On the differential structure of metric measure spaces and applications

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A standard construction...

Given a metric measure space $(X, d, \boldsymbol{m})$ and $p \in (1, \infty)$ there are various (equivalent) definitions of the Sobolev space $W^{1,p}(X, d, \boldsymbol{m})$ of real valued functions on X.

The common feature of these definitions is that for $f \in W^{1,p}(X, d, \boldsymbol{m})$ it is not defined the distributional gradient, but only 'its modulus'.

...the goal of this talk

To show that despite the lack of a smooth structure it is possible to speak about differentials and gradients of Sobolev functions.

More precisely, we will define the action of differentials on gradients.

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- Reminders on analysis on metric measure spaces
- Differentials and gradients
  - The case of normed spaces
  - The abstract case
- An application: Laplacian comparison estimate

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Variational definition of $|\nabla f|$ on $\mathbb{R}^d$

Let $f : \mathbb{R}^d \to \mathbb{R}$ be smooth.

Then $|\nabla f|$ is the minimum continuous function G for which

$$|f(\gamma_1) - f(\gamma_0)| \leq \int_0^1 G(\gamma_t)|\dot{\gamma}_t|\, dt$$

holds for any smooth curve γ

Test plans

Let (X, d) be complete and separable and $\boldsymbol{m}$ a Radon measure on it.

For $t \in [0,1]$ the evaluation map $\mathrm{e_t} : \mathrm{C}([0,1],\mathrm{X}) \to \mathrm{X}$ is defined by

$$\mathrm{e_t}(\gamma) := \gamma_\mathrm{t}$$

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Let $\boldsymbol{\pi} \in \mathscr{P}(C([0,1], X))$. We say that $\boldsymbol{\pi}$ is a test plan provided:

- for some $C > 0$ it holds

$$\mathrm{e_t}_\sharp \boldsymbol{\pi} \leq \mathrm{C}\boldsymbol{m}, \qquad \forall \mathrm{t} \in [0, 1].$$

- it holds

$$\iint_0^1 |\dot{\gamma}_t|^2 \, dt \, d\boldsymbol{\pi} < \infty$$

The Sobolev class $S^2(X, d, \boldsymbol{m})$

We say that $f : X \to \mathbb{R}$ belongs to $S^2(X, d, \boldsymbol{m})$ provided there exists $G \in L^2(X, \boldsymbol{m})$ such that

$$\int \big|f(\gamma_1) - f(\gamma_0)\big| \, d\boldsymbol{\pi}(\gamma) \leq \int\int_0^1 G(\gamma_t)|\dot{\gamma}_t| \, dt \, d\boldsymbol{\pi}(\gamma)$$

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Any such G is called 'weak upper gradient' of f.

It turns out that there exists a minimal G in the $\boldsymbol{m}$-a.e. sense. We will denote it by $|Df|_w$

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Differentials

Given $f : \mathbb{R}^d \to \mathbb{R}$ smooth, its differential $Df : \mathbb{R}^d \to T^*\mathbb{R}^d$ is intrinsically defined by

$$Df(x)(v) := \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}, \qquad \forall x \in \mathbb{R}^d,\ v \in T_x\mathbb{R}^d$$

Gradients

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Rmk. 1 Uniqueness follows by the strict convexity of the norm.

Rmk. 2 ∇f depends linearly on f only if the norm comes from a scalar product.

The non strictly convex case

If $\|\cdot\|$ is not strictly convex, uniqueness of gradients is not anymore granted.

Example $\mathbb{R}^2$ with the L^∞ norm and $f(x_1, x_2) := x_1$. All the vectors $v = (1, v_2)$ with $v_2 \in [-1, 1]$ can be called gradients of f.

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And then the two functions

$$D^+ f(\nabla g) := \max_{v \in \nabla g} Df(v), \qquad D^- f(\nabla g) := \min_{v \in \nabla g} Df(v).$$

A useful identity

$$D^+ f(\nabla g)(x) = \inf_{\varepsilon > 0} \frac{\|D(g + \varepsilon f)\|_*^2(x) - \|Dg\|_*^2(x)}{2\varepsilon}$$

$$D^- f(\nabla g)(x) = \sup_{\varepsilon < 0} \frac{\|D(g + \varepsilon f)\|_*^2(x) - \|Dg\|_*^2(x)}{2\varepsilon}.$$

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The object $D^{\pm} f(\nabla g)$

For $f, g \in S^2$, the functions $D^{\pm} f(\nabla g) : X \to \mathbb{R}$ are defined by

$$D^{+} f(\nabla g) := \inf_{\varepsilon > 0} \frac{|D(g + \varepsilon f)|_w^2 - |Dg|_w^2}{2\varepsilon}$$

$$D^{-} f(\nabla g) := \sup_{\varepsilon < 0} \frac{|D(g + \varepsilon f)|_w^2 - |Dg|_w^2}{2\varepsilon}$$

Calculus rules

Chain rule For $f, g \in S^2$, $\varphi : \mathbb{R} \to \mathbb{R}$ Lipschitz, $\boldsymbol{m}$-a.e. it holds

$$D^{\pm}(\varphi \circ f)(\nabla g) = \varphi' \circ f \, D^{\pm \mathrm{sign}(\varphi' \circ f)} f(\nabla g),$$
$$D^{\pm} f(\nabla(\varphi \circ g)) = \varphi' \circ g \, D^{\pm \mathrm{sign}(\varphi' \circ g)} f(\nabla g).$$

