Gradient estimates of q-harmonic functions of fractional Schrödinger operator

Tadeusz Kulczycki

Institute of Mathematics and Computer Science
Wrocław University of Technology

X_t - the symmetric α-stable process in $\mathbb{R}^d$, $\alpha \in (0, 2)$.
$q : \mathbb{R}^d \to \mathbb{R}$ belongs to the Kato class $\mathcal{J}^\alpha$.

- We say that a Borel function u on $\mathbb{R}^d$ is *q-harmonic* in an open set $D \subset \mathbb{R}^d$ iff

$$u(x) = E^x \left[\exp\left(\int_0^{\tau_W} q(X_s)\, ds \right) u(X_{\tau_W}) \right], \quad x \in W,$$

for every open bounded set W, with $\overline{W} \subset D$.

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  for every open bounded set W, with $\overline{W} \subset D$.
- If u is q-harmonic in open set $D \subset \mathbb{R}^d$ then u is a weak solution of

$$\Delta^{\alpha/2} u + qu = 0, \quad \text{on} \quad D. \tag{1}$$

  $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$ - the fractional Laplacian.
  $\Delta^{\alpha/2} + q$ - the fractional Schrödinger operator.

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- If $D \subset \mathbb{R}^d$ is an open bounded set and (D, q) is gaugeable then a weak solution of (1) is a q-harmonic function on D after a modification on a set of Lebesgue measure zero.

- If u is q-harmonic in D then it is continuous in D.

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Theorem 1

Let $\alpha \in (0, 1]$, $d \in \mathbb{N}$ and $D \subset \mathbb{R}^d$ be an open bounded set. Assume that $q : D \to \mathbb{R}$ is Hölder continuous with Hölder exponent $\eta > 1 - \alpha$. Let u be q-harmonic in D. If u is nonnegative in $\mathbb{R}^d$ then $\nabla u(x)$ exists for any $x \in D$ and we have

$$|\nabla u(x)| \leqslant c \frac{u(x)}{\delta_D(x) \wedge 1}, \quad x \in D,$$

where $\delta_D(x) = \text{dist}(x, \partial D)$ and $c = c(\alpha, d, q)$.

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where $\delta_D(x) = \mathrm{dist}(x, \partial D)$ and $c = c(\alpha, d, q)$.

- The existence of $\nabla u(x)$ and similar gradient estimates are well known: for $\alpha = 2$, M. Cranston, Z. Zhao (1990), for $\alpha \in (1,2)$, K. Bogdan, T. K., A. Nowak (2002).

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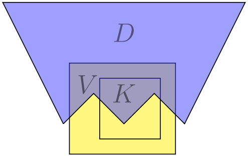
- The results for $\alpha \in (1,2]$ were shown under the assumption that $q \in \mathcal{J}^{\alpha-1}$. The case $\alpha \in (0,1]$ is different. In the proof of Theorem 1 probabilistic ideas are used (see e.g. B. Böttcher, R. Schilling, J. Wang (2011)).

One may ask whether it is possible to weaken the assumption in Theorem 1 that q is Hölder continuous with Hölder exponent $\eta > 1 - \alpha$. It occurs that the exponent $\eta = 1 - \alpha$ is critical in the following sense.

Proposition 2

For any $\alpha \in (0, 1]$, $d \in \mathbb{N}$ and any open bounded set $D \subset \mathbb{R}^d$ there exists $q : D \to [0, \infty)$ which is $1 - \alpha$ Hölder continuous, a function $u : \mathbb{R}^d \to [0, \infty)$ which is q-harmonic in D and a point $z \in D$ such that $\nabla u(z)$ does not exist.

The proof of this proposition is based on the estimates of the Green function of the killed Brownian motion subordinated by the $\alpha/2$-stable subordinator, obtained by R. Song (2004).



When a q-harmonic function u vanishes continuously near some part of the boundary of D and $D \subset \mathbb{R}^d$ is a bounded Lipschitz domain then the estimates obtained in Theorem 1 are sharp near that part of the boundary.

Theorem 3

Let $\alpha \in (0,1]$, $d \in \mathbb{N}$, $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $q : D \to \mathbb{R}$ be Hölder continuous with Hölder exponent $\eta > 1 - \alpha$. Let $V \subset \mathbb{R}^d$ be open and let K be a compact subset of V. Then there exist constants $c = c(D, V, K, \alpha, q)$ and $\varepsilon = \varepsilon(D, V, K, \alpha, q)$ such that for every function $u : \mathbb{R}^d \to [0, \infty)$ which is bounded on V, q-harmonic in $D \cap V$ and vanishes in $D^c \cap V$ we have

$$c^{-1} \frac{u(x)}{\delta_D(x)} \leqslant |\nabla u(x)| \leqslant c \frac{u(x)}{\delta_D(x)}, \quad x \in K \cap D, \quad \delta_D(x) < \varepsilon.$$

Similar result was obtained for $\alpha = 2$ by R. Bañuelos, M. Pang (1999) and for $\alpha \in (1,2)$ by K. Bogdan, T. K., A. Nowak (2002).

