Dirichlet Heat Kernel Estimates for Relativistic Stable Processes

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6th International Conference on Stochastic Analysis and Its Applications
This talk is based on the following joint papers with Zhen-Qing Chen and Panki Kim.


**CKS5** Dirichlet heat kernel estimates for subordinate Brownian motions with Gaussian components. In preparation.
Outline
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1. Introduction
2. Relativistic stable processes
3. Estimates in General smooth open sets
4. Dirichlet heat kernel estimates for subordinate Brownian motions
5. Dirichlet Heat kernel estimates in half-space-like open set
6. Dirichlet Heat kernel estimates in exterior open set
Suppose that $X$ is a symmetric Markov process on (a subset of) $\mathbb{R}^d$ with transition density $p(t, x, y)$ and generator $\mathcal{L}$. $p(t, x, y)$ is also the fundamental solution of $\partial_t u = \mathcal{L}u$ and so it is also called the heat kernel of $\mathcal{L}$. In general, there is no explicit formula for $p(t, x, y)$. Thus establishing sharp two-sided estimates for $p(t, x, y)$ is a fundamental problem.

Two-sided heat kernel estimates for diffusions in $\mathbb{R}^d$ have a long history and many beautiful results have been established. Among the main contributors are: D. G. Aronson, J. Nash, E. B. Davies.

Due to the complication near the boundary, two-sided estimates on the transition density of killed diffusions in a domain $D$ (equivalently, the Dirichlet heat kernel) have been established only recently. See Davies (87), Zhang (02) for the case of bounded $C^{1,1}$ domains.
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The infinitesimal generator of a discontinuous Markov process in $\mathbb{R}^d$ is no longer a differential operator but rather a non-local (or integro-differential) operator $\mathcal{L}$. For instance, the infinitesimal generator of a rotationally symmetric $\alpha$-stable process in $\mathbb{R}^d$ with $\alpha \in (0, 2)$ is a fractional Laplacian operator $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$.

Recently in [CKS10, JEMS], we obtained sharp two-sided estimates for the heat kernel of the fractional Laplacian $\Delta^{\alpha/2}$ in $D$ with zero exterior condition (or equivalently, the transition density function of the symmetric $\alpha$-stable process killed upon exiting $D$) for any $C^{1,1}$ open set $D \subset \mathbb{R}^d$ with $d \geq 1$. As far as we know, this was the first time sharp two-sided estimates were established for Dirichlet heat kernels of non-local operators.
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Since then, studies on this topic have been growing rapidly. The ideas of [CKS10, JEMS] have been adapted to establish two-sided heat kernel estimates of other discontinuous Markov processes, like censored stable processes [CKS10, PTRF] in open subsets of $\mathbb{R}^d$.

In this talk, I will present sharp two-sided estimates on the Dirichlet heat kernels of relativistic stable processes in $C^{1,1}$ domains of $\mathbb{R}^d$. 
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2 Relativistic stable processes

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For $\alpha \in (0, 2]$, a symmetric $\alpha$-stable process $X$ on $\mathbb{R}^d$ is a Lévy process such that for any $t \geq 0$ and $\xi \in \mathbb{R}^d$

$$
\mathbb{E} \left[ \exp \left( i \xi \cdot (X_t - X_0) \right) \right] = \exp \left( -t|\xi|^\alpha \right).
$$

When $\alpha = 2$, it reduces to a Brownian motion.

The infinitesimal generator of a symmetric $\alpha$-stable process $Y$ in $\mathbb{R}^d$ is the fractional Laplacian $\Delta^{\alpha/2}$, which can be written as

$$
\Delta^{\alpha/2} u(x) = \lim_{\epsilon \downarrow 0} \int_{\{y \in \mathbb{R}^d: |y-x| > \epsilon\}} (u(y) - u(x)) \frac{A(d, \alpha)}{|x-y|^{d+\alpha}} \, dy,
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where $A(d, \alpha) := \alpha 2^{\alpha-1} \pi^{-d/2} \Gamma \left( \frac{d+\alpha}{2} \right) \Gamma \left( 1 - \frac{\alpha}{2} \right)^{-1}$. 


