Motions in a random flow

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Model

- Motion in a random incompressible flow

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\begin{aligned}
&\frac{dx(t)}{dt} = \vec{u}(t, x(t)), \quad t \geqslant 0, \\
&x(0) = 0, \\
&\textstyle\sum_{j=1}^{d} \partial_{x_j} u_j(t, x) \equiv 0.
\end{aligned}
\tag{1}
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- $\vec{u}(t, x)$ (*(Eulerian) velocity field of the fluid*) **random vector field**
- basic model of transport in a turbulent flow of fluid

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Basic question

Statistics of a tracer

Knowing the statistics of the flow describe the behavior of the particle

Law of large numbers

Stokes drift

$$v_* = \lim_{t \to +\infty} \frac{x(t)}{t}$$

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Central limit theorem

$[x(t) - v_* t]/\sqrt{t} \Rightarrow N(0, D)$, where

$$D_{ij} = \lim_{t\to+\infty} \frac{1}{t}\mathbb{E}\left[(x_i(t) - v_{*,i}t)(x_j(t) - v_{*,j}t)\right].$$

turbulent diffusivity

usually the assumptions of **strong mixing in time** is made;

Kraichnan, Gawedzki-Kupiainen (flow is white noise in time), T.K.-Papanicolaou 97' (Gaussian, finite dependence range in time), Carmona-Xu 97', Fannjiang-T.K. 99', L. Koralov, 99' (Markovian in time +spectral gap), T.K.-S. Olla 05' (O-U flow with weaker mixing assumptions).

Weakly coupled case

$$\frac{dx(t)}{dt} = \varepsilon\vec{u}(t, x(t)), \quad t \geqslant 0,$$

$$x(0) = x_0, \quad \varepsilon \ll 1$$

Suppose that $\vec{u}(t, x)$ is a random vector field over a probability space $(\Omega, \mathcal{V}, \mathbb{P})$

- time-space stationary
- of zero mean

$$\langle \vec{u}(0, 0) \rangle = 0.$$

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Convergence

Long time behavior of the tracer $x_\varepsilon(t) := x(t/\varepsilon^2)$.

$$\frac{dx_\varepsilon(t)}{dt} = \frac{1}{\varepsilon}\vec{u}\left(\frac{t}{\varepsilon^2}, x_\varepsilon(t)\right), \quad t \geqslant 0,$$

Martingale argument.

Suppose that $f \in C_0^\infty(\mathbb{R}^d)$, $t_i = i\varepsilon^\gamma$, $s < t$

$$\mathbb{E}\left[f(x_\varepsilon(t)) - f(x_\varepsilon(s))\Big|\mathcal{V}_s\right] \approx \sum_{[s\varepsilon^{-\gamma}]}^{[t\varepsilon^{-\gamma}]} \mathbb{E}\left[\Delta f(x_\varepsilon(t_i))\Big|\mathcal{V}_s\right]$$

$$= \frac{1}{\varepsilon}\sum_{[s\varepsilon^{-\gamma}]}^{[t\varepsilon^{-\gamma}]} \int_{t_i}^{t_{i+1}} \mathbb{E}\left[\nabla f(x_\varepsilon(\rho_1)) \cdot \vec{u}\left(\frac{\rho_1}{\varepsilon^2}, x_\varepsilon(\rho_1)\right)\Big|\mathcal{V}_s\right] d\rho_1$$

