# On Green function of subordinate Brownian motion

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## References

This talk is based on the following papers.

**KK** Kang and K, On estimate of Poisson kernel for symmetric Lévy process. (preprint, 2012).

**KM1** K and Mimica, Harnack inequalities for subordinate Brownian motions, *Electronic Journal of Probability*, **17**, #37, (2012)

**KM2** K and Mimica, Green function estimates for subordinate Brownian motions : stable and beyond. (preprint, 2012).

**KSV** K, Song and Vondraček, Two-sided Green function estimates for the killed subordinate Brownian motions, *Proc. London Math. Soc.* **104** (2012), 927–958.

Theses results were announced in the extended Kansai Probability seminar on February 10, 2012.

# Outline

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# Symmetric $\alpha$-stable process

Let $X = (X_t, \mathbb{P}_x)$ be a symmetric (rotationally invariant) $\alpha$-stable process in $\mathbb{R}^d$, $\alpha \in (0,2)$.

$X$ is a pure jump Lévy process with characteristic exponent: $\Phi(\theta) = |\theta|^\alpha$, $\theta \in \mathbb{R}^d$,

$$\mathbb{E}[\exp\{i\theta(X_t - X_0)\}] = \exp\{-t\Phi(\theta)\}$$

The Lévy density of $X$ is

$$J(x) := c(d,\alpha)|x|^{-(d+\alpha)}.$$

The infinitesimal generator of a symmetric $\alpha$-stable process $X$ in $\mathbb{R}^d$ is the fractional Laplacian

$$\Delta^{\alpha/2} f(x) := -(-\Delta)^{\alpha/2} f(x) = \lim_{t \downarrow 0} \frac{1}{t}(\mathbb{E}_x[f(X_t)] - f(x)).$$

The fractional Laplacian can be written in the form

$$\Delta^{\alpha/2} u(x) = \lim_{\varepsilon \downarrow 0} \int_{\{y \in \mathbb{R}^d : |y-x| > \varepsilon\}} (u(y) - u(x)) J(x,y) dy.$$

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# Green function for Symmetric $\alpha$-stable process

For $D \subset \mathbb{R}^d$ open subset of $\mathbb{R}^d$, $\tau_D := \inf\{t > 0 : X_t \notin D\}$ and $X^D$ is the killed process.

Let $G_D(x, y)$ be the Green function of $X^D$: the density of the occupation measure

$$G_D(x, dy) = G_D(x, y)dy = \mathbb{E}_x \int_0^{\tau_D} \mathbf{1}_{(X_t \in dy)}\, dt = \mathbb{E}_x \int_0^{\infty} \mathbf{1}_{(X_t^D \in dy)}\, dt$$

$G_D(x, y) = \int_0^\infty p_D(t, x, y)dt$ where $p_D$ is the transition density of $X^D$.

Analytically speaking, if $\Delta^{\alpha/2}|_D$ is the restriction of $\Delta^{\alpha/2}$ to $D$ with zero exterior condition, then $G_D(\cdot, y)$ is the solution of $(\Delta^{\alpha/2}|_D)u = -\delta_y$.

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# $C^{1,1}$-open sets

$D \subset \mathbb{R}^d$ $(d \geq 2)$ open, is said to be a $C^{1,1}$ open set if there exist a localization radius $R$ and a constant $\Lambda > 0$ such that for every $z \in \partial D$, there is a $C^{1,1}$-function $\psi = \psi_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\psi(0) = 0$, $\nabla\psi(0) = (0, \dots, 0)$, $\|\nabla\psi\|_\infty \leq \Lambda$, $|\nabla\psi(x) - \nabla\psi(w)| \leq \Lambda|x - w|$, and an orthonormal coordinate system $CS_z$: $y = (y_1, \cdots, y_{d-1}, y_d) := (\widetilde{y}, y_d)$ with its origin at $z$ such that

$$B(z, R) \cap D = \{y = (\widetilde{y}, y_d) \in B(0, R) \text{ in } CS_z : y_d > \psi(\widetilde{y})\}.$$

The pair $(R, \Lambda)$ is called the characteristics of the $C^{1,1}$ open set $D$.

# Estimates of Green function for Symmetric $\alpha$-stable processc

$\delta_D(x) = \text{dist}(x, \partial D)$ and $f \asymp g$ means that there is a constant $c > 0$ such that $c^{-1} \leq f/g \leq c$.

If $d > \alpha$ and $D \subset \mathbb{R}^d$ is a bounded $C^{1,1}$-open set, then for all $x, y \in D$,

$$G_D(x,y) \asymp \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{|x-y|^\alpha}\right)\frac{1}{|x-y|^{d-\alpha}}$$

(Chen & Song (1998) and Kulczycki (1997))

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# Relativistic $\alpha$-stable process and its Green function estimates

Relativistic $\alpha$-stable process is a Lévy process with characteristic exponent

$$\Phi(\theta) = (|\theta|^2 + m^{2/\alpha})^{\alpha/2} - m\,, \theta \in \mathbb{R}^d\,, \quad m > 0\,,$$

and infinitesimal generator $m - (-\Delta + m^{2/\alpha})^{\alpha/2}$.

When $\alpha = 1$, the infinitesimal generator reduces to the free relativistic Hamiltonian

$$m - \sqrt{-\Delta + m^2}.$$

Here the kinetic energy of a relativistic particle is $\sqrt{-\Delta + m^2} - m$, instead of $-\Delta$ for a nonrelativistic particle.

Green function estimates in $C^{1,1}$ bounded open sets – Ryznar (2002), Chen and Song (2003):

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# Mixtures and its Green function estimates

Mixtures $X$ is a Lévy process with characteristic exponent

$$\Phi(\theta) = |\theta|^\alpha + |\theta|^\beta\,,\ \theta \in \mathbb{R}^d\,, \quad 0 < \beta < \alpha \le 2\,,$$

and infinitesimal generator $-(-\Delta)^{\beta/2} - (-\Delta)^{\alpha/2}$. $X$ is sum of independent symmetric $\alpha$ and $\beta$ stable.

When $0 < \beta < \alpha < 2$, Green function estimates in $C^{1,1}$ bounded opens sets – Chen, K & Song (2009)

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Case $\alpha = 2$ - sum of independent Brownian motion and symmetric $\beta$-stable. Same Green function estimates ($d \ge 3$), Chen, K, Song, Vondraček (2010)

Same Green function estimates for BM, stable, relativistic stable and mixtures. What is common?