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Let \( p(t, x, y) \) be the transition density of \( X \). When \( d > \alpha \), the potential density (also called the Green function) of \( X \)

\[
G(x, y) = \int_0^\infty p(t, x, y)dt = C(d, \alpha) \frac{1}{|x - y|^{d-\alpha}}
\]

which is the Riesz kernel.

Symmetric stable processes have some nice properties. For example it satisfies the following scaling property: For any \( a > 0 \), \( \{a^{-1/\alpha}(X_{at} - X_0) : t \geq 0\} \) has the same law as \( \{X_t - X_0 : t \geq 0\} \). In terms of the transition density, this means

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p(t, x, y) = a^d p(at, a^{1/\alpha} x, a^{1/\alpha} y).
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However, a symmetric \( \alpha \)-stable process, for \( \alpha \in (0, 2) \), always have infinite variance. When \( \alpha \in (0, 1] \), it also have infinite mean.
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For any $m \geq 0$, a relativistic $\alpha$-stable process $X^m$ on $\mathbb{R}^d$ with weight $m$ is a Lévy process such that for any $t \geq 0$ and $\xi \in \mathbb{R}^d$

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\mathbb{E} [\exp (i\xi \cdot (X^m_t - X^m_0))] = \exp \left( -t \left( (|\xi|^2 + m^{2/\alpha})^{\alpha/2} - m \right) \right).
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When $m = 0$, $X^0$ is simply a (rotationally) symmetric $\alpha$-stable process on $\mathbb{R}^d$. The infinitesimal generator of $X^m$ is

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When $\alpha = 1$, the infinitesimal generator reduces to

$$m - \sqrt{-\Delta} + m^2.$$ 

This operator was used by E. Lieb and his followers in studying the stability of matter.

Let $p^m(t, x, y)$ be the transition density of $X^m$. The function, called the 1-potential density of $X^1$:

$$\int_0^\infty e^{-t} p^1(t, x, y) dt$$

is the Bessel kernel.
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The Lévy measure of $X^m$ has a density

$$J^m(x, y) = A(d, \alpha)|x - y|^{-d-\alpha}\psi(m^{1/\alpha}|x - y|)$$

where

$$\psi(r) := \int_0^\infty s^{d+\alpha-1}e^{-\frac{s}{4}-\frac{r^2}{s}}ds,$$

which is decreasing and is a smooth function of $r^2$ satisfying $\psi(r) \leq 1$ and

$$\psi(r) \asymp \phi(r) := e^{-r}(1 + r^{(d+\alpha-1)/2}) \quad \text{on } [0, \infty).$$

For $m > 0$, $X^m$ has moments of all orders, and it even has some exponential moments. In a small scale, $X^m$ behaves like $X^0$, while in a larger scale, $X^m$ behaves like Brownian motion.
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For $m > 0$, $X^m$ does not satisfy any scaling invariance property. However, it does satisfy some sort of approximate scaling property.

Two-sided estimates on $p(t, x, y)$ is classical. But two-sided estimates on $p^m(t, x, y)$ is more recent.
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For any $m, c > 0$, we define a function $\widetilde{\Psi}_{d, \alpha, m, c}(t, x, y)$ on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ by

$$
\begin{align*}
\widetilde{\Psi}_{d, \alpha, m, c}(t, x, y) :=
\begin{cases}
t^{-d/\alpha} \land t J^m(x, y), & \forall t \in (0, 1/m]; \\
m^{d/\alpha - d/2} t^{-d/2} \exp \left(-c^{-1}(m^{1/\alpha} |x - y| \land m^{2/\alpha - 1} \frac{|x - y|^2}{t})\right), & \forall t \in (1/m, \infty).
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**Theorem [Chen-Kim-Kumagai], [CKS1]**

$$
c_1^{-1} \widetilde{\Psi}_{d, \alpha, m, 1/c_1}(t, x, y) \leq p^m(t, x, y) \leq c_1 \widetilde{\Psi}_{d, \alpha, m, c_1}(t, x, y).
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$$
Relativistic stable process in $D$

For any open set $D$, we use $\tau^m_D := \inf\{ t > 0 : X^m_t \notin D \}$ to denote the first exit time from $D$ by $X^m$, and $X^{m,D}$ to denote the subprocess of $X^m$ killed upon exiting $D$ (or, the killed relativistic stable process in $D$ with mass $m$). We will use $p^m_D(t, x, y)$ to denote the transition density of $X^{m,D}$. 
Theorem [CKS1]

