Trace asymptotics

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Będlewo, September 2012

The spectral counting function for Dirichlet Laplacia in Domains $D \subset \mathbb{R}^d$, $d \geqslant 2$:

$$N_D(\lambda) = card\{\lambda_k | \lambda_k < \lambda\}$$

H. Weyl's (1912) law ($|D|$= volume of D):

$$\lim_{\lambda\to\infty} \lambda^{-d/2} N_D(\lambda) = \frac{|D|}{(4\pi)^{d/2}\Gamma(d/2+1)} = C_d|D|$$

Weyl 1913 conjecture, Ivri (Melrose) 1980 Theorem ($|\partial D|$= surface area):

$$N_D(\lambda) = C_d|D|\lambda^{d/2} - C_d'|\partial D|\lambda^{(d-1)/2} + o(\lambda^{(d-1)/2}), \quad \text{as} \quad \lambda \to \infty,$$

What type of smoothness on D? Smooth enough! Melrose says "I can do it for smooth manifolds with Lipschitz metrics" ($C^{1,1}$ domains)

Weyl two–term implies Trace two-term (small time)

$Z_t(D) = \int_0^\infty e^{-\lambda t} N_D(d\lambda) = \sum_{k=1}^\infty e^{-\lambda_k t} = \int_D p_t^D(x,x)\,dx$

$$\Rightarrow \; (4\pi t)^{d/2} Z_t(D) = |D| - \frac{\sqrt{\pi}}{2}\,|\partial D|\,t^{1/2} + o(t^{1/2}), \qquad t \downarrow 0$$

Kac (1951): Trace Asymtotics imply Weyl's Law. Via Karamata tauberian

μ is a measure on $[0,\infty)$, $\lim_{t\to 0} t^{\gamma}\int_0^{\infty} e^{-t\lambda}d\mu(\lambda) = A, \quad \gamma > 0$. Then $\lim_{a\to\infty} a^{-\gamma}\mu[0,a) = \frac{A}{\Gamma(\gamma+1)}$ **But not Ivri (not second term asymptotic)**

Question: Does Ivri hold for Lipschitz boundaries? Unknown.

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Small time two-term trace asymptotics have been proved for Lipschitz. Here is a little history: M. Kac (1966) for some Polygons in the plane (and conjectured the third term found by McKean-Singer), J. Brossard and R. Carmona (1986) for C^1 boundaries, M. van den Berg (1987) for $C^{1,1}$ boundaries and finally R. Brown 1993 for Lipschitz boundaries.

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Question 1: Does Ivri hold for fractional Laplacian, i.e., symmetric stable processes? Unknown.
Question 2: What about two terms trace asymtotics? Yes, same as Laplacian.

Remark: There are very recent two term eigenvalue Cesàro means asymptotics due to L. Geisinger and R. Frank. Such results are "between" Trace and Weyl

The rotationally symmetric α-process, $0 < \alpha < 2$, $X = \left\{X_t, t \geqslant 0, P_x, x \in \mathbb{R}^d\right\}$ is a Lévy process with $E_x(e^{i\xi\cdot(X_t - X_0)}) = e^{-t|\xi|^\alpha}, \quad x \in \mathbb{R}^d, \ \xi \in \mathbb{R}^d$.

Transition probabilities

$$p_t^{(\alpha)}(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-t|\xi|^\alpha} d\xi = \int_0^\infty \frac{1}{(4\pi s)^{d/2}} e^{\frac{-|x|^2}{4s}} \eta_t^{\alpha/2}(s)\, ds,$$

$\eta_t^{\alpha/2}(s)$ density for $\alpha/2$-stable subordinator.

$$\begin{aligned} p_t^{(\alpha)}(x) &= t^{-d/\alpha} p_1^{(\alpha)}(t^{-1/\alpha}x) \leqslant t^{-d/\alpha} p_1^{\alpha}(0) = \\ &= t^{-d/\alpha}\frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha} = p_t^{\alpha}(0) \ \ (= p_t^{\alpha}(0,0)) \end{aligned}$$

and

$$C_{\alpha,d}^{-1}\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \leqslant p_t^{(\alpha)}(x-y) \leqslant C_{\alpha,d}\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right),$$

From this, first order trace asymtotics follows (Blumenthal and Getoor 1959) and hence a Weyl's Law via Kac's tauberian argument.

