Dirichlet Heat Kernel Estimates for Relativistic Stable Processes

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References

This talk is based on the following joint papers with Zhen-Qing Chen and Panki Kim.

CKS1 Sharp Heat Kernel Estimates for Relativistic Stable Processes in Open Sets. *Ann. Probab.* **40 (1)** (2012), 213–244.

CKS2 Global heat kernel estimates for relativistic stable processes in half-space-like open sets. *Potential Anal.*, **36** (2012) 235–261.

CKS3 Global heat kernel estimate for relativistic stable processes in exterior open sets. *J. Funct. Anal.*, **263** (2012), 448–475.

CKS4 Dirichlet heat kernel estimates for rotationally symmetric Lévy processes. In preparation.

CKS5 Dirichlet heat kernel estimates for subordinate Brownian motions with Gaussian components. In preparation.

Outline

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Suppose that X is a symmetric Markov process on (a subset of) $\mathbb{R}^d$ with transition density $p(t,x,y)$ and generator $\mathcal{L}$. $p(t,x,y)$ is also the fundamental solution of $\partial_t u = \mathcal{L}u$ and so it is also called the heat kernel of $\mathcal{L}$. In general, there is no explicit formula for $p(t,x,y)$. Thus establishing sharp two-sided estimates for $p(t,x,y)$ is a fundamental problem.

Two-sided heat kernel estimates for diffusions in $\mathbb{R}^d$ have a long history and many beautiful results have been established. Among the main contributors are: D. G. Aronson, J. Nash, E. B. Davies.

Due to the complication near the boundary, two-sided estimates on the transition density of killed diffusions in a domain D (equivalently, the Dirichlet heat kernel) have been established only recently. See Davies (87), Zhang (02) for the case of bounded $C^{1,1}$ domains.

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The infinitesimal generator of a discontinuous Markov process in $\mathbb{R}^d$ is no longer a differential operator but rather a non-local (or integro-differential) operator $\mathcal{L}$. For instance, the infinitesimal generator of a rotationally symmetric α-stable process in $\mathbb{R}^d$ with $\alpha \in (0, 2)$ is a fractional Laplacian operator $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$.

Recently in [CKS10, JEMS], we obtained sharp two-sided estimates for the heat kernel of the fractional Laplacian $\Delta^{\alpha/2}$ in D with zero exterior condition (or equivalently, the transition density function of the symmetric α-stable process killed upon exiting D) for any $C^{1,1}$ open set $D \subset \mathbb{R}^d$ with $d \geq 1$. As far as we know, this was the first time sharp two-sided estimates were established for Dirichlet heat kernels of non-local operators.

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Since then, studies on this topic have been growing rapidly. The ideas of [CKS10, JEMS] have been adapted to establish two-sided heat kernel estimates of other discontinuous Markov processes, like censored stable processes [CKS10, PTRF] in open subsets of $\mathbb{R}^d$.

In this talk, I will present sharp two-sided estimates on the Dirichlet heat kernels of relativistic stable processes in $C^{1,1}$ domains of $\mathbb{R}^d$.

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In this talk, I will present sharp two-sided estimates on the Dirichlet heat kernels of relativistic stable processes in $C^{1,1}$ domains of $\mathbb{R}^d$.

Outline

For $\alpha \in (0,2]$, a symmetric α-stable process X on $\mathbb{R}^d$ is a Lévy process such that for any $t \geq 0$ and $\xi \in \mathbb{R}^d$

$$\mathbb{E}\left[\exp\left(i\xi \cdot (X_t - X_0)\right)\right] = \exp\left(-t|\xi|^\alpha\right).$$

When $\alpha = 2$, it reduces to a Brownian motion.