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## subordinate Brownian motions

All processes are subordinate Brownian motions:
$W = (W_t, \mathbb{P}_x)$ $d$-dimensional Brownian motion, $S = (S_t)$ an independent subordinator with the Laplace exponent

$$\phi(\lambda) = b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t})\, \mu(dt)\,,$$

where $b \geq 0$ and $\mu$ is called Lévy measure on $(0, \infty)$ satisfying $\int_{(0,\infty)} (1 \wedge t)\, \mu(dt) < \infty$

Then $X_t := W_{S_t}$ is called a subordinate Brownian motion.

– a Lévy process with characteristic exponent $\Phi(\theta) = \phi(|\theta|^2)$ and infinitesimal generator $-\phi(-\Delta)$.

# Bernstein and complete Bernstein function

A $C^\infty$ function $\phi : (0,\infty) \to [0,\infty)$ is called a Bernstein function if $(-1)^n D^n \phi \le 0$ for every positive integer $n$.

Every Bernstein function has a representation $\phi(\lambda) = a + b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t})\, \mu(dt)$ where $a, b \ge 0$ and $\mu$ is called Lévy measure on $(0,\infty)$ satisfying $\int_{(0,\infty)} (1 \wedge t)\, \mu(dt) < \infty$.

Thus, Laplace exponent of a subordinator is a a Bernstein function. Conversely, for every Bernstein function $\phi$ satisfying $\phi(0+) = 0$, there exists a subordinator with the Laplace exponent $\phi$ .

A Bernstein function $\phi$ is called a complete Bernstein function if $\mu$ has a completely monotone density $\mu(t)$, i.e., $(-1)^n D^n \mu \ge 0$ for every non-negative integer $n$.

The class of complete Bernstein functions has been used throughout the literature in many branches of mathematics but under various names and for very different reasons, e.g. as Pick or Nevanlinna functions in (complex) interpolation theory, Löwner or operator monotone function in functional analysis, or as class (S) in the Russian literature on complex function theory in a half-plane.

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# What is common?

Stable: $\phi(\lambda) = \lambda^{\alpha/2}$, relativistic stable: $\phi(\lambda) = (\lambda + m^{2/\alpha})^{\alpha/2} - m$, mixtures: $\phi(\lambda) = \lambda^{\beta/2} + \lambda^{\alpha/2}$.
In all cases $\phi$ is a complete Bernstein function (i.e., $\mu$ has a CM density, $\mu(dt) = \mu(t)dt$), and

$$\lim_{\lambda \to \infty} \frac{\phi(\lambda)}{\lambda^{\alpha/2}} = 1.$$

Throughout the talk we use notation $f(t) \asymp g(t)$ as $t \to \infty$ (resp. $t \to 0+$) if the quotient $f(t)/g(t)$ stays bounded between two positive constants as $t \to \infty$ (resp. $t \to 0+$).

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# Result of K, Song & Vondraček (Proc. London Math. Soc. 2012)

Recall that $X = (X_t, \mathbb{P}_x)$ is a subordinate Brownian motion in $\mathbb{R}^d$ determined by the Laplace exponent $\phi$ of the subordinator $S$.

**Assumptions in K, Song & Vondraček (12)**

$\phi$ is a complete Bernstein function of the form

$$\phi(\lambda) \asymp \lambda^{\alpha/2}\ell(\lambda), \quad \lambda \to \infty, \qquad 0 < \alpha < 2,$$

where $\ell$ is a slowly varying at $\infty$, and additional hypothesis in case $d \leq 2$ (which implies that $X$ is transient).

Examples:

- Stables, relativistic stables, mixtures: $\lim_{\lambda\to\infty} \ell(\lambda) = 1$;
- $\lambda^{\alpha/2}(\log(1+\log(1+\lambda^{\gamma/2})^{\delta/2}))^{\beta/2}$, $\alpha, \gamma, \delta \in (0,2), \beta \in (0, 2-\alpha]$;
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# A Lemma on Laplace transform

Suppose that

$$\psi(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\, dt,$$

where $f$ is a nonnegative decreasing function. Then

$$f(t) \le \left(1 - e^{-1}\right)^{-1} t^{-1} \psi(t^{-1}), \quad t > 0.$$

If, furthermore, there exist $\delta \in (0,1)$ and $a, t_0 > 0$ such that

$$\psi(r\lambda) \le a r^{-\delta} \psi(\lambda), \quad r \ge 1, t \ge 1/t_0,$$

then there exists $c = c(w, f, a, t_0, \delta) > 0$ such that

$$f(t) \ge c t^{-1} \psi(t^{-1}), \quad t \le t_0.$$

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where $f$ is a nonnegative decreasing function. Then

$$f(t) \le \left(1 - e^{-1}\right)^{-1} t^{-1}\psi(t^{-1}), \quad t > 0.$$

If, furthermore, there exist $\delta \in (0,1)$ and $a, t_0 > 0$ such that

$$\psi(r\lambda) \le ar^{-\delta}\psi(\lambda), \quad r \ge 1, t \ge 1/t_0,$$

then there exists $c = c(w, f, a, t_0, \delta) > 0$ such that

$$f(t) \ge ct^{-1}\psi(t^{-1}), \quad t \le t_0.$$

$$\phi(\lambda) \asymp \lambda^{\alpha/2}\ell(\lambda)\,, \quad \lambda \to \infty, \qquad 0<\alpha<2\,,$$

**potential density of $S$**

The potential measure $U$ of $S$ has a completely monotone density $u$ and

$$\frac{1}{\phi(\lambda)} = \int_{(0,\infty)} e^{-\lambda t} u(t)\, dt.$$

Thus

$$u(t) \asymp t^{-1}\phi(t^{-1})^{-1} \asymp \frac{t^{\alpha/2-1}}{\ell(t^{-1})}\,, \quad t \to 0+\ .$$

the potential measure of a subordinator with Laplace exponent $\psi(\lambda) := \lambda/\phi(\lambda)$, which is also complete Bernstein, has a completely monotone density $v$ given by

$$v(t) = \mu(t,\infty).$$

Thus

$$\frac{\phi(\lambda)}{\lambda} = \frac{1}{\psi(\lambda)} = \int_{(0,\infty)} e^{-\lambda t} v(t)\, dt = \int_{(0,\infty)} e^{-\lambda t} \mu(t,\infty)\, dt.$$