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Leibniz rule For $f_1, f_2 \in S^2 \cap L^\infty$, and $g \in S^2$ it holds

$$D^{+}(f_1 f_2)(\nabla g) \leq f_1 \, D^{\ \mathrm{sign}(f_1)} f_2(\nabla g) + f_2 \, D^{\ \mathrm{sign}(f_2)} f_1(\nabla g),$$
$$D^{-}(f_1 f_2)(\nabla g) \geq f_1 \, D^{-\mathrm{sign}(f_1)} f_2(\nabla g) + f_2 \, D^{-\mathrm{sign}(f_2)} f_1(\nabla g).$$

Special situations

$(X, d, \boldsymbol{m})$ is infinitesimally strictly convex provided

$$D^+ f(\nabla g) = D^- f(\nabla g) \qquad \boldsymbol{m} - a.e.$$

for any $f, g \in S^2$. In this case the common value will be denoted by $Df(\nabla g)$. For $g \in S^2$ the map

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$(X, d, \boldsymbol{m})$ is infinitesimally Hilbertian provided

$$f \mapsto \int |Df|_w^2 \, d\boldsymbol{m} \qquad \text{is a quadratic form on } S^2$$

In this case it holds

$$D^+ f(\nabla g) = D^- f(\nabla g) = D^+ g(\nabla f) = D^- g(\nabla f), \qquad \boldsymbol{m} - a.e.$$

and we denote these quantities by $\nabla f \cdot \nabla g$.

Gradient vector fields

Gradient vector fields

For $g \in S^2$ and $\boldsymbol{\pi} \in \mathscr{P}(C([0,1],X))$ test plan it holds

$$\varlimsup_{t\downarrow 0} \int \frac{g(\gamma_t) - g(\gamma)}{t}\, d\boldsymbol{\pi} \leq \frac{1}{2}\int |Dg|_w^2(\gamma_0)\, d\boldsymbol{\pi} + \varlimsup_{t\downarrow 0} \frac{1}{2t}\iint_0^t |\dot{\gamma}_s|^2\, ds\, d\boldsymbol{\pi}$$

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Theorem (G. '12, Ambrosio-G.-Savaré, '11). For $g \in S^2$ and $\mu \in \mathscr{P}(X)$ such that $\mu \leq C\boldsymbol{m}$, a plan $\boldsymbol{\pi}$ representing ∇g and such that ${\rm e}_{0\,\sharp}\boldsymbol{\pi} = \mu$ exists.

First order differentiation formula

Let $f, g \in S^2$, and $\boldsymbol{\pi}$ which represents ∇g.
Then

$$\int D^+ f(\nabla g)(\gamma_0)\, d\boldsymbol{\pi} \geq \overline{\lim_{t\downarrow 0}} \int \frac{f(\gamma_t) - f(\gamma_0)}{t}\, d\boldsymbol{\pi}$$
$$\geq \underline{\lim_{t\downarrow 0}} \int \frac{f(\gamma_t) - f(\gamma_0)}{t}\, d\boldsymbol{\pi} \geq \int D^- f(\nabla g)(\gamma_0)\, d\boldsymbol{\pi}$$

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Laplacian comparison

On a Riemannian manifold M with $Ric \geq 0$, dim $\leq N$ it holds

$$\Delta \frac{1}{2} d^2(\cdot, \overline{x}) \leq N$$

in the sense of distributions.

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Does the same hold on abstract spaces?

Definition of distributional Laplacian

Let $(X, d, \boldsymbol{m})$ be infinitesimally strictly convex. We say that $g \in D(\Delta)$ provided:

- $g \in S^2$
- there exists a locally finite Borel measure μ on X such that

$$-\int Df(\nabla g)\, d\boldsymbol{m} = \int f\, d\mu.$$

  for every f Lipschitz in $L^1(|\mu|)$ with $\boldsymbol{m}(\mathrm{supp}(f)) < \infty$.

In this case we put $\Delta g := \mu$

Calculus rules

Chain rule Let $g \in D(\Delta) \cap S^2 \cap C(X)$ and $\varphi \in C^{1,1}(\mathbb{R})$.

Then $\varphi \circ g \in D(\Delta)$ and it holds

$$\Delta(\varphi \circ g) = \varphi' \circ g \, \Delta g + \varphi'' \circ g \, |Dg|_w^2 \boldsymbol{m}$$

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On inf. Hilb. spaces, the Laplacian is linear and satisfies the Leibniz rule: for $g_1, g_2 \in D(\Delta) \cap S^2 \cap C(X)$ it holds $g_1 g_2 \in D(\Delta)$ and

$$\Delta(g_1 g_2) = g_1 \Delta g_2 + g_2 \Delta g_1 + 2\nabla g_1 \cdot \nabla g_2.$$

Laplacian comparison on nonsmooth setting

Theorem (G. '12) Let $(X, d, \boldsymbol{m})$ be an infinitesimally strictly convex $CD(0, N)$ space and $\overline{x} \in \mathrm{supp}(\boldsymbol{m})$.

Put $g := \frac{1}{2} d^2(\cdot, \overline{x})$. Then $g \in D(\Delta)$ and $\Delta g \leq N\boldsymbol{m}$.

Thank you