The infinitesimal generator of a symmetric α-stable process Y in $\mathbb{R}^d$ is the fractional Laplacian $\Delta^{\alpha/2}$, which can be written as

$$\Delta^{\alpha/2}u(x) = \lim_{\varepsilon \downarrow 0} \int_{\{y \in \mathbb{R}^d : |y-x| > \varepsilon\}} (u(y) - u(x)) \frac{\mathcal{A}(d,\alpha)}{|x-y|^{d+\alpha}}\, dy,$$

where $\mathcal{A}(d,\alpha) := \alpha 2^{\alpha-1}\pi^{-d/2}\Gamma(\frac{d+\alpha}{2})\Gamma(1-\frac{\alpha}{2})^{-1}$.

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where $\mathcal{A}(d, \alpha) := \alpha 2^{\alpha-1} \pi^{-d/2} \Gamma(\frac{d+\alpha}{2}) \Gamma(1 - \frac{\alpha}{2})^{-1}$.

Let $p(t,x,y)$ be the transition density of X. When $d > \alpha$, the potential density (also called the Green function) of X

$$G(x,y) = \int_0^\infty p(t,x,y)dt = C(d,\alpha)\frac{1}{|x-y|^{d-\alpha}}$$

which is the Riesz kernel.

Symmetric stable processes have some nice properties. For example it satisfies the following scaling property: For any $a > 0$, $\{a^{-1/\alpha}(X_{at} - X_0) : t \geq 0\}$ has the same law as $\{X_t - X_0 : t \geq 0\}$. In terms of the transition density, this means

$$p(t,x,y) = a^d p(at, a^{1/\alpha}x, a^{1/\alpha}y).$$

However, a symmetric α-stable process, for $\alpha \in (0,2)$, always have infinite variance. When $\alpha \in (0,1]$, it also have infinite mean.

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For any $m \geq 0$, a relativistic α-stable process X^m on $\mathbb{R}^d$ with weight m is a Lévy process such that for any $t \geq 0$ and $\xi \in \mathbb{R}^d$

$$\mathbb{E}\left[\exp\left(i\xi \cdot (X_t^m - X_0^m)\right)\right] = \exp\left(-t\left(\left(|\xi|^2 + m^{2/\alpha}\right)^{\alpha/2} - m\right)\right).$$

When $m = 0$, X^0 is simply a (rotationally) symmetric α-stable process on $\mathbb{R}^d$. The infinitesimal generator of X^m is

$$m - (-\Delta + m^{2/\alpha})^{\alpha/2}.$$

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When $\alpha = 1$, the infinitesimal generator reduces to

$$m - \sqrt{-\Delta + m^2}.$$

This operator was used by E. Lieb and his followers in studying the stability of matter.

Let $p^m(t, x, y)$ be the transition density of X^m. The the function, called the 1-potential density of X^1:

$$\int_0^\infty e^{-t} p^1(t, x, y) dt$$

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The Lévy measure of X^m has a density

$$J^m(x,y) = \mathcal{A}(d,\alpha)|x-y|^{-d-\alpha}\psi(m^{1/\alpha}|x-y|)$$

where

$$\psi(r) := \int_0^\infty s^{\frac{d+\alpha}{2}-1} e^{-\frac{s}{4}-\frac{r^2}{s}}\, ds,$$

which is decreasing and is a smooth function of r^2 satisfying $\psi(r) \le 1$ and

$$\psi(r) \asymp \phi(r) := e^{-r}(1 + r^{(d+\alpha-1)/2}) \qquad \text{on } [0,\infty).$$

For $m > 0$, X^m has moments of all orders, and it even has some exponential moments. In a small scale, X^m behaves like X^0, while in a larger scale, X^m behaves like Brownian motion.

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For $m > 0$, X^m does not satisfy any scaling invariance property. However, it does satisfy some sort of approximate scaling property.

Two-sided estimates on $p(t, x, y)$ is classical. But two-sided estimates on $p^m(t, x, y)$ is more recent.

