Invariance Principle for the Random Conductance Model with dynamic bounded Conductances

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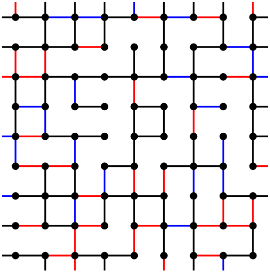
The Static Random Conductance Model

Intuitive description

- Put i.i.d. random conductances (or weights) $\omega_e \in [0, \infty)$ on the edges of the Euclidean lattice $(\mathbb{Z}^d, E_d)$.
- Look at a continuous time Markov chain X_t with jump probabilities proportional to the edge conductances.

$$P_{xy} = \frac{\omega_{xy}}{\sum_{z \sim x} \omega_{xz}}.$$



Bond conductivities: blue $\ll 1$, black ≈ 1, red $\gg 1$.

Definitions

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- **Random walk.** Let $\Omega' = D([0,\infty), \mathbb{Z}^d)$. For each $\omega \in \Omega$ let P_x^ω be the probability law on Ω' which makes the coordinate process $X_t = X_t(\omega')$ a Markov chain with generator

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- **Problems.** What we would like to have:
  - *Gaussian bounds (GB)* on the heat kernel for X.
  - *Quenched functional CLT with diffusivity σ^2:* Let $X_t^{(N)} = N^{-1} X_{N^2 t}$ and W be a BM($\mathbb{R}^d$). Then for $\mathbb{P}$-a.a. ω, under P_0^ω,

$$X^{(N)} \Rightarrow \sigma W.$$

(In particular is $\sigma > 0$?)

Results on the static RCM

- Annealed FCLT: De Masi, Ferrari, Goldstein, Wick (1989).
- "Elliptic": $0 < C_1 \leq \omega_e \leq C_2 < \infty$. GB follow from results of Delmotte (1999). QFCLT proved by Sidoravicius and Sznitman (2004).
- "Supercritical Percolation": $\omega_e \in \{0, 1\}$ (and $p_+ > p_c$.) GB proved by Barlow (2004). QFCLT proved by Sidoravicius and Sznitman (2004), Berger and Biskup (2007), Mathieu and Piatnitski (2007).
- "Bounded above": $\omega_e \in [0, 1]$. Berger, Biskup, Hoffmann, Kozma (2008) showed **GB may fail!** QFCLT holds with $\sigma^2 > 0$: Biskup and Prescott (2007), Mathieu (2007).
- "Bounded below": $\omega_e \in [1, \infty)$. GB and QFCLT proved by Barlow and Deuschel (2010).
- General i.i.d. $\omega_e \geq 0$. QFCLT by A., Barlow, Deuschel, Hambly (2011).

Dynamic Random Conductance Model

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$$\mathcal{L}^\omega_t f(x) = \sum_{y \sim x} \omega_{xy}(t) \left(f(y) - f(x) \right).$$

Denote by $p^\omega(s,x;t,y)$ the transition kernel.

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$$\mathcal{L}^{\omega}_t f(x) = \sum_{y\sim x} \omega_{xy}(t)\,(f(y) - f(x))\,.$$

  Denote by $p^{\omega}(s,x;t,y)$ the transition kernel.
- **Shift.** Let $\tau_{t,x}\omega$ be the environment obtained by shifting ω by t in time and by x in space.

QFCLT's for RWRE with dynamical Environment

- Space-time i.i.d. environment: Boldrighini, Minlos, Pellegrinotti (2004); Rassoul-Agha, Seppäläinen (2005)
- Markovian in time, i.i.d. in space: Bandyopadhyay, Zeitouni (2006)
- Markovian in time, exponential mixing environment: Dolgopyat, Keller, Liverani (2008)
- Continuous-space random walks: Rassoul-Agha, Joseph (2010)
- Ergodic Markovian environment under some coupling conditions: Redig, Völlering (2011)

Assumptions

- **A1: Ergodicity.** The measure $\mathbb{P}$ is invariant and ergodic w.r.t. $(\tau_{t,x})$.
- **A2: Stochastic Continuity.** For any $\delta > 0$ and $f \in L^2(\mathbb{P})$ we have

$$\lim_{h \to 0} \mathbb{P}[|f(\tau_{h,0}\omega) - f(\omega)| \geq \delta] = 0.$$

- **A3: Ellipticity.** There exist positive constants C_l and C_u such that

$$\mathbb{P}\big[C_l \leq \omega_e(t) \leq C_u,\, \forall e \in E_d, t \geq 0\big] = 1.$$

Gaussian Bounds and Annealed Functional CLT

Theorem (Delmotte, Deuschel; 2005)

Under A1-A3, for $\mathbb{P}$-a.e. ω

$$p^\omega(s,x;t,y) \le \frac{c_2}{(t-s)^{d/2}} \exp\left(-c_3 \frac{|x-y|^2}{t-s}\right), \quad \textit{if } |x-y| \le c_1(t-s)$$

and similar lower bounds

Define annealed law $\mathbb{P}^*_{s,x} = \int_\Omega P^\omega_{s,x} \, d\mathbb{P}(\omega)$.

Theorem (A., 2012)

Let $d \ge 1$. Under A1-A3 the law of $X^{(N)}$ converges under $\mathbb{P}^*_{0,0}$ to the law of a Brownian motion on $\mathbb{R}^d$ with a deterministic non-degenerate covariance matrix Σ.