Dirichlet heat kernel

$$\begin{aligned} p_t^{D,\alpha}(x,y) &= p_t^{\alpha}(x-y) - \mathbb{E}_x\left(\tau_D < t, p_{t-\tau_D}^{\alpha}(X(\tau_D),y)\right) \\ &= p_t^{\alpha}(x-y) - r_t(x,y). \end{aligned}$$

$$\begin{aligned} Z_t(D) &= \int_D p_t^{D,\alpha}(x,x)dx = \int_D p_t^{\alpha}(x-y)dx - \int_D r_t(x,x)dx \\ &= p_t(0)\,|D| - \int_D r_t^D(x,x)dx \end{aligned}$$

Need "little" Lemma

$$\lim_{t\to 0} t^{d/\alpha} \int_D r_t(x,x)dx = 0$$

Follows from above heat kernel bounds and

$$\int_D r_t(x,x)dx = \int_{\{x\in D:\ \delta_D(x)\geqslant t^{1/2\alpha}\}} r_t(x,x)dx + \int_{\{x\in D:\ \delta_D(x)< t^{1/2\alpha}\}} r_t(x,x)dx$$

Definition (R-Smooth or $C^{1,1}$ domains)

$D \subset \mathbb{R}^d$ is *R-smooth* if $\forall\ x_0 \in \partial D$ there are two open balls B_1 and B_2 with radii R such that $B_1 \subset D, \quad B_2 \subset \mathbb{R}^d \setminus (D \cup \partial D)$ and $\partial B_1 \cap \partial B_2 = x_0$.

R.B. T. Kulczycki '08, R-smooth domains. Same as van den Berg for $\alpha = 2$

$$\left| t^{d/\alpha} Z_D(t) - C_1(\alpha, d)\,|D| + C_2(\alpha, d)\,|\partial D| t^{1/\alpha} \right| \leqslant \frac{C_3\,|D| t^{2/\alpha}}{R^2}, \quad t > 0.$$

Same as

$$\left| \frac{Z_D(t)}{p_t^\alpha(0)} - |D| + C_2'(\alpha, d)\,|\partial D| t^{1/\alpha} \right| \leqslant \frac{C_3'\,|D| t^{2/\alpha}}{R^2}, \quad t > 0.$$

R.B. T. Kulczycki, B. Siudeja '09 for Lipschitz domains. Same as Brown for $\alpha = 2$

$$\frac{Z_D(t)}{p_t^\alpha(0)} = |D| - C_2{}'(\alpha, d)\,|\partial D| t^{1/\alpha} + o\left(t^{1/\alpha}\right), \quad t \downarrow 0$$

First result holds for relativistic and mixed stables. R.B., J. Mijena, N. Erkan '12

$V \in \mathcal{S}(\mathbb{R}^d)$, "rapidly decreasing" Schwartz function. $H = -\Delta + V$ and $H_0 = \Delta$.

$$\begin{aligned} Tr\left(e^{-tH} - e^{-tH_0}\right) &= \int_{\mathbb{R}^d} \left(p_t^V(x,x) - p_t(x,x)\right) dx \\ &= \frac{1}{(4\pi t)^{d/2}} \left[\sum_{m=1}^{N} C_m(V) t^m + O(t^{N+1})\right], \quad \text{as} \quad t \downarrow 0. \end{aligned}$$

1. $d = 1$, McKean-Moerbeke (1975) give (via KdV) a formula for C_k.
2. $d = 3$, Colin de Verdière (1981) used McKean-Moerbeke techniques and symmetry of some integrals to compute C_1, C_2, C_3, C_4.
3. For all $d \geqslant 1$, R.B.-A. Sá Barreto (1995) a formula is given for C_k in terms of Fourier transforms and C_1, C_2, C_3, C_4, C_5 are computed.