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For any $m, c > 0$, we define a function $\widetilde{\Psi}_{d,\alpha,m,c}(t,x,y)$ on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ by

$$\widetilde{\Psi}_{d,\alpha,m,c}(t,x,y) := \begin{cases} t^{-d/\alpha} \wedge tJ^m(x,y), & \forall t \in (0,1/m]; \\ m^{d/\alpha-d/2} t^{-d/2} \exp\left(-c^{-1}(m^{1/\alpha}|x-y| \wedge m^{2/\alpha-1}\frac{|x-y|^2}{t})\right), & \forall t \in (1/m,\infty). \end{cases}$$

Theorem [Chen-Kim-Kumagai], [CKS1]

$$c_1^{-1}\widetilde{\Psi}_{d,\alpha,m,1/C_1}(t,x,y) \le p^m(t,x,y) \le c_1\widetilde{\Psi}_{d,\alpha,m,C_1}(t,x,y).$$

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Relativistic stable process in D

For any open set D, we use $\tau_D^m := \inf\{t > 0 : X_t^m \notin D\}$ to denote the first exit time from D by X^m, and $X^{m,D}$ to denote the subprocess of X^m killed upon exiting D (or, the killed relativistic stable process in D with mass m). We will use $p_D^m(t, x, y)$ to denote the transition density of $X^{m,D}$.

Outline

Theorem [CKS1]

Suppose that D is a $C^{1,1}$ open set. (i) For any $m \in (0, M]$ and $(t, x, y) \in (0, T] \times D \times D$,

$$\frac{1}{C_1}\left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha} \wedge \frac{t\phi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}}\right)$$
$$\leq p_D^m(t,x,y) \leq$$
$$C_1\left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha} \wedge \frac{t\phi(m^{1/\alpha}|x-y|/(16))}{|x-y|^{d+\alpha}}\right),$$

where $\phi(r) = e^{-r}(1 + r^{(d+\alpha-1)/2})$.
(ii) Suppose in addition that D is bounded. for any $m \in (0, M]$ and $(t, x, y) \in [T, \infty) \times D \times D$,

$$p_D^m(t,x,y) \asymp e^{-t\,\lambda_1^{\alpha,m,D}}\,\delta_D(x)^{\alpha/2}\,\delta_D(y)^{\alpha/2},$$

where $\lambda_1^{\alpha,m,D} > 0$ is the smallest eigenvalue of the restriction of $(m^{2/\alpha} - \Delta)^{\alpha/2} - m$ in D with zero exterior condition.

Our estimates are uniform in m in the sense that the constants are independent of $m \in (0, M]$. Letting $m \downarrow 0$ recovers the below sharp heat kernel estimates for symmetric α-stable processes obtained in [CKS, JEMS10].

Difficulties and Ingredients

- two-sided estimates on p^m.
- the approximate scaling property
- the Lévy density of X^m does not have a simple form and has exponential decay rate as oppose to the polynomial decay rate of the Lévy density of symmetric stable process
- uniform Boundary Harnack principle and parabolic Harnack principle
- There exist positive constants R_0 and $C > 1$ depending only on d and α such that for any $m \in (0, \infty)$, any ball B of radius $r \le R_0 m^{-1/\alpha}$,

$$C^{-1} G_B(x, y) \le G_B^m(x, y) \le C G_B(x, y), \quad x, y \in B.$$

Outline

Stable processes and relativistic stable processes are all examples of subordinate Brownian motions.

Suppose that $S = \{S_t : t \geq 0\}$ is a subordinator with Laplace exponent ϕ:

$$\mathbb{E}e^{-\lambda S_t} = e^{-t\phi(\lambda)} \qquad t, \lambda > 0.$$

Suppose that $B = \{B_t : t \geq 0\}$ is d-dimensional Brownian motion independent of the subordinator S. Then the process $X = \{X_t : t \geq 0\}$ defined by $X_t = B_{S_t}$ is called a subordinate Brownian motion and it is a rotationally symmetric Lévy process.