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Convergence cont'd

$$= \frac{1}{\varepsilon} \sum_{[s\varepsilon^{-\gamma}]}^{[t\varepsilon^{-\gamma}]} \int_{t_i}^{t_{i+1}} \mathbb{E}\left[\nabla f(x_\varepsilon(t_i)) \cdot \vec{u}\left(\frac{\rho_1}{\varepsilon^2}, x_\varepsilon(t_i) \right) \Big| \mathcal{V}_s \right] d\rho_1$$

$$+ \frac{1}{\varepsilon^2} \sum_{[s\varepsilon^{-\gamma}]}^{[t\varepsilon^{-\gamma}]} \int_{t_i}^{t_{i+1}} d\rho_1 \int_{t_i}^{\rho_1} \mathbb{E}\left[\nabla^2 f(\cdot) \cdot \vec{u}\left(\frac{\rho_1}{\varepsilon^2}, \cdot \right) \otimes \vec{u}\left(\frac{\rho_2}{\varepsilon^2}, \cdot \right)\Big|_{x_\varepsilon(t_i)} \Big| \mathcal{V}_s \right]$$

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$$+ O(\varepsilon^{2\gamma - 3})$$

We need $\gamma \in (1, 2)$ to make this scheme work!

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Central limit theorem

Khasminskii 66'

Suppose that $\vec{u}(t,x)$ is

1) zero mean, time-space stationary, with incompressible realizations:

$$\nabla \cdot \vec{u}(t,x) = \sum_{j=1}^{d} \partial_i u_j(t,x) \equiv 0,$$

2) sufficiently strongly mixing in t variable,
3) sufficiently smooth with the respective derivatives bounded.

Then, the process $\{x_\varepsilon(t),\ t \geqslant 0\}$ converges in law, as $\varepsilon \to 0+$, to a Brownian motion,covariance matrix $D = [D_{ij}]$, (Kubo formula):

$$D_{ij} = \frac{1}{2}\int_0^{+\infty} \{\mathbb{E}[u_i(t,0)u_j(0,0)] + \mathbb{E}[u_j(t,0)u_i(0,0)]\}dt.$$

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Some historical remarks

Analogous results:

- Borodin 77' (unbounded fields),
- Kesten-Papanicolaou 79' (time independent situation),
- Kunita 86' (flows),
- T.K. 96' (longer time scales).

Isotropic Ornstein-Uhlenbeck flows with spectrum satisfying power law

- $\vec{u}(t,x)$ is zero mean, stationary Gaussian, Markovian in t

$$R_{pq}(t,x) = \langle u_p(t,x)u_q(0,0)\rangle$$
$$= \int e^{ix\cdot k} e^{-\gamma(|k|)t} \hat{R}_{pq}(k) \frac{dk}{|k|^{d-1}}$$

$$\hat{R}_{pq}(k) = r(|k|)\Gamma_{pq}(\hat{k}), \quad p, q = 1, \ldots, d.$$

- factor $\Gamma_{pq}(\hat{k}) := \delta_{pq} - \hat{k}_p\hat{k}_q$, where $\hat{k} = (\hat{k}_1, \ldots, \hat{k}_d) := k/|k|$, ensures incompressibility of the flow
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- $$r(\ell) = \frac{a(\ell)}{\ell^\alpha}, \quad \gamma(\ell) := \ell^\beta,$$

- $a(\cdot)$ is a compactly supported cut-off function, ensures integrability of the spectrum at ∞ and $a(0) > 0$.
- integrability of $r(\ell) = \frac{a(\ell)}{\ell^\alpha}$ at 0: $\alpha < 1$
- mixing rate $\gamma(\ell) := \ell^\beta$ with $\beta \geqslant 0$.
- the decay of the spatial correlations

$$R_{ij}(0,x) \sim |x|^{\alpha-1}$$

Kubo formula

$$D_{pq} = D\delta_{pq}, \quad D = \left(1 - \frac{1}{d}\right)|S_{d-1}| \int_0^{+\infty} \frac{a(\ell)d\ell}{\ell^{\alpha+\beta}}.$$

$D < +\infty$ iff $\alpha + \beta < 1$.

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Central limit theorem when $D < +\infty$

Theorem (Fannjiang-T.K. 99')

Suppose that $\vec{u}(t,x)$ is a Gaussian, Markovian flow as described before. If $D < +\infty$ then $\{x_\varepsilon(t),\ t \geqslant 0\}$ converge in law, as $\varepsilon \to 0+$, do a Brownian motion with the covariance matrix given by the Kubo formula.

