Universität Bielefeld

SFB 701

# Local regularity for nonlocal operators - a robust approach

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## Parabolic Problems - Related Articles

- *A new formulation of Harnack's inequality for nonlocal operators*, C. R. Math. Acad. Sci. Paris, 2011
- (with A. Mimica)
  *Analysis of jump processes with nondegenerate jumping kernels*, arXiv
- (with M. Felsinger)
  *Local regularity for parabolic nonlocal operators*, arXiv
- (with B. Dyda)
  *Local regularity and comparability for symmetric nonlocal Dirichlet forms*, arXiv (new version soon)

# Discussion of the title

# Discussion of the title I

## Nonlocal Operators:

- $(-\Delta)^{\alpha/2}$ or more general generators of Lévy processes
- $(Lu)(x) = p.v. \int\limits_{\mathbb{R}^d} \big(u(y) - u(x)\big) k(x, y) \mathrm{d}x$

## Local Regularity:

- Given $D \subset \mathbb{R}^d$ we establish properties in $D' \Subset D$ which hold uniformly for all functions $u$ which satisfy $Lu = f$ or $Lu \leq f$ in $D$.
- We interpret $Lu = f$ or $Lu \leq f$ in $D$ probabilistically if helpful.

Area strongly influenced by Hilbert's 19th problem and the solution by DeGiorgi/Nash. **Problem:** Given $F \in C^\infty(\mathbb{R}^d)$ convex with bounded 2nd derivatives, can one prove that every minimizer of $I(v) = \int\limits_D F(\nabla v(x)) \mathrm{d}(x)$ is smooth ?

## Discussion of the title II

Robust Approach:

Given $\alpha_0 \in (0, 2)$ assume

$$\mathcal{L} \subset \left\{ L \;\middle|\; \begin{array}{l} L \text{ is an (integro-)differential operator of} \\ \text{differentiability order } \alpha \text{ with } \alpha_0 \leq \alpha < 2 \end{array} \right\} .$$

A (regularity) result for $\mathcal{L}$ is called robust if the corresponding estimates for $L \in \mathcal{L}$ are uniformly continuous for $\alpha \in (\alpha_0, 2)$ where $\alpha$ corresponds to $L$.

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Example: Assume $\mathcal{L} = \{(-\Delta)^{\alpha/2} | \alpha_0 \le \alpha < 2\}$. Then

$$p(t,x,y) \asymp t^{-\alpha/d}\left(\frac{t}{|x-y|^{\alpha+t}}\right) \quad \textbf{cannot} \text{ be robust} .$$

But, for $D \subset \mathbb{R}^3$ a bounded $C^{1,1}$-domain,

$$G_D(x,y) \asymp |x-y|^{\alpha-d}\left(1 \wedge \frac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha}}\right) \quad \textbf{can} \text{ be robust} .$$

# Discussion of the title III

## Why are robust results interesting?

- Detection of joint properties of continuous/discontinuous processes or local/nonlocal operators.
- Important for the investigation of operators like $(-\Delta)^{\alpha(x)}, \alpha_0 \leq \alpha \leq 2$.
- ... not necessarily as a tool for the study of the limit case $\alpha = 2$.

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Example:(Poincaré Inequality) Given $\alpha \in (\alpha_0, 2)$, $p \geq 1$, $\phi$ some admissible weight, there is $C = C(\alpha_0, p, d, \phi) > 0$ such that for $u \in L^p(B_1)$

$$\int_{B_1} |u(x) - u_{B_1}^{\phi}|^p \phi(x)\, dx \leq C(2-\alpha) \iint_{B_1 B_1} \frac{|u(x) - u(y)|^p}{|x-y|^{d+p\alpha/2}} (\phi(y) \wedge \phi(x))\, dy\, dx$$

# Elliptic Problems - Symmetric Nonlocal Dirichlet Forms

# Elliptic Problems I

For $k : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ and $\alpha \in (0, 2)$ we consider the following quadratic forms on $L^2(D) \times L^2(D)$:

$$\mathcal{E}_D^k(u, u) = \int_D \int_D (u(y) - u(x))^2 k(x, y) \, dx \, dy, \tag{1}$$

$$\mathcal{E}_D^\alpha(u, u) = \alpha(2 - \alpha) \int_D \int_D (u(y) - u(x))^2 |x - y|^{-d-\alpha} \, dx \, dy, \tag{2}$$

where $D \subset \mathbb{R}^d$ is some open set.
Given a kernel $k : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ and functions $u, v : \mathbb{R}^d \to \mathbb{R}$ we define the quantity

$$\mathcal{E}^k(u, v) = \iint_{\mathbb{R}^d \mathbb{R}^d} \big(u(y) - u(x)\big)\big(v(y) - v(x)\big) k(x, y) \, dy \, dx \,,$$

if it is finite.