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In this section we will always assume that ϕ is a complete Bernstein function, that is, the Lévy measure μ has a density $\mu(t)$ which is completely monotone ($(-1)^n\mu^{(n)}(t) \ge 0$ for $n = 1, 2, \dots$). X has a transition density $p(t, x, y)$ with respect to the Lebesgue measure.

For any open set $D \subset \mathbb{R}^d$, we will use X^D to denote the process obtained from X by killing it upon exiting from D. The process X^D has a transition density $p^D(t, x, y)$ with respect to the Lebesgue measure on D. The main results of this section are sharp two-sided estimates on $p^D(t, x, y)$.

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For the first result in this section, we will assume the following **(H):** There exist constants $\delta_1, \delta_2 \in (0,1)$, $a_1, a_2 > 0$ and $R_0 > 0$ such that

$$\text{(LSC)} \qquad \phi(\lambda r) \ge a_1 \lambda^{\delta_1} \phi(r), \qquad \lambda \ge 1, r \ge 1/R_0^2$$
$$\text{(USC)} \qquad \phi(\lambda r) \le a_2 \lambda^{\delta_2} \phi(r), \qquad \lambda \ge 1, r \ge 1/R_0^2.$$

Note that it follows from (USC) that ϕ has no drift.

Definition

Suppose $R > 0$ and $\kappa \in (0,1)$. An open set $D \subset \mathbb{R}^d$ is called κ-fat if there is $R > 0$ such that for every $x \in \overline{D}$ and all $r \in (0, R]$, $D \cap B(x,r)$ contains a ball of radius κr. The pair (R, κ) is called the characteristics of the κ-fat open set D.

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Theorem

Suppose that D is a bounded κ-fat open set in $\mathbb{R}^d$.

- For every $T > 0$, there exist $c_i = c_i(R, \kappa, T, d, \phi) > 1, i = 1, 2$ such that for $0 < t \le T, x, y \in \mathbb{R}^d$,

  $$p_D(t, x, y) \le c_1 \mathbb{P}_x(\tau_D > t)\mathbb{P}_y(\tau_D > t)p(t, c_2^{-1}x, c_2^{-1}y)$$

  and

  $$c_1^{-1}\mathbb{P}_x(\tau_D > t)\mathbb{P}_y(\tau_D > t)p(t, c_2 x, c_2 y) \le p_D(t, x, y).$$

- For every $T > 0$, there is a constant $c_3 \ge 1$ depending only on $\text{diam}(D), T, R, \kappa, d$ and ϕ so that for all $(t, x, y) \in [T, \infty) \times D \times D$,

  $$\begin{aligned} p_D(t, x, y) &\ge c_3^{-1}\, \mathbb{P}_x(\tau_D > 1)\mathbb{P}_y(\tau_D > 1)e^{-t\lambda_1} \\ p_D(t, x, y) &\le c_3\, \mathbb{P}_x(\tau_D > 1)\mathbb{P}_y(\tau_D > 1)e^{-t\lambda_1}, \end{aligned}$$

  where $-\lambda_1 < 0$ is the largest eigenvalue of the generator of X^D.

We can actually prove the first part of the above theorem without the boundedness assumption on D, but with a little extra condition on ϕ which we think is not necessary.

In the above theorem, we do not really need X to be a subordinate Brownian motion. What we really need is that X is a rotationally symmetric, purely discontinuous Lévy process whose Lévy density is comparable to that of a subordinate Brownian motion satisfying the assumptions of the above theorem.