What happens when $D = +\infty$?

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Superdiffusive scaling

Let $x_\varepsilon(t) := x(t/\varepsilon^{2\delta})$. We expect $\delta < 1$.

$$x_\varepsilon(t) = \varepsilon \int_0^{t/\varepsilon^{2\delta}} \vec{u}\,(s, \varepsilon x(s))\, ds.$$

Stationarity of $\vec{u}(s, \varepsilon x(s))$ (Theorem of Port-Stone) $\Rightarrow$

$$\mathbf{E}\left[x_\varepsilon^{(i)}(t) x_\varepsilon^{(j)}(t)\right]$$

$$= \sum_{(i,j)=(p,q),(q,p)} \varepsilon^2 \int_0^{\frac{t}{\varepsilon^{2\delta}}} ds \int_0^s \mathbf{E}\left[u_i(s', \varepsilon x(s')) u_j(0,0)\right] ds'$$

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Superdiffusive scaling cont'd

$$\mathcal{I}_n = \sum_{(i,j)=(p,q),(q,p)} \varepsilon^{n+1} \int_{\Delta_n(t/\varepsilon^{2\delta})} \mathbf{E}\left[W_{n-1,i}(s_1,\cdots,s_n,\mathbf{0})u_j(0,\mathbf{0})\right] ds$$

$$W_0(s_1,x) = \vec{u}(s_1,x)$$

$$W_n(s_1,\cdots,s_{n+1},x) = \left(\vec{u}(s_{n+1},x)\cdot\nabla\right) W_{n-1}(s_1,\cdots,s_n,x) \qquad \text{for } n=1,$$

(iterative convective derivatives)

$$\mathcal{R}_N = \sum_{(i,j)=(p,q),(q,p)} \varepsilon^{N+2} \int_{\Delta_{N+1}(t/\varepsilon^{2\delta})} \mathbf{E}\left[W_{N,i}(s_1,\cdots,s_{N+1},\varepsilon\mathbf{x}(s_{N+1}))u_j(0,0)\right]$$

Superdiffusive scaling cont'd

$$\mathcal{I}_n = \sum_{(i,j)=(p,q),(q,p)} \varepsilon^{n+1} \int_{\Delta_n(t/\varepsilon^{2\delta})} \mathbf{E}\left[W_{n-1,i}(s_1,\cdots,s_n,\mathbf{0})u_j(0,\mathbf{0})\right] ds$$

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Elementary calculations

$$\sum_{(i,j)=(p,q),(q,p)} \varepsilon^2 \int\limits_0^{\frac{t}{\varepsilon^{2\delta}}} ds \int\limits_0^{s} \mathbf{E}\left[u_i(s',0)u_j(0,0)\right] ds'$$

$$= c_d \delta_{pq} \varepsilon^2 \int_0^{+\infty} \frac{a(\ell)[e^{-\ell^\beta t/\varepsilon^{2\delta}} - 1 + \ell^\beta t/\varepsilon^{2\delta}]d\ell}{\ell^{\alpha+2\beta}}$$

(substitution $\ell' := \ell t^{1/\beta}/\varepsilon^{2\delta/\beta}$)

$$\underset{\varepsilon\to 0+}{\longrightarrow} a(0)c(d,\alpha,\beta)\delta_{pq}t^{2H}$$

$$H = \frac{\alpha+2\beta-1}{2\beta}, \quad \delta = \frac{1}{2H}$$

$\alpha + \beta > 1 \Rightarrow 1 > H > 1/2$, $\delta \in (1/2, 1)$

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Superdiffusive fBm limit

Theorem (Fannjiang-T.K. 00')