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# The process

## Lévy measure of $S$

$\mu(t,\infty) \asymp \phi(t^{-1})$ and

$$\mu(t) \asymp t^{-1}\phi(t^{-1}) \asymp \frac{\ell(t^{-1})}{t^{1+\alpha/2}}, \quad t \to 0.$$

This and the fact that $\mu(t)$ completely monotone imply that there exists $c > 0$ such that

$$\mu(t) \le c\mu(2t), \quad t \in (0,2) \quad \text{and} \quad \mu(t) \le c\mu(t+1), \quad t > 1$$

# Lévy density of $X$

The Lévy measure of $X$ has a density $J(x) = j(|x|)$ with

$$j(r) = \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)} \mu(t)\, dt$$

There exists a constant $c > 0$ such that

$$j(r) \le c j(2r)\,, \quad r \in (0,2)$$

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# Green function of $X$ in $\mathbb{R}^d$

$$g(r) = \int_0^\infty (4\pi t)^{-d/2} \exp\left(-\frac{r^2}{4t}\right) u(t)\, dt\,.$$

**Green function estimates for the free process.**

$G(x,y) = g(|x-y|)$ and

$$G(x,y) \asymp \frac{1}{|x-y|^d\, \phi(|x-y|^{-2})} \asymp \frac{1}{|x-y|^{d-\alpha}\ell(|x-y|^{-2})}$$

as $|x-y| \to 0$

# Green function of $X^D$

The transition density and the Green function of $X^D$ are given by

$$p_D(t,x,y) = p(t,x,y) - \mathbb{E}_x\left[p(t-\tau_D, X(\tau_D), y); \tau_D < t\right]$$

and $G_D(x,y) = \int_0^\infty p_D(t,x,y)\,dt$.

Since $X$ is transient, we have the following formula

$$G_D(x,y) = G(x,y) - \mathbb{E}_x[G(X(\tau_D),y)]\,.$$

Also, $G_D(x,y)$ is symmetric and, for fixed $y \in D$, $G_D(\cdot, y)$ is harmonic for $X$ in $D \setminus \{y\}$.

# Result of K, Song and Vondraček (2012)

Theorem: Let $D \subset \mathbb{R}^d$ be a bounded $C^{1,1}$ open set with characteristics $(R, \Lambda)$. Then the Green function $G_D(x,y)$ of $X^D$ satisfies the following estimates:

$$
\begin{aligned}
& G_D(x,y) \\
\asymp \quad & \left(1 \wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(x)^{-2})}}\right) \frac{1}{|x-y|^d\,\phi(|x-y|^{-2})} \\
= \quad & \left(1 \wedge \frac{(\phi(\delta_D(x)^{-2}))^{-1/2}(\phi(\delta_D(x)^{-2}))^{-1/2}}{(\phi(|x-y|^{-2}))^{-1}}\right) \frac{1}{|x-y|^d\,\phi(|x-y|^{-2})}
\end{aligned}
$$

Theorem: (BHP) Let $u$ be a nonnegative function in $\mathbb{R}^d$ that is harmonic in $D \cap B(Q,r)$ with respect to $X$ and vanishes continuously on $D^c \cap B(Q,r)$ ($D$ not necessarily bounded). Then

$$
\frac{u(x)}{u(y)} \le c\, \frac{\phi(\delta_D(x)^{-2})^{-1/2}}{\phi(\delta_D(y)^{-2})^{-1/2}} \qquad \text{for every } x, y \in D \cap B(z, r/2).
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# Conjecture in K, Song and Vondraček (2012)

it is implicitly conjectured in **[KSV]** that for a large class of transient subordinate Brownian motions, Green function $G_D(x,y)$ in $D$ enjoys the following two-sided estimates in terms of $\phi$ and Green function $G(x,y)$ in $\mathbb{R}^d$;

$$c^{-1}\left(1\wedge\frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}}\right)G(x,y)$$

$$\le G_D(x,y)\le c\left(1\wedge\frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}}\right)G(x,y)\,.$$

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Prove this conjecture for larger class of transient subordinate Brownian motions than ones in **[KSV]**.

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Prove this conjecture for larger class of transient subordinate Brownian motions than ones in **[KSV]**.

# Outline

1. Introduction and motivation

2. Main Results

# Assumption

For simplicity, we state assumptions for $d \geq 3$.

**(A-1)** $\phi$ is a complete Bernstein function with the infinite Lévy measure.

**(A-2)** there exist constants $\sigma > 0$, $\lambda_0 > 0$ and $\delta \in (0, 1]$ such that

$$\frac{\phi'(\lambda x)}{\phi'(\lambda)} \leq \sigma x^{-\delta} \quad \text{for all} \quad x \geq 1 \quad \text{and} \quad \lambda \geq \lambda_0.$$

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# Remark

**(i)** **(A-1)** implies that the potential measure of $U$ of $S$ has a decreasing density, i.e., there is a decreasing function $u\colon (0,\infty) \to (0,\infty)$ so that $U(dt) = u(t)\,dt$ .

**(ii)** Since $\phi$ is a complete Bernstein function,

$$\phi(\lambda) = \gamma\lambda + \int_0^\infty (1 - e^{-\lambda t})\,\mu(t)\,dt\,.$$

Note that **(A-2)** implies $\gamma = 0$, by letting $\lambda \to +\infty$ .

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$$\text{If} \quad \delta_1 \in (0,1) \quad \text{and} \quad \frac{\phi(\lambda x)}{\phi(\lambda)} \geq \sigma\, x^{1-\delta_1} \quad \text{for all } x \geq 1 \text{ and } \lambda \geq \lambda_0,$$

then there exists a constant $c > 0$ such that

$$c\phi(\lambda) \leq \lambda\phi'(\lambda) \leq \phi(\lambda) \quad \text{for all } \lambda \geq \lambda_0.$$

Thus, if $\delta, \delta_1 \in (0,1)$ and

$$\sigma\, x^{1-\delta} \geq \frac{\phi(\lambda x)}{\phi(\lambda)} \geq \sigma_1\, x^{1-\delta_1} \quad \text{for all } x \geq 1, \lambda \geq \lambda_0$$

then **(A-2)** hold with $\delta \in (0,1)$

Thus the condition **(A-2)** is more general than assuming that $\phi$ is the class *OR* of *O*-regularly varying functions with its Matuszewska indices contained in $(0,1)$.