# Elliptic Problems II

Assume $\alpha \in (0,2)$, $A, B \geq 1$ and $k : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ be measurable.

For all balls $B \subset \mathbb{R}^d$ with radius smaller or equal one and all functions $u \in C_c^\infty(B)$ : $\quad A^{-1}\, \mathcal{E}_B^k(u,u) \leq \mathcal{E}_B^\alpha(u,u) \leq A\, \mathcal{E}_B^k(u,u)\,.$ (A)

For every $R, \rho \in (0,1)$ there is a nonnegative function $\tau \in C^1(\mathbb{R}^d)$ with $\operatorname{supp}(\tau) \subset \overline{B_{R+\rho}}, \tau(x) \equiv 1$ on $B_R$ and

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \big(\tau(y) - \tau(x)\big)^2 k(x,y)\, dy \leq B\rho^{-\alpha}\,. \quad \text{(B)}$$

# Elliptic Problems III

Fix $\alpha_0 \in (0,2)$, $A \geq 1$, $B \geq 1$. Let $\mathcal{K}(\alpha_0, A, B)$ denote the set of all measurable kernels $k : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ with the property that for each kernel $k$ there is $\alpha \in (\alpha_0, 2)$ such that (A) and (B) hold.

**Theorem (Weak Harnack Inequality, MK07, Dyda-MK11)**

*Let $\alpha_0 \in (0,2)$ and $A \geq 1, B \geq 1$. There are positive reals $p_0, c$ such that for every $k \in \mathcal{K}(\alpha_0, A, B)$ and every $u \in L^\infty(\mathbb{R}^d) \cap H^{\alpha/2}_{loc}(B(1))$ with $u \geq 0$ in $B(1)$ satisfying $\mathcal{E}^k(u,\phi) \geq 0$ for every nonnegative $\phi \in C_c^\infty(B(1))$ the following inequality holds:*

$$c \inf_{B(1/4)} u \geq \Big( \fint_{B(1/2)} u(x)^{p_0}\, dx \Big)^{1/p_0} - c \sup_{x \in B(1/2)} \int_{\mathbb{R}^d \setminus B(1)} u^-(z) k(x,z)\, dz\,.$$

*The constants $p_0, c$ depend only on $d, \alpha_0, A, B$.*

# Parabolic Problems

## Parabolic Problems - Setup

Fix an open domain $\Omega \subset \mathbb{R}^d$, $d \geq 3$, $I \subset \mathbb{R}$, $\alpha_0 \in (0,2)$ and $\gamma, M, \lambda, \Lambda > 0$.
For $t \in I$, $\alpha \in (\alpha_0, 2)$ and $x, y \in \mathbb{R}^d$ assume $k_t(x,y) = k_t(y,x)$ and

$$\frac{\lambda\,(2-\alpha)}{|x-y|^{d+\alpha}} \leq k_t(x,y) \leq \frac{\Lambda\,(2-\alpha)}{|x-y|^{d+\alpha}} \qquad \text{if } |x-y| \leq 1, \qquad \text{(K1)}$$

$$0 \leq k_t(x,y) \leq \frac{M(2-\alpha)}{|x-y|^{d+\gamma}} \qquad \text{if } |x-y| > 1\,. \qquad \text{(K2)}$$

We study (super-)solutions $u \colon I \times \mathbb{R}^d \to \mathbb{R}$ of

$$\partial_t u - Lu = f \qquad \text{in } I \times \Omega,$$

where $$(Lu)(t,x) = p.v. \int_{\mathbb{R}^d} \left(u(t,y) - u(t,x)\right) k_t(x,y)\,\mathrm{d}y.$$

## Parabolic Problems - Special case: $L = -(-\Delta)^{\alpha/2}$

Let $\mathcal{A}_{d,-\alpha} = \frac{2^\alpha \Gamma\left(\frac{d+\alpha}{2}\right)}{\pi^{d/2}\left|\Gamma\left(\frac{-\alpha}{2}\right)\right|}$. Note that

$$\mathcal{A}_{d,-\alpha} \sim \alpha(2-\alpha) \qquad \text{for } \alpha \in (0,2).$$

If $k_t(x,y) = \mathcal{A}_{d,-\alpha} \left|x-y\right|^{-d-\alpha}$, then for $u \in C_c^\infty(\mathbb{R}^d)$

$$(Lu)(x) = \mathcal{A}_{d,-\alpha} \lim_{\varepsilon \to 0} \int\limits_{|y-x|>\varepsilon} \frac{u(y)-u(x)}{\left|y-x\right|^{d+\alpha}} \, \mathrm{d}y$$

$$\widehat{Lu}(\xi) = \left|\xi\right|^\alpha \widehat{u}(\xi).$$

Think: $k_t(x,y) = (2-\alpha) g(t,x,y) \left|x-y\right|^{-d-\alpha}$ with $\lambda \leq g(t,x,y) \leq \Lambda$.