$$(-1)^1 C_1(V) = \int_{\mathbb{R}^d} V(x) dx, \quad (-1)^2 C_2(V) = \frac{1}{2!} \int_{\mathbb{R}^d} V^2(x) dx$$

$$(-1)^3 C_3(V) = \frac{1}{3!} \int_{\mathbb{R}^d} \left(V^3(x) dx + \frac{1}{2} |\nabla V(x)|^2\right) dx$$

$$(-1)^4 C_4(V) = \frac{1}{4!}\int_{\mathbb{R}^d}\left(V^4(x)dx + 2V(x)|\nabla V(x)|^2 + \frac{1}{5}(\Delta V(x))^2\right)dx$$

$$\begin{aligned}(-1)^5 C_5(V) &= \frac{1}{5!}\int_{\mathbb{R}^d}\Big\{V^5(x)dx + \frac{3}{12}|\nabla\Delta V(x)|^2 + 5V^2(x)|\nabla V(x)|^2 \\ &+ \frac{15}{27}V(x)(\Delta V(x))^2 + \frac{4}{9}V(x)\left(\sum_{i,j=1}^{d}\partial_i\partial_j V(x)\right)^2\Big\}dx\end{aligned}$$

$V \geqslant 0 \Rightarrow (-1)^m C_m(V) \geqslant 0, \quad m = 1, 2, 3, 4, 5.$

In Fact: If $\widehat{V} \geqslant 0$ (Fourier transform), then $(-1)^m C_m(V) \geqslant 0$, for all $m \geqslant 1$. Further, under $\widehat{V} \geqslant 0$, if $C_m(V) = 0$ for some $m \geqslant 2$, then $V = 0$.

Main application to scattering theory for $H = -\Delta + V$ ("scattering by potentials"): Under "these assumptions" meromorphic extension of $(H - \lambda^2)^{-1}$ has infinitely many poles. (Precise statement in Theorem 4.1 ins RB & Sá Barreto 1995)

M. van den Berg (1991)–For $\alpha = 2$

$V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \cap$ Hölder $0 < \gamma \leqslant 1$: $|V(x) - V(y)| \leqslant M|x-y|^\gamma$

$$Tr(e^{-tH} - e^{-tH_0}) = \frac{1}{(4\pi t)^{d/2}} \left(-t \int_{\mathbb{R}^d} V(x)dx + \frac{1}{2} t^2 \int_{\mathbb{R}^d} |V(x)|^2 dx + \mathcal{O}(t^3) \right)$$

R.B. S. Yolcu (2011): $H = \Delta^{\alpha/2} + V$, $H_0 = \Delta^{\alpha/2}$, $0 < \alpha \leqslant 2$.

$V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \cap$ Hölder, $0 < \gamma \leqslant 1$ and $\gamma < \alpha$.

$$\left| \frac{\left(Tr(e^{-tH} - e^{-tH_0})\right)}{p_t^{(\alpha)}(0)} + t \int_{\mathbb{R}^d} V(x)dx - \frac{1}{2} t^2 \int_{\mathbb{R}^d} |V(x)|^2 dx \right|$$

$$\leqslant C_{\alpha,\gamma,d} \|V\|_1 \left(\|V\|_\infty^2 e^{t\|V\|_\infty} t^3 + M t^{\gamma/\alpha + 2} \right), \quad \forall t > 0.$$

$$\Rightarrow \frac{Tr(e^{-tH} - e^{-tH_0})}{p_t^{(\alpha)}(0)} = -t \int_{\mathbb{R}^d} V(x)dx + \frac{1}{2} t^2 \int_{\mathbb{R}^d} |V(x)|^2 dx + \mathcal{O}(t^{\gamma/\alpha + 2})$$

Similarly for relativistic $m - \left(m^{2/\alpha} - \Delta\right)^{\alpha/2}$, $m > 0$, and mixed stables $\Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$, $0 < \beta < \alpha < 2$.