# Remark

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If $\delta_1 \in (0,1)$ and $\dfrac{\phi(\lambda x)}{\phi(\lambda)} \geq \sigma\, x^{1-\delta_1}$ for all $x \geq 1$ and $\lambda \geq \lambda_0$,

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## Examples

- Assume that $\phi$ complete Bernstein and

$$\phi(\lambda) \asymp \lambda^{\alpha/2}\ell(\lambda), \ \lambda \to \infty \quad (0 < \alpha < 2)$$

where $\ell$ varies slowly at infinity, For example, $\ell(\lambda) = \log(1+\lambda)$ or $\ell(\lambda) = \log(1+\log(1+\lambda))$.

- (Geometric stable processes, $\delta = 1$ case)

$$\phi(\lambda) = \log(1+\lambda^{\beta/2}), \quad (0 < \beta \le 2, d > \beta).$$

- (Iterated geometric stable processes, $\delta = 1$ case)

$$\phi_1(\lambda) = \log(1+\lambda^{\beta/2}) \quad (0 < \beta \le 2)$$
$$\phi_{n+1} = \phi_1 \circ \phi_n \quad n \in \mathbb{N},$$

with an additional condition $d > 2^{1-n}\beta^n$.

- (Relativistic geometric stable processes, $\delta = 1$ case)

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## Asymptotics of Green function and Lévy density

**[KM1]** K and Mimica, *Electronic Journal of Probability* (2012)

Assume that the potential measure of $S$ has a decreasing density and that **(A-2)** holds. Then

$$G(x,y) = g(|x-y|) \asymp \frac{\phi'(|x-y|^{-2})}{|x-y|^{d+2}\phi(|x-y|^{-2})^2} \text{ as } |x-y| \to 0$$

For geometric stable processes, $g(r) \asymp \frac{1}{r^d (\log r)^2}$ as $r \to 0$.

If the Lévy measure $\mu$ of $S$ has a deceasing density and **(A-2)** holds,

$$J(x,y) = j(|x-y|) \asymp \frac{\phi'(|x-y|^{-2})}{|x-y|^{d+2}} \text{ as } |x-y| \to 0$$

For geometric stable processes, $j(r) \asymp \frac{1}{r^d}$ as $r \to 0$.
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# (Scale invariant) Harnack Inequality

**[KM1]**

Suppose $d \geq 1$ and $X$ is a subordinate Brownian motion satisfying **(A-1)–(A-2)**. There exists a constant $c > 0$ such that for all $x_0 \in \mathbb{R}^d$ and $r \in (0,1)$

$$h(x_1) \leq c\, h(x_2) \quad \text{for all} \quad x_1, x_2 \in B(x_0, \tfrac{r}{2})$$

and for every non-negative function $h\colon \mathbb{R}^d \to [0,\infty)$ which is harmonic in $B(x_0, r)$.

# Remark

- A non-scale invariant Harnack inequality was proved for geometric stable and iterated geometric stable processes by Šikić,Song and Vondraček (PTRF 2006).

- Using theory of fluctuation of one-dimensional Lévy processes, a scale invariant Harnack inequality was proved for geometric stable process in $d = 1$ by Grzywny and Ryznar (2011)

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A very successful technique for proving Harnack inequality for stable-like Markov jump processes was developed in Bass & Levin (02). The proof relied on an estimate of Krylov and Safonov type: $\mathbb{P}_x(T_A < \tau_{B(0,r)}) \geq c\,\frac{|A|}{|B(0,r)|}$ for any $r \in (0,1)$, $x \in B(0,\frac{r}{2})$ where $T_A = \tau_{A^c}$ denotes the first hitting time of the set $A$.

Although this technique is quite general and can be applied to a much larger class of Markov jump processes, there are situations when it is not applicable even to a rotationally invariant Lévy process.

The Krylov-Safonov type estimate was indispensable in the proof of the Harnack inequality in Šikić and Song and Vondraček (PTRF 2006). Contrary to the case of stable-like processes, this estimate is not uniform in $r \in (0,1)$. For example, for a geometric stable process it is possible to find a sequence of radii $(r_n)$ and closed sets $A_n \subset B(0,r_n)$ such that $r_n \to 0$, $\frac{|A_n|}{|B(0,r_n)|} \geq \frac{1}{4}$ and

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# Uniform Boundary Harnack Principle

**[KM2]**

Suppose $d \geq 1$. There exists a constant $c = c(\phi, d) > 0$ such that For every $z_0 \in \mathbb{R}^d$, every open set $D \subset \mathbb{R}^d$, every $r \in (0,1)$ and for any nonnegative functions $u, v$ in $\mathbb{R}^d$ which are regular harmonic in $D \cap B(z_0, r)$ with respect to $X$ and vanish in $D^c \cap B(z_0, r)$, we have

$$\frac{u(x)}{v(x)} \leq c\, \frac{u(y)}{v(y)} \quad \text{for all } x, y \in D \cap B(z_0, r/2).$$

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**(i)** The proof is similar to the one of

P. Kim, R. Song, and Z. Vondraček, *Uniform boundary Harnack principle for rotationally symmetric Lévy processes in general open sets*, to appear in Science in China (2012),

which is motived by a earlier work by Bogdan, Kulczycki and Kwaśnicki (2008).

**(ii)** Very recently, the uniform boundary Harnack principle is obtained for a large class of Markov processes in

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# The role of fluctuation theory

$B = (B_t)$ one-dimensional Brownian motion, $S$ independent subordinator with Laplace exponent $\phi$, $Z_t := B(S_t)$ 1-dim SBM. $Z$ is a symmetric Lévy process.

$\overline{Z}_t := \sup\{Z_s : 0 \le s \le t\}$, $L$ the local time at zero of the reflected process $\overline{Z} - Z$, $H_t := Z(L_t^{-1})$ the ladder height process of $Z$:

$H$ is a subordinator with Laplace exponent

$$\chi(\lambda) = \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\log(\phi(\lambda^2\theta^2))}{1+\theta^2}\, d\theta\right)$$

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$B = (B_t)$ one-dimensional Brownian motion, $S$ independent subordinator with Laplace exponent $\phi$, $Z_t := B(S_t)$ 1-dim SBM. $Z$ is a symmetric Lévy process.