If $D = +\infty$ $(\alpha + \beta > 1)$ then $\{\varepsilon x(t/\varepsilon^{2\delta}),\ t \geqslant 0\}$ converge in law, as $\varepsilon \to 0+$, to a fractional Brownian motion with the **Hurst exponent**

$$H = \frac{\alpha + 2\beta - 1}{2\beta}$$

and the covariance matrix given by $D_{pq} = D\delta_{pq}$,

$$D = 2a(0)|S_{d-1}|\left(1 - \frac{1}{d}\right)\int_0^{+\infty} \frac{[e^{-\ell^\beta} - 1 + \ell^\beta]d\ell}{\ell^{\alpha+2\beta}}.$$

The relative motion of two particles

$x_i(t)$, $i = 1, 2$ two particles satisfying

$$\frac{dx_i(t)}{dt} = \varepsilon \vec{u}(t, x_i(t)), \quad t \geqslant 0,$$

$$x(0) = x_i, \quad \varepsilon \ll 1, \quad i = 1, 2$$

$z_\varepsilon(t) := x_2(t/\varepsilon^2) - x_1(t/\varepsilon^2)$, $z := x_2 - x_1$.

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Two particle motion - description

$$\dot{x}_\varepsilon(t) = \frac{1}{\varepsilon}\vec{u}\left(\frac{t}{\varepsilon^2}, x_\varepsilon(t)\right), \quad x_\varepsilon(0) = 0, \tag{2}$$

$$\dot{z}_\varepsilon(t) = \frac{1}{\varepsilon}\vec{v}\left(\frac{t}{\varepsilon^2}, x_\varepsilon(t), z_\varepsilon(t)\right), \quad z_\varepsilon(0) = z,$$

$$\vec{v}(t,x,z) := \vec{u}\,(t, x+z) - \vec{u}\,(t,x)\,.$$

Relative velocity

$$\vec{v}(t,x,z) = \sqrt{2}\int_{-\infty}^{t}\int e^{-|k|^\beta (t-s)} e^{ik\cdot x}(e^{ik\cdot z} - 1)|k|^{\beta/2} w(ds, dk). \tag{3}$$

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$w(dt, dk)$ – $\mathbb{C}^d$-valued, space-time Gaussian noise:

$$w^*(dt, dk) = w(dt, -dk),$$

$$\mathbb{E}\left[w_i(dt, dk)w_j^*(dt', dk')\right] = \hat{R}_{ij}(k)\delta(t - t')\delta(k - k')dtdt'dkdk',$$

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Martingale argument

$$\Delta t_i = \varepsilon^{\gamma},\ \gamma \in (1,2)$$

$$f(z_\varepsilon(t)) - f(z_\varepsilon(s)) = \sum_i [f(z_\varepsilon(t_{i+1})) - f(z_\varepsilon(t_i))] \tag{5}$$

$$= \frac{1}{\varepsilon}\sum_{p=1}^{d}\sum_i \int_{t_i}^{t_{i+1}} \partial_p f(z_\varepsilon(s)) v_p\left(\frac{s}{\varepsilon^2}, z_\varepsilon(s), x_\varepsilon(s)\right) ds$$

$$= \frac{1}{\varepsilon}\sum_{p=1}^{d}\sum_i \int_{t_i}^{t_{i+1}} \partial_p f(z_\varepsilon(t_{i-1})) v_p\left(\frac{s}{\varepsilon^2}, z_\varepsilon(t_{i-1}), x_\varepsilon(t_{i-1})\right) ds \tag{6}$$

$$+\frac{1}{\varepsilon}\sum_{p=1}^{d}\sum_i \int_{t_i}^{t_{i+1}} \left\{\int_{t_{i-1}}^{s} \frac{d}{d\rho}\left[\partial_p f(z_\varepsilon(\rho)) v_p\left(\frac{s}{\varepsilon^2}, z_\varepsilon(\rho), x_\varepsilon(\rho)\right)\right] d\rho\right\} ds$$