Suppose that $D$ is a $C^{1,1}$ open set. (i) For any $m \in (0, M]$ and $(t, x, y) \in (0, T] \times D \times D$,

$$
\frac{1}{C_1} \left( 1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left( t^{-d/\alpha} \wedge \frac{t\phi(m_1^{1/\alpha} |x - y|)}{|x - y|^{d+\alpha}} \right) 
\leq p_D^m(t, x, y) \leq C_1 \left( 1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left( t^{-d/\alpha} \wedge \frac{t\phi(m_1^{1/\alpha} |x - y|/16)}{|x - y|^{d+\alpha}} \right),
$$

where $\phi(r) = e^{-r}(1 + r^{(d+\alpha-1)/2})$.

(ii) Suppose in addition that $D$ is bounded. for any $m \in (0, M]$ and $(t, x, y) \in [T, \infty) \times D \times D$,

$$
p_D^m(t, x, y) \asymp e^{-t \lambda_1^{\alpha, m, D}} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2},
$$

where $\lambda_1^{\alpha, m, D} > 0$ is the smallest eigenvalue of the restriction of $(m^2/\alpha - \Delta)^{\alpha/2} - m$ in $D$ with zero exterior condition.
Our estimates are uniform in $m$ in the sense that the constants are independent of $m \in (0, M]$. Letting $m \downarrow 0$ recovers the below sharp heat kernel estimates for symmetric $\alpha$-stable processes obtained in [CKS, JEMS10].
Difficulties and Ingredients

- Two-sided estimates on $p^m$.
- The approximate scaling property.
- The Lévy density of $X^m$ does not have a simple form and has exponential decay rate as oppose to the polynomial decay rate of the Lévy density of symmetric stable process.
- Uniform Boundary Harnack principle and parabolic Harnack principle.

There exist positive constants $R_0$ and $C > 1$ depending only on $d$ and $\alpha$ such that for any $m \in (0, \infty)$, any ball $B$ of radius $r \leq R_0 m^{-1/\alpha}$,

$$C^{-1} G_B(x, y) \leq G^m_B(x, y) \leq C G_B(x, y), \quad x, y \in B.$$
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Stable processes and relativistic stable processes are all examples of subordinate Brownian motions.

Suppose that \( S = \{S_t : t \geq 0\} \) is a subordinator with Laplace exponent \( \phi \):
\[
\mathbb{E} e^{-\lambda S_t} = e^{-t \phi(\lambda)} \quad t, \lambda > 0.
\]

Suppose that \( B = \{B_t : t \geq 0\} \) is \( d \)-dimensional Brownian motion independent of the subordinator \( S \). Then the process \( X = \{X_t : t \geq 0\} \) defined by \( X_t = B_{S_t} \) is called a subordinate Brownian motion and it is a rotationally symmetric Lévy process.
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In this section we will always assume that $\phi$ is a complete Bernstein function, that is, the Lévy measure $\mu$ has a density $\mu(t)$ which is completely monotone ($(-1)^n \mu^{(n)}(t) \geq 0$ for $n = 1, 2, \ldots$). $X$ has a transition density $p(t, x, y)$ with respect to the Lebesgue measure.

For any open set $D \subset \mathbb{R}^d$, we will use $X^D$ to denote the process obtained from $X$ by killing it upon exiting from $D$. The process $X^D$ has a transition density $p^D(t, x, y)$ with respect to the Lebesgue measure on $D$. The main results of this section are sharp two-sided estimates on $p^D(t, x, y)$. 
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For the first result in this section, we will assume the following (H):
There exist constants $\delta_1, \delta_2 \in (0, 1)$, $a_1, a_2 > 0$ and $R_0 > 0$ such that

\[ (\text{LSC}) \quad \phi(\lambda r) \geq a_1 \lambda^{\delta_1} \phi(r), \quad \lambda \geq 1, \ r \geq 1/R_0^2 \]
\[ (\text{USC}) \quad \phi(\lambda r) \leq a_2 \lambda^{\delta_2} \phi(r), \quad \lambda \geq 1, \ r \geq 1/R_0^2. \]

Note that it follows from (USC) that $\phi$ has no drift.