Follows from two simple facts:

1. $$\int_{\mathbb{R}^d} E^t_{x,x}\left(\int_0^t V(X_s)ds\right)dx = t\int_{\mathbb{R}^d} V(x)dx$$

2. $$E^t_{0,0}\left(\int_0^t |X_s|^\gamma ds\right) \leqslant C_{\alpha,\gamma,d}\, t^{\gamma/\alpha+1},$$

$$\begin{aligned}
E^t_{0,0}\left(\int_0^t |X_s|^\gamma ds\right) &= \int_0^t E^t_{0,0}(|X_s|^\gamma)ds = \int_0^t \int_{\mathbb{R}^d} \frac{p_s^{(\alpha)}(0,y)p_{t-s}^{(\alpha)}(y,0)}{p_t^{(\alpha)}(0,0)}|y|^\gamma dyds \\
&= \int_0^{t/2} \int_{\mathbb{R}^d} \frac{p_s^{(\alpha)}(0,y)p_{t-s}^{(\alpha)}(y,0)}{p_t^{(\alpha)}(0,0)}|y|^\gamma dyds \\
&+ \int_{t/2}^t \int_{\mathbb{R}^d} \frac{p_s^{(\alpha)}(0,y)p_{t-s}^{(\alpha)}(y,0)}{p_t^{(\alpha)}(0,0)}|y|^\gamma dyds \\
&= 2\int_0^{t/2} \int_{\mathbb{R}^d} \frac{p_s^{(\alpha)}(0,y)p_{t-s}^{(\alpha)}(y,0)}{p_t^{(\alpha)}(0,0)}|y|^\gamma dyds.
\end{aligned}$$

For all $0 < s < t/2$ and all $y \in \mathbb{R}^d$,

$$p^{(\alpha)}_{t-s}(y,0) \leqslant p^{(\alpha)}_{t-s}(0,0) \leqslant p^{(\alpha)}_{t/2}(0,0).$$

$$\frac{p^{(\alpha)}_{t/2}(0,0)}{p^{(\alpha)}_{t}(0,0)} = 2^{d/\alpha}$$

$$\begin{aligned}
E^t_{0,0}\left(\int_0^t |X_s|^\gamma ds\right) &\leqslant 2^{d/\alpha+1}\int_0^{t/2}\int_{\mathbb{R}^d} p^{(\alpha)}_s(0,y)|y|^\gamma dyds \\
&= 2^{d/\alpha+1}\int_0^{t/2} E_0(|X_s|^\gamma)ds \\
&= 2^{d/\alpha+1}\int_0^{t/2} s^{\gamma/\alpha} E_0(|X_1|^\gamma)ds \\
&= \frac{2^{d/\alpha+1}}{2^{\gamma/\alpha+1}} E_0(|X_1|^\gamma)\frac{t^{\gamma/\alpha+1}}{\gamma/\alpha+1} \\
&= C_{\alpha,\gamma,d}\, t^{\gamma/\alpha+1}
\end{aligned}$$

$$p_t^H(x,y) = p_t^{(\alpha)}(x,y) E_{x,y}^t \left(e^{-\int_0^t V(X_s)ds} \right)$$

$$\begin{aligned} Tr(e^{-tH} - e^{-tH_0}) &= \int_{\mathbb{R}^d} (p_t^H(x,x) - p_t^{(\alpha)}(x,x))dx \\ &= p_t^{(\alpha)}(0) \int_{\mathbb{R}^d} E_{x,x}^t \left(e^{-\int_0^t V(X_s)ds} - 1 \right) dx, \end{aligned}$$

$$Use : \left| e^{-z} - 1 + z - \frac{z^2}{2} \right| \leqslant C|z|^3 e^{|z|}$$

$$\begin{aligned} & \left| \frac{Tr(e^{-tH} - e^{-tH_0})}{p_t^{(\alpha)}(0)} + \int_{\mathbb{R}^d} E_{x,x}^t \left(\int_0^t V(X_s)ds \right) dx \right. \\ - & \left. \frac{1}{2} \int_{\mathbb{R}^d} E_{x,x}^t \left(\left[\int_0^t V(X_s)ds \right]^2 \right) dx \right| \leqslant Ct^2 \|V\|_\infty^2 e^{t\|V\|_\infty} t \|V\|_1 \\ = & \quad Ct^3 |V\|_\infty^2 e^{t\|V\|_\infty} \|V\|_1 \end{aligned}$$