$$= J_1 + J_2 \tag{7}$$

Martingale argument - conditioning

$$\mathbb{E}\left[J_1\middle|\mathcal{F}_s\right]$$

$$=\mathbb{E}\left[\frac{1}{\varepsilon}\sum_{p=1}^{d}\sum_{i}\int_{t_i}^{t_{i+1}}\partial_p f(z_\varepsilon(t_{i-1}))\bar{v}_p\left(\frac{s}{\varepsilon^2},\frac{t_{i-1}}{\varepsilon^2},z_\varepsilon(t_{i-1}),x_\varepsilon(t_{i-1})\right)ds\middle|\mathcal{F}_s\right]$$

$$\approx 0,$$

as $\varepsilon \to 0+$.

Martingale argument - conditioning

$$\mathbb{E}\left[J_2\middle|\mathcal{F}_s\right]=\mathbb{E}\left[\sum_{p,q=1}^{d}\sum_i \partial^2_{pq}f(z_\varepsilon(t_i))c_{pq}(z_\varepsilon(t_i))\Delta t_i\middle|\mathcal{F}_s\right]$$

$$\approx\mathbb{E}\left[\sum_{p,q=1}^{d}\int_s^t \partial^2_{pq}f(z_\varepsilon(u))c_{pq}(z_\varepsilon(u))du\middle|\mathcal{F}_s\right],$$

as $\varepsilon\to 0+$.

Formula for diffusivity

$$c_{pq}(z):=\int_0^\infty \mathbb{E}\left[v_p(t,0,z)v_q(0,0,z)\right]dt$$

$$=\int\frac{1-\cos(k\cdot z)}{|k|^{\alpha+\beta+d-1}}\Gamma_{pq}(k)dk,\qquad z\in\mathbb{R}^d,\ p,q=1,\ldots,d.$$

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Convergence result

Theorem, (T.K., Novikov, Ryzhik 12')

Suppose that $\alpha+\beta>1$ and $\alpha+2\beta<3$. Then $\{z_\varepsilon(t),\ t\geqslant 0\}$ converge in law over $C[0,+\infty)$, as $\varepsilon\to 0+$ to the diffusion with the generator

$$Lf(z)=\sum_{p,q=1}^{d} c_{pq}(z)\partial^2_{p,q}f(z),\quad f\in C_0^\infty(\mathbb{R}^d). \tag{8}$$

Related results

Time independent case

$$\frac{dx(t)}{dt} = v + \varepsilon \vec{u}(x(t)), \quad x(0) = 0. \tag{9}$$

Here $v \neq 0$, $\varepsilon \ll 1$, $\vec{u}(x)$ stationary, zero mean, $R(x) = [R_{ij}(x)]$ the covariance matrix, with div-free realizations.

$$y_\varepsilon(t) = y(t/\varepsilon^2) := x\left(t/\varepsilon^2\right) - vt/\varepsilon^2,$$

Theorem (Kesten-Papanicolaou 79')

Suppose that $\vec{u}(t,x)$ is mixing at sufficiently fast rate. Then, $\{y_\varepsilon(t),\ t \geqslant 0\}$ converge in law to a Brownian motion with the covariance matrix $D = [D_{ij}]$

$$D_{ij} = \frac{1}{2}\int_0^\infty \left(R_{ij}(\mathbf{v}t) + R_{ji}(\mathbf{v}t)\right) dt, \quad i,j = 1,\ldots,d. \tag{10}$$

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Gaussian drifts

Covariance

$$R_{ij}(x) = \int_{\mathbb{R}^d} e^{ik\cdot x}\hat{R}_{ij}(k)dk, \tag{11}$$

the power-energy spectrum:

$$\hat{R}_{ij}(k) = \frac{1_{[0,K]}(|k|)}{|k|^{\alpha+d-1}}\Gamma_{ij}(\hat{k}), \tag{12}$$

where $\alpha < 1$. The rate of decay of the correlations

$$R_{ij}(x) \sim |x|^{\alpha-1}, \quad \text{for } |x| \gg 1. \tag{13}$$

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Result

Theorem (T.K. Ryzhik 07')

