmonic extension of \( u \) to \( V_1 \), that is, \( \upsilon = u \) on \( V_0 \), and for all \( x \in V_1 \setminus V_0 \),

\[ \sum_{y \sim x} (\upsilon(y) - \upsilon(x)) = 0. \]

Condition: If \( u \) is not constant, then \( \upsilon \) is not constant on every \( V_l \).

It is conjectured that every level \( l \) gasket is nondegenerate.

(This is a problem of linear algebras.)

level 4 gasket
7. On the nondegeneracy condition

The nondegeneracy condition is summarized as follows:

\( u \): function on \( V_0 \)

\( \exists v \): the harmonic extension of \( u \) to \( V_1 \), that is, \( v = u \) on \( V_0 \), and for all \( x \in V_1 \setminus V_0 \),

\[ \sum_{y \sim x} (v(y) - v(x)) = 0. \]

Condition: If \( u \) is not constant, then \( v \) is not constant on every 

It is conjectured that every level \( l \) gasket is nondegenerate.

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7. On the nondegeneracy condition

The nondegeneracy condition is summarized as follows:

\( u \): function on \( V_0 \)

\( \exists ! \nu \): the harmonic extension of \( u \) to \( V_1 \), that is, \( \nu = u \) on \( V_0 \), and for all \( x \in V_1 \setminus V_0 \),

\[ \sum_{y \sim x} (\nu(y) - \nu(x)) = 0. \]

**Condition**: If \( u \) is not constant, then \( \nu \) is not constant on every \( \bigtriangleup \bigtriangleup \).

It is conjectured that every level \( l \) gasket is nondegenerate.

(This is a problem of linear algebras.)
7. On the nondegeneracy condition

The nondegeneracy condition is summarized as follows:

- **$u$: function on $V_0$**

- **$\exists! \, v$: the harmonic extension of $u$ to $V_1$, that is,**
  
  \[ v = u \text{ on } V_0, \text{ and for all } x \in V_1 \setminus V_0, \]
  \[ \sum_{y \sim x} (v(y) - v(x)) = 0. \]

**Condition:** If $u$ is not constant, then $v$ is not constant on every

It is conjectured that every level $l$ gasket is nondegenerate.

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\[ V_0 = \{ \bullet \} \]
\[ V_1 = \{ \bullet , \bullet \} \]