$\overline{Z}_t := \sup\{Z_s : 0 \le s \le t\}$, $L$ the local time at zero of the reflected process $\overline{Z} - Z$, $H_t := Z(L_t^{-1})$ the ladder height process of $Z$:

$H$ is a subordinator with Laplace exponent

$$\chi(\lambda) = \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\log(\phi(\lambda^2\theta^2))}{1+\theta^2}\,d\theta\right)$$

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Let $V$ be the renewal function of the ladder height process: $V(t) = V(0,t)$. Since $\chi$ is a complete BF, $V$ has a CM (potential) density $v$. In particular, $V$ is $C^\infty$ on $(0,\infty)$.

The key fact (Silverstein 1980): $V$ is invariant, hence harmonic, for the killed process $Z^{(0,\infty)}$.

Asymptotic behavior of $V$ at zero follows from the asymptotic behavior of $\chi$ at infinity: Using

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# A technical assumption

**(A-3)** If $0 < \delta \leq 1/2$, we further assume that there exist constants $\sigma_1 > 0$ and $\delta_1 \in [\delta, 2\delta)$ such that

$$\frac{\phi(\lambda x)}{\phi(\lambda)} \geq \sigma_1 x^{1-\delta_1} \quad \text{for all} \quad x \geq 1 \quad \text{and} \quad \lambda \geq \lambda_0 \, .$$

Remark:

- This condition is only for $0 < \delta \leq 1/2$.
- **(A-2)** implies that for every $\varepsilon > 0$ there is a constant $c = c(\varepsilon) > 0$ so that

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# Green function estimates [KM2]

Suppose that $X = (X_t : t \geq 0)$ is a transient subordinate Brownian motion whose characteristic exponent is given by $\Phi(\theta) = \phi(|\theta|^2)$, $\theta \in \mathbb{R}^d$ satisfying **(A-1)–(A-3)** (with some additional assumption when $d \leq 2$).

Then for every bounded $C^{1,1}$ open set $D$ in $\mathbb{R}^d$ with characteristics $(R, \Lambda)$, the Green function $G_D(x, y)$ of $X$ in $D$ satisfies the following estimates:

$$
\begin{aligned}
& G_D(x,y) \\
& \asymp \left(1 \wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}}\right) \frac{\phi'(|x-y|^{-2})}{|x-y|^{d+2}\phi(|x-y|^{-2})^2} \\
& \asymp \left(1 \wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}}\right) G(x,y).
\end{aligned}
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# Corollary [KM2]

Suppose that $X=(X_t:\ t\geq 0)$ is a transient subordinate Brownian motion whose characteristic exponent is given by $\Phi(\theta)=\phi(|\theta|^2)$, $\theta\in\mathbb{R}^d$, where $\phi:(0,\infty)\to[0,\infty)$ is a complete Bernstein function such that

$$c_1\,x^{\alpha/2}\leq\frac{\phi(\lambda x)}{\phi(\lambda)}\leq c_2\,x^{\beta/2}\ \text{ for all }\ x\geq 1\ \text{ and }\ \lambda\geq\lambda_1\,.$$

for some constants $c_1,c_2,\lambda_1>0$, $\alpha,\beta\in(0,2)$ and $\alpha\leq\beta$. We further assume that $2\beta-\alpha<1$ (with some additional assumption when $d\leq 2$). .

Then for every bounded $C^{1,1}$ open set $D$ in $\mathbb{R}^d$ with characteristics $(R,\Lambda)$, there exists $c=c(\mathrm{diam}(D),R,\Lambda,\phi,d)>1$ such that the Green function $G_D(x,y)$ of $X$ in $D$ satisfies the following estimates:

$$G_D(x,y)\asymp\left(1\wedge\frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}}\right)\frac{1}{|x-y|^d\,\phi(|x-y|^{-2})}.$$

This extends the main result of **[KSV]**.

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# Green function estimates – conclusion

$$
\begin{aligned}
G_D(x,y) &\asymp \left(1 \wedge \frac{V(\delta_D(x))}{V(|x-y|)}\right)\left(1 \wedge \frac{V(\delta_D(y))}{V(|x-y|)}\right) G(x,y) \\
&\asymp \left(1 \wedge \frac{V(\delta_D(x))V(\delta_D(y))}{V(|x-y|)^2}\right) G(x,y) \\
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# Decay rate of harmonic function near boundary [KM2]

**Boundary Haranck inequality with the explicit decay rate**

Suppose that $d \geq 1$ and that $D$ is a (possibly unbounded) $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics $(R, \Lambda)$. Then there exists $c = c(R, \Lambda, \phi, d) > 0$ such that for $r \in (0, (R \wedge 1)/4]$, $z \in \partial D$ and any nonnegative function $u$ in $\mathbb{R}^d$ that is harmonic in $D \cap B(z, r)$ with respect to $X$ and vanishes continuously on $D^c \cap B(z, r)$, we have

$$\frac{u(x)}{u(y)} \leq c \, \frac{\phi(\delta_D(x)^{-2})^{-1/2}}{\phi(\delta_D(y)^{-2})^{-1/2}} \qquad \text{for every } x, y \in D \cap B(z, r/2).$$

In the case of geometric stable process,

$$\frac{u(x)}{u(y)} \leq c \, \frac{(\log(\delta_D(x)^{-1}))^{-1/2}}{(\log(\delta_D(y)^{-1}))^{-1/2}} \qquad \text{for every } x, y \in D \cap B(z, r/2).$$

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# Poisson Kernel Estimates [KK]: Assumption

$D$ is a bounded open set with $d_D := \text{diam}(D) < M$ for some $M > 0$. $X$ is a purely discontinuous rotationally symmetric Lévy process with Lévy exponent $\Phi(|\xi|)$.

We assume the function $\Phi : [0,\infty) \to [0,\infty)$ satisfies the following properties.