Suppose that $t > 0$, $\alpha < 0$ and $\rho > 0$. Then,

$$\lim_{\varepsilon \to 0+} \mathbb{E}\left|y\left(t/\varepsilon^{2(1-\rho)}\right)\right|^2 = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0+} \mathbb{E}\left|y\left(t/\varepsilon^{2(1+\rho)}\right)\right|^2 = +\infty.$$

When $\alpha \in (0,1)$ we have

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Result without the assumption of weak coupling

$$\frac{dx(t)}{dt} = \vec{u}(x(t)), \quad x(0) = 0, \tag{14}$$

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Instead of Gaussian - "Poisson shots type" fields (Nieznaj 11')

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Instead of Gaussian - **"Poisson shots type"** fields (Nieznaj 11')

Motion in a random Hamiltonian field

$$\begin{aligned} \frac{dx(t)}{dt} &= \nabla_k H_\varepsilon(x(t), k(t)), \\ \frac{dk(t)}{dt} &= -\nabla_x H_\varepsilon(x(t), k(t)), \\ x(0) &= x_0, \quad k(0) = k_0. \end{aligned} \tag{15}$$

$$H_\varepsilon(x, k) := \frac{|k|^2}{2} + \varepsilon V(x),\ \varepsilon \ll 1 \quad \text{(weak coupling regime)}$$

$V(x)$ random potential

Result on a particle diffusion approximation

Theorem (Kesten-Papanicolaou, 80' ($d \geqslant 3$), T.K., L.Ryzhik, 06' ($d = 2$))

If $V(x)$ is **strictly stationary, sufficiently strongly mixing and** $d \geqslant 2$ then $\{k(t/\varepsilon^2),\ t \geqslant 0\}$converges in law, as $\varepsilon \to 0+$, to a diffusion $\{\bar{k}(t),\ t \geqslant 0\}$ on a sphere $S_{|k_0|} := [|k| = |k_0|]$. Moreover $\varepsilon^2 x(t/\varepsilon^2)$ converge to

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Motion of two particles

Suppose that $d \geqslant 3$.

$$\frac{dx_i(t)}{dt} = \nabla_k H_\varepsilon(x_i(t), k_i(t)), \tag{16}$$
$$\frac{dk_i(t)}{dt} = -\nabla_x H_\varepsilon(x_i(t), k_i(t)),$$
$$x_i(0) = x_i^{(0)}, \quad k_i(0) = k_i^{(0)}, \; i = 1, 2..$$

$$H_\varepsilon(x, k) := \frac{|k|^2}{2} + \varepsilon V(x)$$

Motion of two particles cont'd

- if $|x_1^{(0)} - x_2^{(0)}| \gg O(\varepsilon^2)$ then the particles would experience approx. two different random media. The limit should be two copies of motions based on independent diffusions (Bal, T.K., Ryzhik 03')
- $\tilde{x}_2^{(0)} = x + \varepsilon^2 y$; separation of the initial momentum:

  - if $|k_1^{(0)} - k_2^{(0)}| \gg O(1)$ then the limit should consist of two copies of independent diffusions,

  - if $k_2^{(0)} = k + \varepsilon^{2/3} p$ we should have a nontrivial limit for the separation process in times $t \sim \varepsilon^{-4/3}$ (work in progress).