$$\left|E_{x,x}^t\left[\int_0^t V(X_s)ds\right]^2 - t^2V^2(x)\right| = \left|E_{x,x}^t\left[\int_0^t V(X_s)ds\right]^2 - \left[\int_0^t V(x)ds\right]^2\right|$$

$$= \left|E_{x,x}^t\left(\left[\int_0^t V(X_s)ds\right]^2 - \left[\int_0^t V(x)ds\right]^2\right)\right|$$

$$= E_{0,0}^t\left(\left[\int_0^t (V(X_s+x)-V(x))ds\right]\left[\int_0^t V(X_s+x)+V(x))ds\right]\right)$$

$$\leqslant ME_{0,0}^t\left(\left[\int_0^t |X_s|^\gamma ds\right]\left[\int_0^t (|V(X_s+x)|+|V(x)|)\,ds\right]\right).$$

$$\left|\int_{\mathbb{R}^d} E_{x,x}^t\left(\left[\int_0^t V(X_s)ds\right]^2\right)dx - t^2\int_{\mathbb{R}^d}|V(x)|^2dx\right|$$

$$\leqslant 2tM\|V\|_1 E_{0,0}^t\left(\int_0^t |X_s|^\gamma ds\right)$$

$$\leqslant M\|V\|_1 C_{\alpha,\gamma,d}\, t^{\gamma/\alpha+2} \quad \text{(Fact 2)}$$

General formula–R.B. Sá Barreto (1995), $\alpha = 2$

$$I_j = \{(\lambda_1, ..., \lambda_j) : 0 < \lambda_j < \lambda_{j-1} < ... < \lambda_1 < 1\}.$$

For any integer $N \geqslant 1$,

$$\frac{Tr(e^{-tH_V} - e^{-tH_2})}{p_t^{(2)}(0)} = \sum_{m=1}^{N} c_m(V)t^m + \mathcal{O}(t^{N+1}),$$

as $t \downarrow 0$, with

$$C_1(V) = -\int_{\mathbb{R}^d} V(\theta)d\theta, \;\; C_m(V) = (-1)^m \sum_{j+n=m, j\geqslant 2} C_{n,j}^{(2)}(V),$$

$$C_{n,j}^{(2)}(V) = \frac{(2\pi)^d}{(2\pi)^{jd} n!} \int_{I_j} \int_{\mathbb{R}^{(j-1)d}} \left\{L_j^{(2)}(\lambda, \theta)\right\}^n \widehat{V}(-\sum_{i=1}^{j-1} \theta_i) \prod_{i=1}^{j-1} \widehat{V}(\theta_i) d\theta_i d\lambda_i d\lambda_j,$$

$$L_j^{(2)}(\lambda, \theta) = \sum_{k=1}^{j-1} (\lambda_k - \lambda_{k+1}) \left|\sum_{i=1}^{k} \theta_i\right|^2 - \left|\sum_{k=1}^{j-1} (\lambda_k - \lambda_{k+1}) \sum_{i=1}^{k} \theta_i\right|^2.$$

L. Acuña–Valverde 2012 (grad student at Purdue) has a "general" expansion for $0 < \alpha < 2$, similar but not as clean. From it, one can compute "coefficients."

Example (For $1 < \alpha < 2$, $d \geqslant 3$)

$$\frac{Tr(e^{-tH_V} - e^{-tH_\alpha})}{p_t^{(\alpha)}(0)} = -t\int_{\mathbb{R}^d} V(\theta)d\theta + \frac{t^2}{2!}\int_{\mathbb{R}^d} V^2(\theta)d\theta - \left\{ \frac{t^3}{3!}\int_{\mathbb{R}^d} V^3(\theta)d\theta + \mathcal{L}_{d,\alpha}\, t^{2+\frac{2}{\alpha}} \int_{\mathbb{R}^d} |\nabla V(\theta)|^2\, d\theta \right\} + \mathcal{O}(t^4), \quad t \downarrow 0.$$

$$\mathcal{L}_{d,\alpha} = \frac{C_{d,\alpha} K_1(d,\alpha)}{(2\pi)^d}, \quad C_{d,\alpha} = \frac{\pi^{d/2}}{p_1^{(\alpha)}(0)},$$

$$K_1(d,\alpha) = \int_0^1 \int_0^{\lambda_1} E\left[\frac{S^*_{1-w} S^*_w}{(S^*_{1-w} + S^*_w)^{1+\frac{d}{2}}}\right] dw d\lambda_1.$$

Note: As $\alpha \uparrow 2$, get the 3rd term expansion for the Laplacian.