**Definition**

Suppose $R > 0$ and $\kappa \in (0, 1)$. An open set $D \subset \mathbb{R}^d$ is called $\kappa$-fat if there is $R > 0$ such that for every $x \in \overline{D}$ and all $r \in (0, R]$, $D \cap B(x, r)$ contains a ball of radius $\kappa r$. The pair $(R, \kappa)$ is called the characteristics of the $\kappa$-fat open set $D$. 
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Theorem

Suppose that \( D \) is a bounded \( \kappa \)-fat open set in \( \mathbb{R}^d \).

For every \( T > 0 \), there exist \( c_i = c_i(R, \kappa, T, d, \phi) > 1, i = 1, 2 \) such that for \( 0 < t \leq T, x, y \in \mathbb{R}^d \),

\[
p_D(t, x, y) \leq c_1 \mathbb{P}_x(\tau_D > t) \mathbb{P}_y(\tau_D > t)p(t, c_2^{-1} x, c_2^{-1} y)
\]

and

\[
c_1^{-1} \mathbb{P}_x(\tau_D > t) \mathbb{P}_y(\tau_D > t)p(t, c_2 x, c_2 y) \leq p_D(t, x, y).
\]

For every \( T > 0 \), there is a constant \( c_3 \geq 1 \) depending only on \( \text{diam}(D), T, R, \kappa, d \) and \( \phi \) so that for all \( (t, x, y) \in [T, \infty) \times D \times D \),

\[
p_D(t, x, y) \geq c_3^{-1} \mathbb{P}_x(\tau_D > 1) \mathbb{P}_y(\tau_D > 1)e^{-t\lambda_1}
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\]

where \(-\lambda_1 < 0\) is the largest eigenvalue of the generator of \( X^D \).
We can actually prove the first part of the above theorem without the boundedness assumption on $D$, but with a little extra condition on $\phi$ which we think is not necessary.

In the above theorem, we do not really need $X$ to be a subordinate Brownian motion. What we really need is that $X$ is a rotationally symmetric, purely discontinuous Lévy process whose Lévy density is comparable to that of a subordinate Brownian motion satisfying the assumptions of the above theorem.
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In the above theorem, we do not really need $X$ to be a subordinate Brownian motion. What we really need is that $X$ is a rotationally symmetric, purely discontinuous Lévy process whose Lévy density is comparable to that of a subordinate Brownian motion satisfying the assumptions of the above theorem.
Theorem ($C^{1,1}$ case)

Suppose that

$$\phi(\lambda) \asymp \lambda^{\alpha/2} \ell(\lambda), \quad \lambda \to \infty,$$

where $\alpha \in (0, 2)$ and $\ell$ is a positive function which is slowing varying at infinity. Let $D$ be a bounded $C^{1,1}$ open subset of $\mathbb{R}^d$ with characteristics $(R_0, \Lambda_0)$.

(1) For every $T > 0$, there exist $c_j = c_j(R_0, \Lambda_0, T, d, \alpha, \ell) \geq 1$, $j = 1, 2$, such that for all $(t, x, y) \in (0, T] \times D \times D$,

$$c_1^{-1} \left(1 \wedge \frac{1}{t \phi(\delta_D^{-2}(x))} \right)^{1/2} \left(1 \wedge \frac{1}{t \phi(\delta_D^{-2}(y))} \right)^{1/2} \left(\frac{1}{(\Phi^{-1}(t))^d} \wedge tJ(c_2 x, c_2 y) \right) \leq p_D(t, x, y) \leq c_1 \left(1 \wedge \frac{1}{t \phi(\delta_D^{-2}(x))} \right)^{1/2} \left(1 \wedge \frac{1}{t \phi(\delta_D^{-2}(y))} \right)^{1/2} \left(\frac{1}{(\Phi^{-1}(t))^d} \wedge tJ(x/C_2, y/C_2) \right),$$

where $\Phi(r) = \frac{1}{\phi(r^{-2})}$. 