Quenched Functional CLT

- **A4: Time-Mixing.** There exists $p_1 > 1$ such that for all bounded φ, ψ of the form $\varphi(\omega) = \tilde{\varphi}(\omega(t_1))$ and $\psi(\omega) = \tilde{\psi}(\omega(t_2))$, $|t_1 - t_2| \geq 1$, for some $\tilde{\varphi}, \tilde{\psi}$ depending on finitely many variables we have

$$|\mathbb{E}[\varphi\psi] - \mathbb{E}[\varphi]\mathbb{E}[\psi]| \leq c|t_1 - t_2|^{-p_1} \|\varphi\|_{L^\infty(\mathbb{P})} \|\psi\|_{L^\infty(\mathbb{P})}.$$

- **A5: Space-Mixing.** Let $d \geq 3$. There exists $p_2 > 2d/(d-2)$ such that for all φ, ψ of the form $\varphi(\omega) = \tilde{\varphi}(\omega(t_0))$ and $\psi(\omega) = \tilde{\psi}(\omega(t_0))$ for some $\tilde{\varphi}, \tilde{\psi}$ depending on finitely many variables we have

$$|\mathbb{E}[\varphi(\omega)\psi(\tau_{0,x}\omega)] - \mathbb{E}[\varphi]\mathbb{E}[\psi]| \leq c|x|^{-p_2} \|\varphi\|_{L^\infty(\mathbb{P})} \|\psi\|_{L^\infty(\mathbb{P})}.$$

Theorem (A.,2012)

Let $d \geq 3$. Under A1-A5, $\mathbb{P}$-a.s. $X^{(N)}$ converges (under $P^\omega_{0,0}$) in law to a Brownian motion on $\mathbb{R}^d$ with a deterministic non-degenerate covariance matrix Σ.

Overview of proofs

- Basic idea: Homogenization. Use the process of 'the environment seen from the particle'

$$\eta_t := \tau_{t,X_t}\omega, \qquad t \geq 0,$$

to construct the time-dependent corrector function χ such that

$$X_t = M_t + \chi(t, X_t, \omega).$$

- Problem: η is not reversible!
- To control the corrector one needs to show

$$\lim_{n\to\infty} n^{-1/2} \max_{k\leq n} |\chi(k, X_k, \omega)| = 0 \qquad \text{in } P_{0,0}^{\omega}\text{-probability.}$$

- This follows from some fractional ergodic theorems by Derriennic and Lin if for some $\delta > 0$

$$\mathbb{E}E_{0,0}^{\omega}\left[|\chi(n, X_n, \omega)|^2\right] = O(n^{1-\delta}).$$

Local Limit Theorem

For the Gaussian heat kernel with diffusion matrix Σ write

$$k_t(x) = \frac{1}{\sqrt{(2\pi t)^d \det \Sigma}} \exp(-x \cdot \Sigma^{-1} x/2t), \qquad k_t(x,y) = k_t(0, y-x).$$

Combine the GB and the FCLT for X, following arguments from Barlow, Hambly (2009), to obtain

Theorem

Let $T > 0$. Under A1-A3 we have

$$\lim_{N\to\infty} \sup_{x\in\mathbb{R}^d} \sup_{t\geq T} \Big| N^d \mathbb{E}[p^\omega(0,0;N^2 t, \lfloor Nx \rfloor)] - k_t(x) \Big| = 0.$$

and under A1-A5,

$$\lim_{N\to\infty} \sup_{x\in\mathbb{R}^d} \sup_{t\geq T} \Big| N^d p^\omega(0,0;N^2 t, \lfloor Nx \rfloor) - k_t(x) \Big| = 0, \qquad \mathbb{P}\text{-a.s.}$$

Application: $\nabla\phi$-Interface Models

- Ginzburg-Landau interface models describe the separation of two thermodynamical phases.
- The interface is specified by a field of height variables $\phi_t(x)$, $x \in \mathbb{Z}^d$, $t \geq 0$, given by

$$d\phi_t(x) = -\sum_{y:|x-y|=1} V'(\phi_t(x) - \phi_t(y))\, dt + \sqrt{2} dw_t(x),$$

  with
  - $\{w(x), x \in \mathbb{Z}^d\}$ collection of independent Brownian motions,
  - potential $V \in C^2(\mathbb{R}, \mathbb{R}_+)$ even and strictly convex.
- In $d \geq 3$ there exists an ergodic Gibbs measure μ which is reversible for the dynamics.

Space-Time Covariances

- Helffer-Sjöstrand representation:

$$\mathrm{cov}_\mu(\phi_0(0), \phi_t(y)) = \int_0^\infty \mathbb{E}_\mu p^\phi(0, 0; t + s, y)\, ds,$$

where $p^\phi(s, x; t, y)$ is the transition kernel of a RW with generator

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- By the local limit theorem

$$N^{d+2}\,\mathrm{cov}_\mu(\phi_0(0), \phi_{N^2 t}(\lfloor Ny \rfloor)) = N^d \int_0^\infty \mathbb{E}_\mu p^\phi(0,0; N^2(t+s), \lfloor Ny \rfloor)\, ds$$

$$\xrightarrow{N\to\infty} \int_0^\infty k_{t+s}(y)\, ds.$$

Conclusion, Outlook and open questions

- For the static RCM we have a QFCLT in the case of general i.i.d. conductances.
- For the dynamic RCM we have a QFCLT under ellipticity and mixing assumptions.

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  - in the static case with stationary, ergodic conductances under some moment conditions? **Yes:** A., Deuschel, Slowik (in preparation)
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Thank you!