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Theorem ($C^{1.1}$ case)

Suppose that

$$\phi(\lambda) \asymp \lambda^{\alpha/2}\ell(\lambda), \qquad \lambda \to \infty,$$

where $\alpha \in (0,2)$ and ℓ is a positive function which is slowing varying at infinity. Let D be a bounded $C^{1,1}$ open subset of $\mathbb{R}^d$ with characteristics (R_0, Λ_0).
(1) For every $T > 0$, there exist $c_j = c_j(R_0, \Lambda_0, T, d, \alpha, \ell) \geq 1$, $j = 1, 2$, such that for all $(t, x, y) \in (0, T] \times D \times D$,

$$c_1^{-1}\left(1 \wedge \frac{1}{t\phi(\delta_D^{-2}(x))}\right)^{1/2}\left(1 \wedge \frac{1}{t\phi(\delta_D^{-2}(y))}\right)^{1/2}\left(\frac{1}{(\Phi^{-1}(t))^d} \wedge tJ(c_2x, c_2y)\right)$$
$$\leq p_D(t,x,y) \leq$$
$$c_1\left(1 \wedge \frac{1}{t\phi(\delta_D^{-2}(x))}\right)^{1/2}\left(1 \wedge \frac{1}{t\phi(\delta_D^{-2}(y))}\right)^{1/2}\left(\frac{1}{(\Phi^{-1}(t))^d} \wedge tJ(x/C_2, y/c_2)\right),$$

where $\Phi(r) = \frac{1}{\phi(r^{-2})}$.

Theorem (Cont)

(2) For every $T > 0$, there is a constant $c_3 \geq 1$ depending only on $\mathrm{diam}(D), R_0, \Lambda_0, d, \alpha, \ell$ and T so that for all $(t, x, y) \in [T, \infty) \times D \times D$,

$$
\begin{aligned}
p_D(t,x,y) &\geq c_3^{-1} e^{-\lambda_1 t} \frac{1}{\sqrt{\phi(\delta_D^{-2}(x))}} \frac{1}{\sqrt{\phi(\delta_D^{-2}(y))}} \\
p_D(t,x,y) &\leq c_3 e^{-\lambda_1 t} \frac{1}{\sqrt{\phi(\delta_D^{-2}(x))}} \frac{1}{\sqrt{\phi(\delta_D^{-2}(y))}},
\end{aligned}
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where $-\lambda_1 < 0$ is the largest eigenvalue of the generator of X^D.

Again, we get get rid of the boundedness assumption on D if we can assume a little extra condition ϕ which we think is not necessary. We can also deal with the case when the subordinate Brownian motion X has a Gaussian component when D is a $C^{1,1}$ open set.

Theorem (Cont)

(2) For every $T > 0$, there is a constant $c_3 \geq 1$ depending only on diam$(D), R_0, \Lambda_0, d, \alpha, \ell$ and T so that for all $(t, x, y) \in [T, \infty) \times D \times D$,

$$
\begin{aligned}
p_D(t,x,y) &\geq c_3^{-1} e^{-\lambda_1 t} \frac{1}{\sqrt{\phi(\delta_D^{-2}(x))}} \frac{1}{\sqrt{\phi(\delta_D^{-2}(y))}} \\
p_D(t,x,y) &\leq c_3 e^{-\lambda_1 t} \frac{1}{\sqrt{\phi(\delta_D^{-2}(x))}} \frac{1}{\sqrt{\phi(\delta_D^{-2}(y))}},
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Outline

A half-space is any set which, after isometry, can be written as $\{(x_1, \dots, x_d) : x_d > 0\}$.

An open set D is said to be half-space-like if, after isometry, $H_a \subset D \subset H_b$ for some real numbers $a > b$. Here for any real number a, $H_a := \{(x_1, \dots, x_d) : x_d > a\}$. H_0 will be simply written as H.

For any $m, c > 0$, define

$$\Psi_{d,\alpha,m,c}(t,x,y) := \begin{cases} t^{-d/\alpha} \wedge \frac{t\phi(c^{-1}m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}} & t \in (0, 1/m], \\ m^{d/\alpha-d/2}t^{-d/2}\exp\left(-c^{-1}(m^{1/\alpha}|x-y| \wedge m^{2/\alpha-1}\frac{|x-y|^2}{t})\right) & t \in (1/m, \infty), \end{cases}$$

where $\phi(r) = e^{-r}\left(1 + r^{(d+\alpha-1)/2}\right)$.