**(P1)** $\Phi \in C^1$ is an increasing function with $\Phi(0) = 0$ and $\lim_{t\to\infty} \Phi(t) = \infty$.

**(P2)** There exists constant $C_0 \geq 1$ such that

$$\Phi(t\lambda) \leq C_0 \lambda^2 \Phi(t) \qquad \text{for all } \lambda \geq 1, t > 0.$$

**(P3)** There exists constant $C_1 > 0$ such that

$$\Phi'(t\lambda) \leq C_1 \lambda \Phi'(t) \qquad \text{for all } \lambda \geq 1, t > 0.$$

**(P4)** There exists a increasing function $\Psi : ((5M)^{-1}, \infty) \to (0,\infty)$ and a constant $C_2 \geq 1$ such that

$$C_2^{-1}\Psi(\lambda) \leq \lambda^{1+d}\frac{\Phi'(\lambda)}{\Phi(\lambda)} \leq C_2\Psi(\lambda) \quad \lambda \in ((5M)^{-1}, \infty).$$

# Poisson Kernel Estimates [KK]: Assumption (continue)

We assume that Green function $G_D(x,y)$ and Lévy density $j(|x|)$ have following estimation.

**(G)**

$$C_3\left(1\wedge\frac{\Phi(|x-y|^{-1})}{\sqrt{\Phi(\delta_D(x)^{-1})}}\right)^{1/2}\left(1\wedge\frac{\Phi(|x-y|^{-1})}{\sqrt{\Phi(\delta_D(y)^{-1})}}\right)^{1/2}\frac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})^2}$$

$$\leq G_D(x,y)$$

$$\leq C_4\left(1\wedge\frac{\Phi(|x-y|^{-1})}{\sqrt{\Phi(\delta_D(x)^{-1})}}\right)^{1/2}\left(1\wedge\frac{\Phi(|x-y|^{-1})}{\sqrt{\Phi(\delta_D(y)^{-1})}}\right)^{1/2}\frac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})^2}.$$

**(J1)** There exist positive constants $C_5(M)$ and $C_6(M)$ such that

$$C_5\frac{\Phi'(r^{-1})}{r^{d+1}}\leq j(r)\leq C_6\frac{\Phi'(r^{-1})}{r^{d+1}},\qquad r\in(0,10M).$$

**(J2)** There exists $C_7>0$ such that

$$j(r)\leq C_7 j(r+1),\qquad \forall r>1.$$

By the result of Ikeda and Watanabe (Lévy system) we have

$$\mathbb{P}_x(X_{\tau_D} \in F) = \int_F \int_D G_D(x,y) j(|z-y|)\, dy\, dz$$

for any $F \subset \overline{D}^c$. We define the Poisson kernel of the set $D$ by

$$K_D(x,z) = \int_D G_D(x,y) j(|z-y|)\, dy,$$

so that $\mathbb{P}_x(X_{\tau_D} \in F) = \int_F K_D(x,z)\, dz$ for any $F \subset \overline{D}^c$.

Note that if $U$ is a Lipschitz open set

$$\mathbb{P}_x(X_{\tau_U} \in \partial U) = 0 \qquad \text{and} \qquad \mathbb{P}_x(X_{\tau_U} \in dz) = K_U(x,z)dz \quad \text{on } U^c.$$

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# Poisson Kernel Estimates [KK]

Let $D$ be a bounded open set which satisfies cone condition with cone characteristic constant $(R, \eta)$ and $d_D := \text{diam}(D) < M$ for some $M > 0$. Also, a function $\Phi : [0, \infty) \to [0, \infty)$ satisfies **(P1)-(P4)** and **(G)**, **(J1)** and **(J2)** hold. Then, there exists $c = c(C_0, C_1, C_2, C_3, C_4, C_5, C_6, C_7, R/d_D, \eta, M, d) > 1$ such that the following inequality holds for every $x \in D$ and $z \in \overline{D}^c$

$$
\begin{aligned}
& c^{-1} \frac{\Phi(\delta_D(z)^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2} \Phi(|x-z|^{-1})(1 + \Phi(d_D^{-1})^{1/2} \Phi(\delta_D(z)^{-1})^{-1/2})} j(|x-z|) \\
\le & K_D(x,z) \\
\le & c \frac{\Phi(\delta_D(z)^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2} \Phi(|x-z|^{-1})(1 + \Phi(d_D^{-1})^{1/2} \Phi(\delta_D(z)^{-1})^{-1/2})} j(|x-z|)
\end{aligned}
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This gives uniform estimate for $K_{B(0,r)}(x,z)$ for small $r$ from uniform estimate of $G_{B(0,r)}(x,z)$ for small $r$.

# Poisson Kernel Estimates [KK]

## Poisson Kernel Estimates on bounded $C^{1,1}$ open set

Suppose that $X = (X_t : t \geq 0)$ is a transient subordinate Brownian motion whose characteristic exponent is given by $\Phi(\theta) = \phi(|\theta|^2)$, $\theta \in \mathbb{R}^d$ satisfying **(A-1)–(A-3)**. then for every bounded $C^{1,1}$ open set $D$ in $\mathbb{R}^d$ with characteristics $(R, \Lambda)$

$$K_D(x,z) \asymp \frac{\phi(\delta_D(z)^{-2})^{1/2}}{\phi(\delta_D(x)^{-2})^{1/2}\phi(|x-z|^{-2})(1+\phi(\delta(z)^{-2})^{-1/2})} j(|x-z|).$$

- $\alpha$-stable: Chen & Song (97) (Jakubowski (02) for Lipschitz and Michalik (06) for cone).
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- If $\alpha, \beta \in (0,2)$ and $c_1 x^{\alpha/2} \leq \phi(\lambda x)/\phi(\lambda) \leq c_2 x^{\beta/2}$ for all $x \geq 1$ and $\lambda \geq \lambda_1$ ($2\beta - \alpha < 1$ if $d \geq 2$) then for $z \in \overline{D}^c$ with $d_D(z) \leq 2d_D$

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Remarks on some parts of the proof

# Asymptotical properties [KM1]

We assume that $f\colon (0,\infty) \to (0,\infty)$ is a differentiable function satisfying

$$|f(\lambda+\varepsilon) - f(\lambda)| = \int_0^\infty \left(e^{-\lambda t} - e^{-(\lambda+\varepsilon)t}\right) \nu(t)\, dt\,, \tag{1}$$

for all $\lambda > 0$, $\varepsilon \in (0,1)$ and a decreasing function $\nu\colon (0,\infty) \to (0,\infty)$.