Example (For $\frac{4}{3} < \alpha < 2$, $d \geqslant 5$)

$$\frac{Tr(e^{-tH_V} - e^{-tH_\alpha})}{p_t^{(\alpha)}(0)} = -t\int_{\mathbb{R}^d} V(\theta)d\theta + \frac{t^2}{2!}\int_{\mathbb{R}^d} V^2(\theta)d\theta$$

$$- \quad \left\{\frac{t^3}{3!}\int_{\mathbb{R}^d} V^3(\theta)d\theta + \mathcal{L}_{d,\alpha}t^{2+\frac{2}{\alpha}}\int_{\mathbb{R}^d} |\nabla V(\theta)|^2\right\}$$

$$+ \quad \left\{\frac{t^4}{4!}\int_{\mathbb{R}^d} V^4(\theta)d\theta + \mathcal{M}_{d,\alpha}t^{3+\frac{2}{\alpha}}\int_{\mathbb{R}^d} V(\theta)\,|\nabla V(\theta)|^2\,d\theta\right.$$

$$+ \quad \left.\mathcal{N}_{d,\alpha}t^{2+\frac{2\cdot 2}{\alpha}}\int_{\mathbb{R}^d} |\Delta V(\theta)|^2\,d\theta\right\} + \mathcal{O}(t^5), \quad t \downarrow 0.$$

NOTE 1: $1 < 2 < 3 < 2 + \frac{2}{\alpha} < 4 < 3 + \frac{2}{\alpha} < 2 + \frac{2\cdot 2}{\alpha} < 5$

NOTE 2: As $\alpha \uparrow 2$, get the 4th term expansion for the Laplacian.

Example (For $\alpha = 1$, $d \geqslant 4$)

$$\frac{Tr(e^{-H_V t} - e^{-H_1 t})}{p_t^{(1)}(0)} = -\, t \int_{\mathbb{R}^d} V(\theta) d\theta + \frac{t^2}{2!} \int_{\mathbb{R}^d} V^2(\theta) d\theta - \frac{t^3}{3!} \int_{\mathbb{R}^d} V^3(\theta) d\theta$$
$$+ \frac{t^4}{4!} \left(\int_{\mathbb{R}^d} V^4(\theta) d\theta + 4! \mathcal{L}_{d,1} \int_{\mathbb{R}^d} |\nabla V(\theta)|^2 d\theta \right)$$
$$- \frac{t^5}{5!} \left(\int_{\mathbb{R}^d} V^5(\theta) d\theta + 5! \mathcal{M}_{d,1} \int_{\mathbb{R}^d} V(\theta) \left|\nabla V(\theta)\right|^2 d\theta \right) + \mathcal{O}(t^6),$$

as $t \downarrow 0$.

$$K_2(d,\alpha) = \int_0^1 \int_0^{\lambda_1} E\left[\frac{(S^*_{1-(\lambda_1-\lambda_2)} S^*_{\lambda_1-\lambda_2})^2}{(S^*_{1-(\lambda_1-\lambda_2)} + S^*_{\lambda_1-\lambda_2})^{1+\frac{d}{2}}} \right] d\lambda_2\, d\lambda_1.$$

$$K_3(d,\alpha) = \int_0^1 \int_0^{\lambda_1} \int_0^{\lambda_2} E\left[\frac{S^*_{1-(\lambda_1-\lambda_3)} S^*_{\lambda_1-\lambda_2} + S^*_{1-(\lambda_1-\lambda_3)} S^*_{\lambda_2-\lambda_3} + S^*_{\lambda_1-\lambda_2} S^*_{\lambda_2-\lambda_3}}{(S^*_{1-(\lambda_1-\lambda_3)} + S^*_{\lambda_1-\lambda_2} + S^*_{\lambda_2-\lambda_3})^{1+d/2}} \right]$$