Applications

Let us consider the eigenvalue problem for the fractional Schrödinger operator on D with zero exterior condition

$$\begin{aligned} \Delta^{\alpha/2}\varphi + q\varphi &= -\lambda\varphi \qquad \text{on } D, \\ \varphi &= 0 \qquad \text{on } D^c. \end{aligned}$$

It is well known that for the above problem there exists a sequence of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ satisfying

$$\lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \dots, \qquad \lim_{n\to\infty} \lambda_n = \infty,$$

and a sequence of corresponding eigenfunctions $\{\varphi_n\}_{n=1}^{\infty}$, which can be chosen so that they form an orthonormal basis in $L^2(D)$. All φ_n are bounded and continuous on D and φ_1 is strictly positive on D.

Corollary 4

Assume that $\alpha \in (0,2)$, $D \subset \mathbb{R}^d$ is an open bounded set, $q \in \mathcal{J}^{\alpha-1}$ when $\alpha \in (1,2)$, or q is Hölder continuous on D with Hölder exponent $\eta > 1-\alpha$ when $\alpha \in (0,1]$. Then $\nabla\varphi_n(x)$ exist for any $n \in \mathbb{N}$, $x \in D$ and we have

$$|\nabla\varphi_1(x)| \leqslant c\frac{\varphi_1(x)}{\delta_D(x) \wedge 1}, \qquad x \in D,$$

$$|\nabla\varphi_n(x)| \leqslant \frac{c_n}{\delta_D(x) \wedge 1}, \qquad x \in D.$$

Furthermore, if additionally $D \subset \mathbb{R}^d$ is a bounded Lipschitz domain then there exists $\varepsilon = \varepsilon(D,q,\alpha)$ such that

$$|\nabla\varphi_1(x)| \geqslant c\frac{\varphi_1(x)}{\delta_D(x)}, \qquad x \in D, \quad \delta_D(x) \leqslant \varepsilon.$$

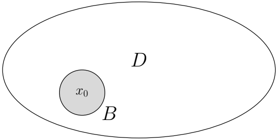
Corollary 5

Let $\alpha \in (0,1)$, $d \in \mathbb{N}$ and $D \subset \mathbb{R}^d$ be an open bounded set. Assume that $q : D \to \mathbb{R}$ is Hölder continuous with Hölder exponent $\eta > 1 - \alpha$ and either u is nonnegative on $\mathbb{R}^d$ or $\|u\|_\infty < \infty$. If u is a weak solution of

$$\Delta^{\alpha/2} u + qu = 0, \quad \text{on} \quad D. \tag{2}$$

then (after a modification on a set of Lebesgue measure zero) u is continuous on D and it is a strong solution of (2).

The idea of the proof. ($d > \alpha$)



$$B = B(x_0, r),$$
$$r = (\delta_D(x_0)/2) \wedge c,$$

$$u(x) = E^x u(X_{\tau_B}) + G_B(qu)(x), \quad x \in B$$

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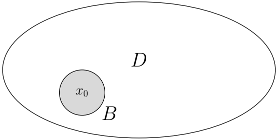

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- $f(x) = E^x u(X_{\tau_B})$ - α-harmonic function in B.

$$|\nabla f(x_0)| \leqslant c_1 \frac{f(x_0)}{\delta_B(x_0)} \leqslant c_2 \frac{u(x_0)}{r}.$$

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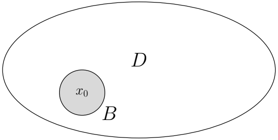

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$$|\nabla f(x_0)| \leqslant c_1 \frac{f(x_0)}{\delta_B(x_0)} \leqslant c_2 \frac{u(x_0)}{r}.$$

- $G_B(qu)(x) = \int_B G_B(x, y) q(y) u(y)\, dy.$

$$\frac{\partial}{\partial x_i} G_B(qu)(x) \stackrel{?}{=} \int_B \frac{\partial}{\partial x_i} G_B(x, y) q(y) u(y)\, dy.$$

For $\alpha \in (0, 1]$ $\frac{\partial}{\partial x_i} G_B(x, y)$ is not integrable.
$\frac{\partial}{\partial x_i} G_B(x, y) \simeq \frac{\partial}{\partial x_i} |x - y|^{\alpha - d} \simeq \frac{(x_i - y_i)}{|x - y|} |x - y|^{\alpha - d - 1}.$
(the difference between cases $\alpha \in (0, 1]$ and $\alpha \in (1, 2)$)