**Lemma A**

Suppose (1) holds. Then for all $t > 0$,

$$\nu(t) \le (1 - 2e^{-1})^{-1}\, t^{-2} |f'(t^{-1})|\,.$$

# Asymptotical properties [KM1]

$$|f(\lambda+\varepsilon)-f(\lambda)| = \int_0^\infty \left(e^{-\lambda t} - e^{-(\lambda+\varepsilon)t}\right) \nu(t)\,dt\,, \quad (1)$$

## Lemma B

Assume that (1) holds and $|f'|$ is decreasing and there exist $c_1 > 0$, $\lambda_0 > 0$ and $\delta > 0$ such that

$$\left|\frac{f'(\lambda x)}{f'(\lambda)}\right| \leq c_1 x^{-\delta} \text{ for all } \lambda \geq \lambda_0 \text{ and } x \geq 1\,.$$

Then there is a constant $c_2 = c_2(c_1, \lambda_0, \delta) > 0$ such that

$$\nu(t) \geq c_2 t^{-2} |f'(t^{-1})| \text{ for any } t \leq 1/\lambda_0\,.$$

# Lévy density $\mu$ of subordinator $S$ [KM1]

$$|f(\lambda+\varepsilon)-f(\lambda)| = \int_0^\infty \left(e^{-\lambda t} - e^{-(\lambda+\varepsilon)t}\right) \nu(t)\,dt\,, \quad (1)$$

**Proposition 1**

If If the Lévy measure $\mu$ of $S$ has a deceasing density $t \to \mu(t)$ and **(A-2)** holds, then Lévy density $\mu$ of subordinator $S$ satisfies

$$\mu(t) \asymp t^{-2}\phi'(t^{-1}),\ t \to 0+\ .$$

**Proof.** Note that

$$\phi(\lambda+\varepsilon)-\phi(\varepsilon) = \int_0^\infty \left(e^{-\lambda t} - e^{-\lambda(t+\varepsilon)}\right) \mu(t)\,dt$$

for any $\lambda > 0$ and $\varepsilon > 0$ and thus the condition (1) holds with $f = \phi$ and $\nu = \mu$. Since $\phi$ is a Bernstein function, it follows that $\phi' \geq 0$ and $\phi'$ is decreasing. □

# Potential density $u$ of subordinator $S$ [KM1]

## Proposition 2

If the potential measure $U$ of $S$ has a decreasing density $u$ and **(A-2)** holds, then potential density $u$ of subordinator $S$ satisfies

$$u(t) \asymp t^{-2} \frac{\phi'(t^{-1})}{\phi(t^{-1})^2}, \quad t \to 0+ \ .$$

**Proof.** Since $\int_0^\infty e^{-\lambda t} u(t)\, dt = \frac{1}{\phi(\lambda)} =: \psi(\lambda)$. (1) is satisfied with $f = \frac{1}{\phi} = \psi$ and $\nu = u$.

For $\lambda \geq \lambda_0$ and $x \geq 1$, **(A-2)** implies

$$\left| \frac{\psi'(\lambda x)}{\psi'(\lambda)} \right| = \left( \frac{\phi(\lambda)}{\phi(\lambda x)} \right)^2 \frac{\phi'(\lambda x)}{\phi'(\lambda)} \leq \frac{\phi'(\lambda x)}{\phi'(\lambda)} \leq c x^{-\delta}.$$

Since $\phi$ is a Bernstein function, $\phi' \geq 0$ and $\phi'$ is a decreasing function. Thus $|f'| = \frac{\phi'}{\phi^2}$ is also a decreasing function. □

## Asymptotical properties for SBM [KM1]

Let $\eta\colon (0,\infty) \to (0,\infty)$ be a decreasing function satisfyng the following conditions:

**(a)** there exists a decreasing function $\psi\colon (0,\infty) \to (0,\infty)$ such that $\lambda \mapsto \lambda^2\psi(\lambda)$ is increasing and satisfies

$$\eta(t) \asymp t^{-2}\psi(t^{-1}),\ t \to 0+;$$

**(b)**

$$\int_1^\infty t^{-d/2}\eta(t)\,dt < \infty.$$

### Lemma A3

If

$$I(r) = \int_0^\infty (4\pi t)^{-d/2} \exp\left(-\frac{r^2}{4t}\right)\eta(t)\,dt$$

exists for $r \in (0,1)$ small enough, then

$$I(r) \asymp r^{-d-2}\psi(r^{-2}),\ r \to 0+\ .$$

## Asymptotical properties for $j$ [KM1]

We have

$$j(r) \asymp r^{-d-2}\phi'(r^{-2}),\ \ r \to 0+\ .$$

**Proof.** Recall

$$j(r) = \int_0^\infty (4\pi t)^{-d/2} \exp\left(-\frac{r^2}{4t}\right) \mu(t)\, dt\,.$$

and $\mu(t) \sim t^{-2}\phi'(t^{-1}),\ \ t \to 0+$.
Use Lemma A3 with $\eta = \mu$ and $\psi = \phi'$.

$$\int_1^\infty t^{-d/2}\mu(t)\, dt \le \int_1^\infty \mu(t)\, dt = \mu(1,\infty) < \infty,$$

since $\mu$ is a Lévy measure. □

## Asymptotical properties for $g$ [KM1]

We have

$$g(r) \asymp r^{-d-2}\frac{\phi'(r^{-2})}{\phi(r^{-2})^2}, \quad r \to 0+ \,.$$

**Proof.** Recall that

$$g(r) = \int_0^\infty (4\pi t)^{-d/2} \exp\left(-\tfrac{r^2}{4t}\right) u(t)\, dt \,.$$

and $u(t) \sim t^{-2}\frac{\phi'(t^{-1})}{\phi(t^{-1})^2}, \; t \to 0+.$
Use Lemma A3 with $\eta = u$ and $\psi = \frac{\phi'}{\phi^2}$.
Note that $X$ being transient implies that

$$\int_1^\infty t^{-d/2} u(t)\, dt < \infty \,.$$

$\square$

Need a preliminary estimate of the Green function $G_{B(0,r)}(x,y)$ of the ball $B(0,r)$ when $y$ is near its boundary.