Theorem (Cont)

(2) For every $T > 0$, there is a constant $c_3 \geq 1$ depending only on $\text{diam}(D), R_0, \Lambda_0, d, \alpha, \ell$ and $T$ so that for all $(t, x, y) \in [T, \infty) \times D \times D$,

$$p_D(t, x, y) \geq c_3^{-1} e^{-\lambda_1 t} \frac{1}{\sqrt{\phi(\delta_D^{-2}(x))}} \frac{1}{\sqrt{\phi(\delta_D^{-2}(y))}}$$

$$p_D(t, x, y) \leq c_3 e^{-\lambda_1 t} \frac{1}{\sqrt{\phi(\delta_D^{-2}(x))}} \frac{1}{\sqrt{\phi(\delta_D^{-2}(y))}},$$

where $-\lambda_1 < 0$ is the largest eigenvalue of the generator of $X^D$.

Again, we get get rid of the boundedness assumption on $D$ if we can assume a little extra condition $\phi$ which we think is not necessary. We can also deal with the case when the subordinate Brownian motion $X$ has a Gaussian component when $D$ is a $C^{1,1}$ open set.
Theorem (Cont)

(2) For every $T > 0$, there is a constant $c_3 \geq 1$ depending only on $\text{diam}(D)$, $R_0$, $\Lambda_0$, $d$, $\alpha$, $\ell$ and $T$ so that for all $(t, x, y) \in [T, \infty) \times D \times D$,

$$\begin{align*}
p_D(t, x, y) & \geq c_3^{-1} e^{-\lambda_1 t} \frac{1}{\sqrt{\phi(\delta_D^{-2}(x))}} \frac{1}{\sqrt{\phi(\delta_D^{-2}(y))}} \\
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\end{align*}$$

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Outline

1. Introduction
2. Relativistic stable processes
3. Estimates in General smooth open sets
4. Dirichlet heat kernel estimates for subordinate Brownian motions
5. Dirichlet Heat kernel estimates in half-space-like open set
6. Dirichlet Heat kernel estimates in exterior open set
A half-space is any set which, after isometry, can be written as \( \{(x_1, \ldots, x_d) : x_d > 0\} \).

An open set \( D \) is said to be half-space-like if, after isometry, \( H_a \subset D \subset H_b \) for some real numbers \( a > b \). Here for any real number \( a \), \( H_a := \{(x_1, \ldots, x_d) : x_d > a\} \). \( H_0 \) will be simply written as \( H \).

For any \( m, c > 0 \), define

\[
\Psi_{d,\alpha,m,c}(t, x, y) := \begin{cases} 
    t^{-d/\alpha} \wedge \frac{t\phi(c^{-1}m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}} & t \in (0, 1/m], \\
    m^{d/\alpha-d/2}t^{-d/2} \exp \left(-c^{-1}(m^{1/\alpha}|x-y| \wedge m^{2/\alpha-1}|x-y|^2) \right) & t \in (1/m, \infty),
\end{cases}
\]

where \( \phi(r) = e^{-r} \left(1 + r^{(d+\alpha-1)/2}\right) \).
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\end{cases}
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where \( \phi(r) = e^{-r} \left(1 + r^{(d+\alpha-1)/2}\right) \).
Theorem [CKS2]

Suppose $D$ is a half-space-like $C^{1,1}$ open set. For any $M > 0$, there exist $C_i > 1 \geq 1$, $i = 1, 2$, such that for all $m \in (0, M]$, 

(i) if $t \in (0, 1/m]$

$$C_1^{-1} \left( \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \left( \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \Psi_{d,\alpha,m,2}(t, x, y) \leq p^m_D(t, x, y)$$

$$\leq C_1 \left( \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \left( \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \Psi_{d,\alpha,m,1/C_2}(t, x, y)$$

(ii) if $t > 1/m$

$$C_1^{-1} \left( \frac{(2-\alpha)/2\alpha \delta_D(x) + \delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \left( \frac{(2-\alpha)/2\alpha \delta_D(y) + \delta_D(y)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \Psi_{d,\alpha,m,2}(t, x, y)$$

$$\leq p^m_D(t, x, y) \leq C_1 \left( \frac{(2-\alpha)/2\alpha \delta_D(x) + \delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \left( \frac{(2-\alpha)/2\alpha \delta_D(y) + \delta_D(y)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \times \Psi_{d,\alpha,m,1/C_2}(t, x, y).$$
These estimates are new even when $D$ is the upper half space $H$. Observe that although $H$ is invariant under scaling, global two-sided estimates on $p^m_H(t, x, y)$ can not be derived through a scaling argument from the short time estimates which hold only for $m \in (0, M]$ and $t \in (0, T]$.