Theorem (Acuña–Valverde 2012): Assume integer M is $1 \leqslant M < \frac{d+\alpha}{2}$. Then for all $J \geqslant 2$, there exists bounded function $R_{J+1}^{(\alpha)}(t)$, $0 < t < 1$, such that

$$\frac{Tr(e^{-tH_V} - e^{-tH_\alpha})}{p_t^{(\alpha)}(0)} = -t\int_{\mathbb{R}^d} V(\theta)d\theta - \sum_{j=2}^{J}\sum_{n=0}^{M-1}(-1)^{n+j}C_{n,j}^{(\alpha)}(V)t^{\frac{2n}{\alpha}+j} + t^{\Phi_{J+1}(M)}R_{J+1}^{(\alpha)}(t),$$

$$\Phi_{J+1}(M) = \min\left\{J+1, 2+\frac{2M}{\alpha}\right\}.$$

$$C_{n,j}^{(\alpha)}(V) \quad = \quad \frac{C_{d,\alpha}}{(2\pi)^{jd}n!}\int_{I_j}\int_{\mathbb{R}^{(j-1)d}} E\left[S_{1,\frac{\alpha}{2}}^{-d/2}\left\{L_j^{(\alpha)}(\lambda,\theta)\right\}^n\right]\widehat{V}(-\sum_{i=1}^{j-1}\theta_i)\prod_{i=1}^{j-1}\widehat{V}(\theta_i)d\theta_i d\lambda_i d\lambda_j,$$

$$L_j^{(\alpha)}(\lambda,\theta) = \sum_{k=1}^{j-1} S^*_{\lambda_k-\lambda_{k+1}}\left|\sum_{i=1}^{k}\theta_i\right|^2 - \frac{1}{S_{1,\frac{\alpha}{2}}}\left|\sum_{k=1}^{j-1} S^*_{\lambda_k-\lambda_{k+1}}\sum_{i=1}^{k}\theta_i\right|^2.$$

Furthermore,

$$C_{n,j}^{(\alpha)}(V) \to C_{n,j}^{(2)}(V), \quad \text{as}, \quad \alpha \uparrow 2.$$

A couple of questions:

Existence of resonances in potential scattering for fractional Laplacian ???

As first step, need to do Chapter IV "Trace formulae and scattering poles" in Richard Melrose's 1994 "Stanford Lecture Notes" book.

$D \subset \mathbb{R}^2$. By McKean–Singer (1967) for the Laplacian, $\alpha = 2$

$$\lim_{t \to 0} \left\{ Z_D(t) - (4\pi t)^{-1} \left(|D| - \frac{\sqrt{\pi t}}{2} |\partial D| \right) \right\} = \frac{(1-r)}{6},$$

r number of holes in D.

Question

Is there a non-local version of this (for $0 < \alpha < 2$)?

Dziękuję bardzo.
Bardzo się cieszę, że znowu jestem w Będlewie.

Thank you!

Theorem

Assume that $M \geqslant 1$ is an integer satisfying $M < \frac{d+\alpha}{2}$. Then, given $J \geqslant 2$, there exists a bounded function $R^{(\alpha)}_{J+1}(t)$, $0 < t < 1$, such that

$$\frac{Tr(e^{-tH_V} - e^{-tH_\alpha})}{p^{(\alpha)}_t(0)} = -t\int_{\mathbb{R}^d} V(\theta)d\theta + \sum_{j=2}^{J}\sum_{n=0}^{M-1}(-1)^{n+j}C^{(\alpha)}_{n,j}(V)t^{\frac{2n}{\alpha}+j} + t^{\Phi_{J+1}(M)}R^{(\alpha)}_{J+1}(t) \tag{1}$$

where $\Phi_{J+1}(M) = \min\left\{J+1, 2+\frac{2M}{\alpha}\right\}$, *and the constants* $C^{(\alpha)}_{n,j}(V)$ *are given by*