To be more precise, there are a function $\xi\colon (0,1)\to(0,\infty)$ and constants $c_1, c_2 > 0$ and $0<\kappa_1<\kappa_2<1$ such that for every $r\in(0,1)$,

$$c_1\xi(r)r^{-d}\,\mathbb{E}_y\tau_{B(0,r)} \le G_{B(0,r)}(x,y) \le c_2\xi(r)r^{-d}\,\mathbb{E}_y\tau_{B(0,r)}, \qquad (2)$$

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The function $\xi$ that appeared in (2) is of the form $\xi(r)=\frac{r^{-2}\phi'(r^{-2})}{\phi(r^{-2})}$. Note that for many cases,

$$\xi(r)\asymp 1 \quad \text{as} \quad r\to 0+,$$

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# Boundary estimate

$D$ a bounded Lipschitz domain. Fix $z_0 \in D$ and let

$$g(x) := G_D(x, z_0) \wedge c$$

where $c$ is an appropriate constant. By using the methods of Bogdan (1997) and Hansen (2005), and the BHP, one proves

**Theorem**: Let $D$ be a bounded Lipschitz domain. Then

$$G_D(x,y) \asymp \frac{g(x)g(y)}{g(A)^2} \frac{\phi'(|x-y|^{-2})}{|x-y|^d \, \phi(|x-y|^{-2})^2}, \quad A \in \mathcal{B}(x,y).$$

where

$$\mathcal{B}(x,y) := \begin{cases} \{A \in D : & \delta_D(A) > \kappa(\delta_D(x) \vee \delta_D(y) \vee |x-y|), \\ & |x-A| \vee |y-A| < 5(\delta_D(x) \vee \delta_D(y) \vee |x-y|)\} \\ & \text{if } \delta_D(x) \vee \delta_D(y) \vee |x-y| < \varepsilon_1 \\ \{z_0\} & \text{if } \delta_D(x) \vee \delta_D(y) \vee |x-y| \geq \varepsilon_1 . \end{cases}$$

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# Harmonic function in the half-space

$$\mathcal{A}f(x) := \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} (f(y) - f(x)) j(|x-y|)\, dy$$

$\mathcal{A}|_{C_0^2}$ coincides with the infinitesimal generator of $X$.

Let $\mathbb{R}^d_+ := \{x = (\widetilde{x}, x_d) \in \mathbb{R}^d : x_d > 0\}$ be the half-space, and define

$$w(x) := V((x_d)^+) = V(\delta_{\mathbb{R}^d_+}(x)).$$

Theorem: $w$ is harmonic w.r.t. $X$ in $\mathbb{R}^d_+$ and, for any $r > 0$, regular harmonic in $\mathbb{R}^{d-1} \times (0, r)$ w.r.t. $X$.

Theorem: $\mathcal{A}w(x)$ is well defined for all $x \in \mathbb{R}^d_+$ and $\mathcal{A}w(x) = 0$ for all $x \in \mathbb{R}^d_+$.

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# "Test function" method

$D \subset \mathbb{R}^d$ a bounded $C^{1,1}$ open set with characteristics $(R, \Lambda)$. Fix $Q \in \partial D$ and define

$$h(y) := V(\delta_D(y))\mathbf{1}_{D\cap B(Q,R)}(y)\,.$$

**Key technical lemma:**

There exist $C = C(\alpha, \Lambda, R)$ and $R_2 \leq R/4$ (independent of $Q \in \partial D$) such that $\mathcal{A}h$ is well defined in $D \cap B(Q, R_2)$ and

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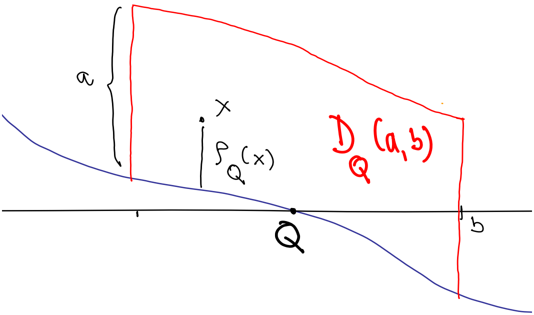
# Exit time probability and expectation

Define $\rho_Q(x) := x_d - \psi_Q(x)$ where $(\widetilde{x}, x_d)$ are the coordinates of $x$ in $CS_Q$. For $a, b > 0$ define the "box"

$$D_Q(a,b) := \{y \in D : \; 0 < \rho_Q(y) < a, |\widetilde{y}| < b\}.$$

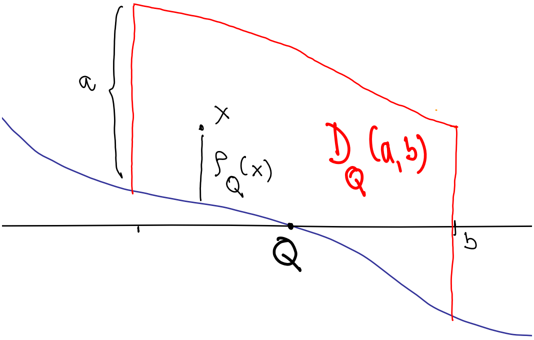


**Lemma:** $\exists\ C_1, C_2$ and $R_3$ such that for every $r \leq R_3$ and every $x \in D_Q(r,r)$

$$\mathbb{P}_x(X_{\tau_{D_Q(r,r)}} \in D) \geq C_1 V(\delta_D(x))$$

and

$$\mathbb{E}_x\left[\tau_{D_Q(r,r)}\right] \leq C_2 V(\delta_D(x)) .$$



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By applying the BHP to harmonic functions $x \mapsto G_D(x, z_0)$ and $x \mapsto \mathbb{P}_x(X_{\tau_{D(r,r)}} \in B(z_0, \epsilon_1/4))$ (for appropriate $r > 0$ and $\epsilon_1 > 0$) and by use of previous lemma one proves

Theorem: Suppose that $D$ is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics $(R, \Lambda)$. Then there exists $C = C(R, \Lambda, \alpha, \mathrm{diam}(D)) > 0$ such that

$$C^{-1}\left(V(\delta_D(x)) \wedge 1\right) \le g(x) \le C\left(V(\delta_D(x)) \wedge 1\right), \qquad x \in D.$$

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Thank you!