## Parabolic Problems - Main Theorems I

### Theorem (Weak Harnack inequality, MF/MK 2012)

*There is a constant* $C = C(d, \alpha_0, \lambda, \Lambda, \gamma, M)$ *such that for every supersolution* $u$ *on* $Q = (-1, 1) \times B_2(0)$ *which is nonnegative in* $(-1, 1) \times \mathbb{R}^d$ *the following inequality holds:*

$$\|u\|_{L^1(U_\ominus)} \leq C \left( \inf_{U_\oplus} u + \|f\|_{L^\infty(Q)} \right), \tag{HI}$$

*where* $U_\oplus = \left(1 - (\frac{1}{2})^\alpha, 1\right) \times B_{1/2}(0)$, $U_\ominus = \left(-1, -1 + (\frac{1}{2})^\alpha\right) \times B_{1/2}(0)$.

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$\alpha$ in the domains $U_\oplus$ and $U_\ominus$ can be replaced 2, i.e. the assertion is still true if

$$U_\oplus = \left(\tfrac{3}{4}, 1\right) \times B_{1/2}(0), \quad U_\ominus = \left(-1, -\tfrac{3}{4}\right) \times B_{1/2}(0).$$

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  - for every $k_t(x, y)$ satisfying (K1), (K2)
    - for every supersolution.

**Example:** For $\alpha_n = 2 - \frac{1}{n}$, $k_t^n(x, y)$ corresponding to $\alpha_n$ and $u_n$ a (super-)solution to

$$\partial_t u_n - L^n u_n = f,$$

the estimate holds uniformly in $n \in \mathbb{N}$.

## Parabolic Problems - Main Theorems II

### Theorem (Hölder regularity)

*There is a constant $\beta = \beta(d, \lambda, \Lambda, \gamma, M, \alpha_0)$ such that for every solution u in $Q = I \times \Omega$ with $f = 0$ and every $Q' \Subset Q$ the following estimate holds:*

$$\sup_{(t,x),(s,y) \in Q'} \frac{|u(t,x) - u(s,y)|}{\left(|x-y| + |t-s|^{1/\alpha}\right)^{\beta}} \leq \frac{\|u\|_{L^\infty(I \times \mathbb{R}^d)}}{D^\beta},$$

*with some constant $D = D(Q, Q') > 0$.*

## Lemma (Bombieri-Giusti)

*Let* $(U(r))_{\theta \le r \le 1}$ *be increasing with* $U(r) \subset \mathbb{R}^{d+1}$*. Fix* $m, c_0 > 0$*,* $\theta \in [1/2, 1]$*,* $\eta \in (0,1)$ *and* $0 < p_0 \le \infty$*. Assume that* $w \colon U(1) \to [0, \infty)$ *is measurable and*

$$\Big( \int\limits_{U(r)} w^{p_0} \Big)^{1/p_0} \le \Big( \frac{c_0}{(R-r)^m \, |U(1)|} \Big)^{1/p - 1/p_0} \Big( \int\limits_{U(R)} w^p \Big)^{1/p} < \infty \tag{BG1}$$

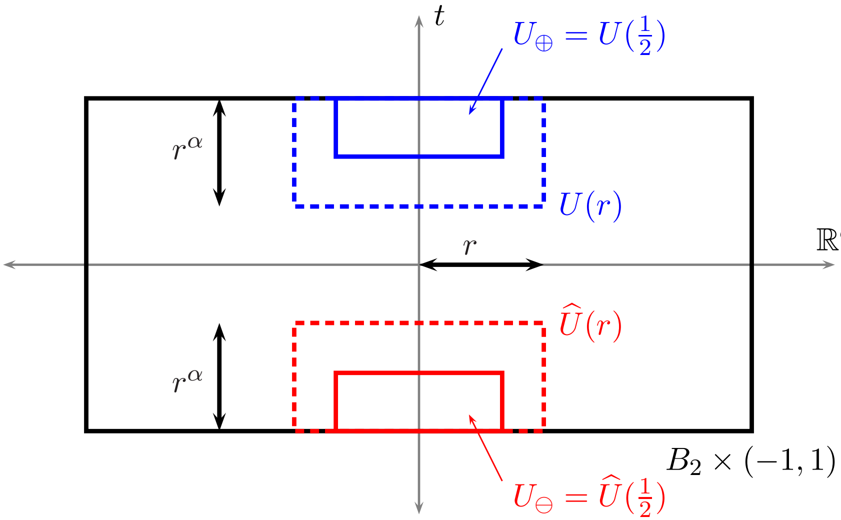
*for all* $r, R \in [\theta, 1], r < R$ *and for all* $p \in (0, 1 \wedge \eta p_0)$*. Additionally suppose*

$$\forall s > 0 \colon \quad |U(1) \cap \{\log w > s\}| \le \frac{c_0}{s} \, |U(1)| \, . \tag{BG2}$$