$$C^{(\alpha)}_{n,j}(V) = \frac{C_{d,\alpha}}{(2\pi)^{jd}n!}\int_{I_j}\int_{\mathbb{R}^{(j-1)d}} E\left[S_{1,\frac{\alpha}{2}}^{-d/2}\left\{L^{(\alpha)}_j(\lambda,\theta)\right\}^n\right]\widehat{V}(-\sum_{i=1}^{j-1}\theta_i)\prod_{i=1}^{j-1}\widehat{V}(\theta_i)d\theta_i d\lambda_i d\lambda_j,$$

$$L^{(\alpha)}_j(\lambda,\theta) = \sum_{k=1}^{j-1}S^*_{\lambda_k-\lambda_{k+1}}\left|\sum_{i=1}^{k}\theta_i\right|^2 - \frac{1}{S_{1,\frac{\alpha}{2}}}\left|\sum_{k=1}^{j-1}S^*_{\lambda_k-\lambda_{k+1}}\sum_{i=1}^{k}\theta_i\right|^2, \text{ and } C_{d,\alpha} = \frac{\pi^{d/2}}{p^{(\alpha)}_1(0)},$$

$S^*_{\lambda_1-\lambda_2}, S^*_{\lambda_2-\lambda_3},\dots,S^*_{\lambda_{j-1}-\lambda_j}, S^*_{1-(\lambda_1-\lambda_j)}$ *independent,*

$$S^*_{1-(\lambda_1-\lambda_j)} + \sum_{k=1}^{j-1}S^*_{\lambda_k-\lambda_{k+1}} = S_{1,\frac{\alpha}{2}}, \quad S^*_l \overset{\mathcal{D}}{=} S_l,\ l \in \left\{1-(\lambda_1-\lambda_j), \lambda_k-\lambda_{k+1}\right\}_{k=1}^{j-1}$$

Theorem

Under the same conditions,

$$\begin{aligned}\frac{Tr(e^{-tH_V}-e^{-tH_\alpha})}{p_t^{(\alpha)}(0)} &= -t\int_{\mathbb{R}^d} V(\theta)d\theta \\ &+ \sum_{\substack{\frac{2n}{\alpha}+j<\Phi_{J+1}(M)\\ 2\leqslant j\leqslant J,\ 0\leqslant n\leqslant M-1}} (-1)^{n+j}C_{n,j}^{(\alpha)}(V)t^{\frac{2n}{\alpha}+j} \\ &+ \mathcal{O}(t^{\Phi_{J+1}(M)}),\quad t\downarrow 0\end{aligned}$$

For $\alpha = 2$ we have no restrictions on J and M. $\Phi_{J+1}(M) = \min\{J+1, M+2\}$. Then, by taking $M = J-1$, $\Phi_{J+1}(J-1) = J+1$.

$$\frac{Tr(e^{-tH_V}-e^{-tH_2})}{p_t^{(2)}(0)} = -t\int_{\mathbb{R}^d} V(\theta)d\theta + \sum_{\substack{n+j<J+1\\ 2\leqslant j\leqslant J,\ 0\leqslant n\leqslant J-2}} (-1)^{n+j}C_{n,j}^{(2)}(V)t^{n+j}+\mathcal{O}(t^{J+1}$$

$$\sum_{\substack{n+j<J+1\\ 2\leqslant j\leqslant J, 0\leqslant n\leqslant J-2}} (-1)^{n+j}C_{n,j}^{(2)}(V)t^{n+j} = \sum_{l=2}^{J} c_l(V)t^l,$$

Theorem (R.B. Sá Barreto 1995)

$V \in \mathcal{S}(\mathbb{R}^d)$, $V \neq 0$ and $|V(x)| \leqslant Ae^{-B|x|^{1+\varepsilon}}$ Let $H = -\Delta + V$ defined on $L^2(\mathbb{R}^n)$, $n \geqslant 3$ odd. Then the meromorphic extension of $(H - \lambda^2)^{-1}$ has infinitely many poles provided that one of the following conditions holds:

- $n = 3$
- $n \leqslant 9$ *and* $V \geqslant 0$
- $\widehat{V} \geqslant 0$.