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Theorem [CKS2]

Suppose D is a half-space-like $C^{1,1}$ open set. For any $M > 0$, there exist $C_i > 1 \geq 1$, $i = 1, 2$, such that for all $m \in (0, M]$,

(i) if $t \in (0, 1/m]$

$$C_1^{-1} \left(\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \left(\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \Psi_{d,\alpha,m,C_2}(t,x,y) \leq p_D^m(t,x,y)$$
$$\leq C_1 \left(\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \left(\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \Psi_{d,\alpha,m,1/C_2}(t,x,y)$$

(ii) if $t > 1/m$

$$C_1^{-1} \left(\frac{m^{(2-\alpha)/2\alpha} \delta_D(x) + \delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \left(\frac{m^{(2-\alpha)/2\alpha} \delta_D(y) + \delta_D(y)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right)$$
$$\times \Psi_{d,\alpha,m,C_2}(t,x,y)$$
$$\leq p_D^m(t,x,y) \leq$$
$$C_1 \left(\frac{m^{(2-\alpha)/2\alpha} \delta_D(x) + \delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \left(\frac{m^{(2-\alpha)/2\alpha} \delta_D(y) + \delta_D(y)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right)$$
$$\times \Psi_{d,\alpha,m,1/C_2}(t,x,y).$$

These estimates are new even when D is the upper half space H. Observe that although H is invariant under scaling, global two-sided estimates on $p_H^m(t,x,y)$ can not be derived through a scaling argument from the short time estimates which hold only for $m \in (0, M]$ and $t \in (0, T]$.

For a fixed half-space-like $C^{1,1}$ open set D with $C^{1,1}$ characteristics (R, Λ_0) and $H_a \subset D \subset H_b$, mD is still a half-space-like $C^{1,1}$ open set but with $C^{1,1}$-characteristics $(mR, \Lambda_0/m)$ and $H_{ma} \subset mD \subset H_{mb}$. So we can not use the scaling property

$$p_D^m(t,x,y) = m^{d/\alpha} p^1_{m^{1/\alpha}D}(mt, m^{1/\alpha}x, m^{1/\alpha}y)$$

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A major part is to derive global sharp two-sided heat kernel estimates for X^m in a half-space.

Then, we use the push-inward technique developed in [Chen-Tokle, PTRF 2011] to extend it to half-space-like $C^{1,1}$ open sets.

Passing $m \downarrow 0$ recovers the sharp heat kernel estimates for symmetric α-stable processes which were previously obtained in [Chen-Tokle, PTRF 2011].

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Outline

An open set D in $\mathbb{R}^d$ is called an exterior open set if D^c is compact.

Theorem [CKS3]

Suppose that $d \geq 3$, $M > 0$ and D is an exterior $C^{1,1}$ open set in $\mathbb{R}^d$. Then there are constants $c_i > 1, i = 1, 2$, such that for every $m \in (0, M]$, $t > 0$ and $(x, y) \in D \times D$,

$$p_D^m(t,x,y) \leq c_1 \left(1 \wedge \frac{\delta_D(x)}{1 \wedge t^{1/\alpha}}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{1 \wedge t^{1/\alpha}}\right)^{\alpha/2} \Psi_{d,\alpha,m,c_2}(t,x,y)$$

and

$$p_D^m(t,x,y) \geq c_1^{-1} \left(1 \wedge \frac{\delta_D(x)}{1 \wedge t^{1/\alpha}}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{1 \wedge t^{1/\alpha}}\right)^{\alpha/2} \Psi_{d,\alpha,m,1/c_2}(t,x,y).$$

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The reason that we assume $d \geq 3$ is that we used the transience of X^m. By Chung-Fuch's criterion for Lévy processes, X^m is transient if and only if $d \geq 3$.

The large time upper bound is relatively easy to establish. The main difficulty is in establishing the large time lower bound.

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Thank you!