*Then there is a constant* $C = C(\theta, \eta, m, c_0, p_0)$ *such that*

$$\Big( \int\limits_{U(\theta)} w^{p_0} \Big)^{1/p_0} \le C \, |U(1)|^{1/p_0} \; .$$

# Proof of the Harnack inequality – Key Lemma



# Proof of the Harnack inequality – Tool 1

Let $u$ be a supersolution and set $\widetilde{u} = u + \|f\|_{L^\infty(Q)}$.

$\widetilde{u}^{-1}$ satisfies (BG1) with $p_0 = \infty$ and $U(r) = (1 - r^\alpha, 1) \times B_r$.

$\widetilde{u}$ satisfies (BG1) with $\widehat{p}_0 = 1$ and $\widehat{U}(r) = (-1, -1 + r^\alpha) \times B_r$.

## Proposition 1

*Let $\frac{1}{2} \le r < R \le 1$. There are constants $C_1, C_2, \omega > 0$ depending on $d, \alpha_0, \lambda, \Lambda, M, \gamma$ such that for every supersolution $u$ in $Q_R = (-R^\alpha, R^\alpha) \times \Omega$, $\Omega \Supset B_R$ with $u \ge \varepsilon > 0$ in $Q_R$:*

$$\forall p > 0: \qquad \sup_{U(r)} \widetilde{u}^{-1} \le \left( \frac{C_1}{(R-r)^{d+\alpha}} \right)^{1/p} \left( \int_{U(R)} \widetilde{u}^{-p}(t,x)\, dx\, dt \right)^{1/p},$$

$$\forall p \in (0,1): \int_{\widehat{U}(r)} \widetilde{u}(t,x)\, dx\, dt \le \left( \frac{C_2}{(R-r)^{\omega}} \right)^{1/p-1} \left( \int_{\widehat{U}(R)} \widetilde{u}^{p}(t,x)\, dx\, dt \right)^{1/p}.$$

$w := e^{-a}\widetilde{u}^{-1}$ satisfies (BG2)
$\widehat{w} := e^{a}\widetilde{u}$ satisfies (BG2)

## Proposition 2

*There is $C = C(d, \alpha_0, \lambda, \Lambda, \gamma, M) > 0$ such that for every supersolution $u$ in $Q = (-1,1) \times B_2(0)$ which satisfies $u \geq \varepsilon > 0$ in $(-1,1) \times \mathbb{R}^d$, there is a constant $a = a(\widetilde{u}) \in \mathbb{R}$ such that:*

$$\forall s > 0\colon\ (dt \otimes dx)\left(Q_{\oplus}(1) \cap \{\log \widetilde{u} < -s - a\}\right) \leq \frac{C\,|B_1|}{s},$$

$$\forall s > 0\colon\ (dt \otimes dx)\left(Q_{\ominus}(1) \cap \{\log \widetilde{u} > s - a\}\right) \leq \frac{C\,|B_1|}{s}.$$

Note $\log \widetilde{u} < -s - a \Leftrightarrow \log w > s$ and $\log \widetilde{u} > s - a \Leftrightarrow \log w > s$.

Both $w = e^{-a}\widetilde{u}^{-1}$ and $\widehat{w} = e^{a}\widetilde{u}$ satisfy the assumptions in the Lemma of Bombieri-Giusti. Hence,

$$\sup_{U(1/2)} w = e^{-a} \sup_{U(1/2)} \widetilde{u}^{-1} \leq C, \quad \text{and}$$

$$\|\widehat{w}\|_{L^1(\widehat{U}(1/2))} = e^{a} \|\widetilde{u}\|_{L^1(\widehat{U}(1/2))} \leq \widehat{C}\,.$$

This yields

$$\|\widetilde{u}\|_{L^1(\widehat{U}(1/2))} \leq C\,\widehat{C} \left( \sup_{U(1/2)} \widetilde{u}^{-1} \right)^{-1},$$

and finally

$$\|u\|_{L^1(U_\ominus)} \leq \|\widetilde{u}\|_{L^1(U_\ominus)} \leq C \left( \sup_{U_\oplus} \widetilde{u}^{-1} \right)^{-1} \leq C \left( \inf_{U_\oplus} u + \|f\|_{L^\infty(Q)} \right).$$

# Parabolic Problems - Related Results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