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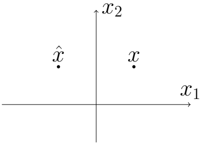
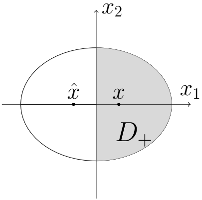


$x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d,$
$\hat{x} = (-x_1, x_2, \ldots, x_d).$



$\hat{D} = \{x \in \mathbb{R}^d : \hat{x} \in D\},$
D - symmetric iff $D = \hat{D}$,
$D_+ = \{(x_1, x_2, \ldots, x_d) \in D : x_1 > 0\}$

$$G_D f(x) - G_D f(\hat{x}) = \int_{D_+} (G_D(x,y) - G_D(\hat{x},y))(f(y) - f(\hat{y}))\, dy.$$

In order to get estimates of $\frac{\partial}{\partial x_1} G_D f(0)$ it is enough to get estimates of $G_D(x,y) - G_D(\hat{x},y)$.

- $X_t = B_{\eta_t}$.
  X_t - the symmetric α-stable process in $\mathbb{R}^d$, B_t - the Brownian motion in $\mathbb{R}^d$, η_t - $\alpha/2$-stable subordinator (starting from 0).

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- $X_t = B_{\eta_t}$.
  X_t - the symmetric α-stable process in $\mathbb{R}^d$, B_t - the Brownian motion in $\mathbb{R}^d$, η_t - $\alpha/2$-stable subordinator (starting from 0).
- $H = \mathbb{R}^d_+ = \{x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_1 > 0\}$.
  B_t^H - the Brownian motion in $\mathbb{R}^d$ killed on exiting H.
  $\tilde{X}_t = (B^H)_{\eta_t}$.
  X_t^D - the process X_t killed on exiting $D \quad \sim \quad p_D(t,x,y)$.
  $\tilde{X}_t^{D_+}$ - the process $\tilde{X}_t$ killed on exiting $D_+ \quad \sim \quad \tilde{p}_{D_+}(t,x,y)$.

Lemma

$$\tilde{p}_{D_+}(t,x,y) = p_D(t,x,y) - p_D(t,\hat{x},y), \quad x,y \in D_+.$$

The idea of the proof. $(d > \alpha)$

Lemma

$\tilde{p}_{D_+}(t,x,y) = p_D(t,x,y) - p_D(t,\hat{x},y), \quad x,y \in D_+.$

$X_t^D \quad \sim \quad G_D(x,y) = \int_0^\infty p_D(t,x,y)\,dt.$
$\tilde{X}_t^{D_+} \quad \sim \quad \tilde{G}_{D_+}(x,y) = \int_0^\infty \tilde{p}_{D_+}(t,x,y)\,dt.$
For symmetric domains $D \subset U$ we have $\tilde{G}_{D_+}(x,y) \leqslant \tilde{G}_{U_+}(x,y)$. In particular we have $\tilde{G}_{D_+}(x,y) \leqslant \tilde{G}_{R_+^d}(x,y)$.
Let $B = B(0,r)$, $x,y \in B_+$. We have

$$\begin{aligned} G_B(x,y) - G_B(\hat{x},y) &= \tilde{G}_{B_+}(x,y) \leqslant \tilde{G}_{R_+^d}(x,y) \\ &= G_{\mathbb{R}^d}(x,y) - G_{\mathbb{R}^d}(\hat{x},y) \\ &= \frac{c_1}{|x-y|^{d-\alpha}} - \frac{c_1}{|\hat{x}-y|^{d-\alpha}} \\ &\leqslant \frac{c_2|x-y|}{|x-y|^{d-\alpha}|\hat{x}-y|}. \end{aligned}$$

The idea of the proof. $(d > \alpha)$

- Let $\alpha \in (0, 1)$, $B = B(0, r)$. Assume that the function $f : B \to \mathbb{R}$ is Borel and bounded on B and satisfies

$$|f(x) - f(\hat{x})| \leqslant A|x|^{\beta}, \qquad x \in B(0, r/2),$$

for some constants $A \geqslant 1$ and $\beta \geqslant 0$.

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for some constants $A \geqslant 1$ and $\beta \geqslant 0$.
- If $\alpha \in (0,1)$ and $\beta \in [0, 1-\alpha)$ then there exists $c = c(d,\alpha,\beta)$ such that for any $x \in B$ we have
$$|G_B f(x) - G_B f(\hat{x})| \leqslant cA|x|^{\beta+\alpha} + c\frac{\sup_{y\in B} f(y)}{r}|x|^{\beta+\alpha}.$$

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- Recall that $u(x) = E^x u(X_{\tau_B}) + G_B(qu)(x)$. Put $f(x) = q(x)u(x)$.
- This allows to obtain estimates of $\frac{\partial}{\partial x_1} G_B(qu)(0)$.