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Propagation of parabolic waves in a random medium

Schrödinger equation with a random potential

$$i\varepsilon\frac{\partial\phi_\varepsilon}{\partial t}+\frac{\varepsilon^2}{2}\Delta\phi_\varepsilon-\gamma V\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)\phi_\varepsilon=0, \tag{17}$$
$$\phi_\varepsilon(0,x)=\phi_0(x/\varepsilon).$$

Here $V(t,x)$ is a random field in the spatial dimension $d\geqslant 1$, $\gamma\ll 1$ is the (small) parameter that measures the relative strength of the (weak) random fluctuations, $\varepsilon\ll 1$ corresponds to the macroscopic/microscopic scales ratio, $\phi_0\in\mathcal{S}(\mathbb{R}^n)$.

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About the potential

$V(t,x)$ is time-space stationary gaussian field, covariance function $R(t,x)=\mathbb{E}[V(t,x)V(0,0)]$ has the spatial power spectrum:

$$\tilde{R}(t,k)=\int e^{-ik\cdot x}R(t,x)dx$$

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where $\hat{R}(k)\in L^1(\mathbb{R}^d)$ equivalently

$$R(t,x)=\int_{\mathbb{R}^{1+d}} e^{i\omega t+ikx}\hat{R}(\omega,k)d\omega dk,$$

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Behavior of the covariance of the wave function

Wigner transform

$$W_\varepsilon(t,x,k) = \int \phi_\varepsilon\left(t, x - \frac{\varepsilon y}{2}\right) \bar{\phi}_\varepsilon\left(t, x + \frac{\varepsilon y}{2}\right) e^{ik\cdot y} \frac{dy}{(2\pi)^n}. \quad (20)$$

It satisfies the following kinetic equation

$$W_t^\varepsilon + k \cdot \nabla_x W^\varepsilon - \mathcal{L}^\varepsilon W^\varepsilon = 0. \quad (21)$$

$$\mathcal{L}^\varepsilon W(x,k) :=$$
$$\frac{i\gamma}{\varepsilon} \int_{\mathbb{R}^d} e^{ip\cdot x} \hat{V}(t/\varepsilon, p) \left[W(x, k - \frac{\varepsilon p}{2}) - W(x, k + \frac{\varepsilon p}{2})\right] \frac{dp}{(2\pi)^d}$$

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Theorem: Limiting behavior of Wigner transform, Ball, T.K., Ryzhik (10) (gaussian), Bal, Papanicoalou, Ryzhik (02') (shot noise type fields)

Suppose that $\gamma(k) \geqslant \gamma_0 > 0$ and $\gamma = \sqrt{\varepsilon}$, then the processes $\{W_\varepsilon(t),\ t \geqslant 0\}$ converge, as $\varepsilon \to 0$, in probability, in the topology of $C([0,+\infty); L^2_w(\mathbb{R}^{2d}))$, to the solution $\overline{W}$ of the following transport equation

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This result can be generalized, see Gomez (11'),

$$\tilde{R}(t,k) = e^{-\gamma(k)|t|}\hat{R}(k), \tag{23}$$

where $\hat{R}(k) \sim |k|^{-d-\alpha+1}$, $\gamma(k) \sim |k|^{\beta}$ equivalently

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Behavior of the wave function

Lack of coherence, we need to compensate for oscillations

Define

$$\hat{\zeta}_\varepsilon(t,\xi) = \frac{1}{\varepsilon^d}\hat{\phi}_\varepsilon(t,\xi/\varepsilon)e^{i|\xi|^2 t/(2\varepsilon)}, \tag{24}$$

We have phase transition!!!

Strongly decorrelating fields

Theorem (T.K., L. Ryzhik 2010)

Assume that the spatial power spectrum satisfies

$$\int \frac{\hat{R}(p)dp}{\gamma(p)} < +\infty \tag{25}$$

and $\gamma = \sqrt{\varepsilon}$. Then, for each $(t,\xi) \in \mathbb{R}^{1+d}$ fixed, $\hat{\zeta}_\varepsilon(t,\xi)$ converges in law, as $\varepsilon \to 0$, to

$$\hat{\zeta}(t,\xi) = e^{-tD_\xi/2}\hat{\phi}_0(\xi) + Z(t,\xi) \tag{26}$$

Here 1) $Z(t,\xi)$ is a centered, complex valued Gaussian random variable, whose variance equals

$$\mathbb{E}|Z(t,\xi)|^2 = \widehat{W}(t,\xi) - e^{-tReD_\xi}|\hat{\phi}_0(\xi)|^2.$$

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Strongly decorrelating fields, cont'd

Theorem, c.d.

