On Harnack inequality for subordinate Brownian motions

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Subordinators

Let S_t be a subordinator i.e. a increasing Lévy process starting from 0.
The Laplace transform of a subordinator is of the form

$$Ee^{-\lambda S_t} = e^{-t\phi(\lambda)}, \qquad \lambda \geqslant 0,$$

where ϕ is called the Laplace exponent of S or a Bernstein function and has the following representation:

$$\phi(\lambda) = a + b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda u})\mu(du),$$

where $a, b \geqslant 0$ and μ is a Lévy measure on $(0, \infty)$ such that $\int \frac{u}{1+u}\mu(du) < \infty$.

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- A function ϕ is a *special Bernstein function* (SBF) if $\frac{\lambda}{\phi(\lambda)}$ is again a Bernstein function.
- If ϕ is a unbounded SBF then there exists a decreasing positive density u on $(0,\infty)$ of a potential measure U.

- Let B_t be a Brownian motion in $\mathbb{R}^d$ with the characteristic function of the form
$$Ee^{i\xi B_t} = e^{-t|\xi|^2}.$$
By $g_t(\cdot)$ we denote the density of B_t. Assume that B_t and S_t are stochastically independent. Then a process $X_t = B_{S_t}$ is a subordinate Brownian motion. The characteristic function of X_t takes the form
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- By Cap we denote $0-$order capacity.

From now on $a = 0$

Proposition

Let $d \geqslant 3$*. For* $r > 0$

$$c_1 \phi(r^{-2})^{-1} \leqslant \int_{B(0,r)} G(y)dy \leqslant c_2 \phi(r^{-2})^{-1},$$

where $c_1 = \frac{(1-2e^{-1})\Gamma\left(\frac{d}{2},\frac{1}{4}\right)}{2\Gamma\left(\frac{d}{2}\right)}$, $c_2 = \left(e + \frac{ed}{2^{d+1}\Gamma\left(\frac{d}{2}+2\right)}\right)$.

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Corollary

Let $d \geqslant 3$. *There exists a constant* $C=C(d)$ *such that, for* $r>0$

$$C^{-1}r^d\phi(r^{-2}) \leqslant \mathrm{Cap}(\overline{B(0,r)}) \leqslant Cr^d\phi(r^{-2}).$$

Proposition

Let $d \geqslant 3$. If for some $\beta > 0$ a function $\phi(\lambda)\lambda^{-\beta}$ is almost increasing on $[R^{-1}, \infty)$, then

$$G(x) \approx \frac{1}{|x|^d \phi(|x|^{-2})}, \qquad |x| \leqslant R.$$

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Theorem

Suppose that ϕ is a SBF. Then $G(x) \approx \frac{1}{|x|^d \phi(|x|^{-2})}$, for $|x| \leqslant R$ iff for some $\beta > 0$ a function $\phi(u)u^{-\beta}$ is increasing on $[R^{-1}, \infty)$.

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- X_t^D -the killed process when exiting the set D.
- The potential of the process X_t^D is called the *Green function* and is denoted by G_D $(d \geqslant 3)$.

$$G_D(x,y) = G(x-y) - E^x G(X_{\tau_D} - y).$$

Theorem

Let $d \geqslant 3$ and ϕ be a SBF. For any open set D

$$G_D(x,y) \geqslant C_1 G(x-y), \qquad 2|x-y| \leqslant \delta_D(x) \vee \delta_D(y), \tag{1}$$

where $C_1 = \frac{\Gamma\left(\frac{d}{2}-1, \frac{1}{4}\right)}{\Gamma\left(\frac{d}{2}-1\right)} \left(1 - e^{-\frac{3}{4}}\right)$.

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Corollary

In the above two cases we have

$$CG(x,y) \leqslant G_{B(0,r)}(x,y) \leqslant G(x,y), \qquad |x|, |y| \leqslant r/3.$$

From now on $b = 0$

- Let u be a Borel measurable function on $\mathbb{R}^d$. We say that h is *harmonic* function in an open set $D \subset \mathbb{R}^d$ if

$$h(x) = E^x h(X_{\tau_B}), \quad x \in B,$$

for every bounded open set B with the closure $\overline{B} \subset D$.

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- We say that h is *regular harmonic* if

$$f(x) = E^x[\tau_D < \infty; h(X_{\tau_D}))], \quad x \in D.$$

Theorem

Let $d \geqslant 1$. Suppose that for some $\beta > 0$ a function $\phi(\lambda)\lambda^{-\beta}$ is almost increasing on $[R^{-1}, \infty)$. Then there exists a constant $C = C(R)$ such that for all $r \in (0, R]$ and any non-negative function h which is harmonic in $B(0, r)$

$$\sup_{x \in B(0,r/3)} h(x) \leqslant C \inf_{x \in B(0,r/3)} h(x).$$

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Theorem

Let $d \geqslant 1$. Suppose that for some $\beta > 0$ a function $\phi(\lambda)\lambda^{-\beta}$ is almost increasing on $[R^{-1}, \infty)$. Then there exist constants $C = C(R)$ and $\gamma > 0$ such that for all $r \in (0, R]$ and any bounded function h which is harmonic in $B(0, r)$

$$|h(x) - h(y)| \leqslant C \sup_z |h(z)| \left(\frac{|x - y|}{r} \right)^{\gamma}, \qquad x, y \in B(0, r/3).$$

Examples

Recall that

$$E^{iz\cdot X_t} = e^{-t\phi(|z|^2)}$$

- Let f be a SBF. Define $\phi(u) = \frac{u}{f(u)}$. If, for $\lambda, u \geqslant 1$,

$$\frac{f'(\lambda u)}{f'(u)} \leqslant c\lambda^{-\beta}.$$

  then the scale-invariant Harnack inequality holds for $r \leqslant 1$.

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  then the scale-invariant Harnack inequality holds for $r \leqslant 1$.
- Let $\phi(u) \approx u^{\alpha}\ell_1(u)$ for $u \geqslant 1$ and $\phi(u) \approx u^{\beta}\ell_2(u)$, for $u \leqslant 1$, where ℓ_1, ℓ_2 are slowly varying functions. If $\alpha, \beta > 0$ then the scale-invariant Harnack inequality holds, for any $0 < r < \infty$.

Proposition

Let $d \geqslant 3$ and for some $\beta > 0$ a function $\phi(\lambda)\lambda^{-\beta}$ be almost increasing on $[R^{-1}, \infty)$. Then there exists a constant $C_5 = C_5(R)$ such that for any $r \leqslant R$ and any compact $A \subset B(0,r)$, $x \in B(0,r)$,

$$P^x(\tau_{A^c} < \tau_{B(0,3r)}) \geqslant C_5 \frac{|A|}{|B(0,r)|}.$$

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