For a fixed half-space-like $C^{1,1}$ open set $D$ with $C^{1,1}$ characteristics $(R, \Lambda_0)$ and $H_a \subset D \subset H_b$, $mD$ is still a half-space-like $C^{1,1}$ open set but with $C^{1,1}$-characteristics $(mR, \Lambda_0/m)$ and $H^m_a \subset mD \subset H^m_b$. So we can not use the scaling property

$$p^m_D(t, x, y) = m^{d/\alpha} p^1_{m^{1/\alpha}D}(mt, m^{1/\alpha}x, m^{1/\alpha}y)$$

to obtain sharp two-sided estimates for $p^m_D(t, x, y)$ that are uniform in $m \in (0, M]$ from that of $p^1_D(t, x, y)$. 
These estimates are new even when $D$ is the upper half space $H$. Observe that although $H$ is invariant under scaling, global two-sided estimates on $p_H^m(t, x, y)$ can not be derived through a scaling argument from the short time estimates which hold only for $m \in (0, M]$ and $t \in (0, T]$.

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$$p_D^m(t, x, y) = m^{d/\alpha} p_D^{1/\alpha}(mt, m^{1/\alpha}x, m^{1/\alpha}y)$$

to obtain sharp two-sided estimates for $p_D^m(t, x, y)$ that are uniform in $m \in (0, M]$ from that of $p_D^{1}(t, x, y)$.
A major part is to derive global sharp two-sided heat kernel estimates for $X^m$ in a half-space.

Then, we use the push-inward technique developed in [Chen-Tokle, PTRF 2011] to extend it to half-space-like $C^{1,1}$ open sets.

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An open set $D$ in $\mathbb{R}^d$ is called an exterior open set if $D^c$ is compact.

**Theorem [CKS3]**

Suppose that $d \geq 3$, $M > 0$ and $D$ is an exterior $C^{1,1}$ open set in $\mathbb{R}^d$. Then there are constants $c_i > 1$, $i = 1, 2$, such that for every $m \in (0, M]$, $t > 0$ and $(x, y) \in D \times D$,

$$p^m_D(t, x, y) \leq c_1 \left( 1 \wedge \frac{\delta_D(x)}{1 \wedge t^{1/\alpha}} \right)^{\alpha/2} \left( 1 \wedge \frac{\delta_D(y)}{1 \wedge t^{1/\alpha}} \right)^{\alpha/2} \Psi_{d, \alpha, m, c_2}(t, x, y)$$

and

$$p^m_D(t, x, y) \geq c_1^{-1} \left( 1 \wedge \frac{\delta_D(x)}{1 \wedge t^{1/\alpha}} \right)^{\alpha/2} \left( 1 \wedge \frac{\delta_D(y)}{1 \wedge t^{1/\alpha}} \right)^{\alpha/2} \Psi_{d, \alpha, m, 1/c_2}(t, x, y).$$
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$$p^m_D(t, x, y) \leq c_1 \left( 1 \wedge \frac{\delta_D(x)}{1 \wedge t^{1/\alpha}} \right)^{\alpha/2} \left( 1 \wedge \frac{\delta_D(y)}{1 \wedge t^{1/\alpha}} \right)^{\alpha/2} \psi_{d, \alpha, m, c_2}(t, x, y)$$

and

$$p^m_D(t, x, y) \geq c_1^{-1} \left( 1 \wedge \frac{\delta_D(x)}{1 \wedge t^{1/\alpha}} \right)^{\alpha/2} \left( 1 \wedge \frac{\delta_D(y)}{1 \wedge t^{1/\alpha}} \right)^{\alpha/2} \psi_{d, \alpha, m, 1/c_2}(t, x, y).$$
The reason that we assume $d \geq 3$ is that we used the transience of $X^m$. By Chung-Fuch’s criterion for Lévy processes, $X^m$ is transient if and only if $d \geq 3$.

The large time upper bound is relatively easy to establish. The main difficulty is in establishing the large time lower bound.
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Thank you!