2) $D_\xi = \int D(p,\xi)dp$ and

$$D(p,\xi) = \frac{2\hat{R}(p)}{(2\pi)^d[\gamma(p) - i(\xi \cdot p - |p|^2/2)]} \tag{27}$$

3) $\widehat{W}(t,\xi)$ is the solution of:

$$\begin{cases} \partial_t \widehat{W}(t,\xi) = \mathcal{L}\widehat{W}(t,\xi), \\ \widehat{W}(0,\xi) = |\hat{\phi}_0(\xi)|^2. \end{cases} \tag{28}$$

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$$\mathcal{L}F(\xi) := \int D(p,\xi)[F(p) - F(\xi)]dp, \tag{29}$$

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Weak decorrelation

the spatial power spectrum:

$$\hat{R}(p) = \frac{a(p)}{|p|^{\alpha+d-1}} \tag{30}$$

and the mixing rate is

$$\gamma(p) = \mu|p|^{\beta} \tag{31}$$

for some $\alpha < 1$, $0 \leqslant \beta \leqslant 1$, $\mu > 0$, and a compactly supported, non-negative, bounded measurable function $a(p)$. We assume that $a(p)$ is continuous at $p = 0$ and $a(0) > 0$.

In order for the previous regime to hold we need to assume that $\alpha + \beta < 1$. We assume that $\alpha + \beta > 1 \Rightarrow \alpha \in (0, 1)$.

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Result

Theorem (T.K., L. Ryzhik 2010)

Let $\gamma := \varepsilon^\kappa$ and $\kappa = 1 - (1-\alpha)/(2\beta)$.
Then, for each $(t,\xi) \in \mathbb{R}^{1+d}$ fixed, $\hat{\zeta}_\varepsilon(t,\xi)$ converges in law to

$$\bar{\zeta}_0(t,\xi) = \hat{\phi}_0(\xi) e^{i\sqrt{\tilde{D}} B_\kappa(t)}, \tag{32}$$

where $B_\kappa(t;\xi)$ is a standard fractional Brownian motion with Hurst exponent κ. $\tilde{D}$ is given by

$$\tilde{D} = \begin{cases} \dfrac{a(0)K_1(\alpha,\beta,\mu)}{\kappa(2\kappa-1)(2\pi)^d}, & \beta < 1, \\[2ex] \dfrac{a(0)K_2(\xi;\alpha,\mu)}{\alpha(2\alpha-1)(2\pi)^d}, & \beta = 1. \end{cases}$$

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Let $\gamma := \varepsilon^\kappa$ and $\kappa = 1 - (1-\alpha)/(2\beta)$.
Then, for each $(t,\xi) \in \mathbb{R}^{1+d}$ fixed, $\hat{\zeta}_\varepsilon(t,\xi)$ converges in law to

$$\bar{\zeta}_0(t,\xi) = \hat{\phi}_0(\xi) e^{i\sqrt{\tilde{D}} B_\kappa(t)}, \tag{32}$$

where $B_\kappa(t;\xi)$ is a standard fractional Brownian motion with Hurst exponent κ. $\tilde{D}$ is given by

$$\tilde{D} = \begin{cases} \dfrac{a(0)K_1(\alpha,\beta,\mu)}{\kappa(2\kappa-1)(2\pi)^d}, & \beta < 1, \\ \\ \dfrac{a(0)K_2(\xi;\alpha,\mu)}{\alpha(2\alpha-1)(2\pi)^d}, & \beta = 1. \end{cases}$$

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