Lévy measure density corresponding to inverse local time

Tomoko Takemura and Matsuyo Tomisaki

2012. 9.11

motivation

We are concerned with Lévy measure density corresponding to the inverse local time at the regular end point for harmonic transform of a one dimensional diffusion process. We show that the Lévy measure density is represented as a Laplace transform of the spectral measure corresponding to an original diffusion process, where the absorbing boundary condition is posed at the end point if it is regular.

	h transform		Itô and McKean	
$\mathbb{D}_{s,m,k}$	$\longleftrightarrow$	$\mathbb{D}_{s_h,m_h,0}$	$\longleftrightarrow$	$\mathbb{D}^*_{s_h,m_h,0}$
absorbing		absorbing		reflecting
				$n^*(\xi)$

Tabel contents

One dimensional diffusion process

- We set

$$\begin{aligned} s &: \text{continuous increasing fnc. on } I = (l_1, l_2),\ -\infty \le l_1 < l_2 \le \infty \\ m &: \text{right continuous increasing fnc. on } I \\ k &: \text{right continuous nondecreasing fnc. on } I \end{aligned}$$

One dimensional diffusion process

- We set

$$\begin{aligned} s &: \text{continuous increasing fnc. on } I = (l_1, l_2),\ -\infty \le l_1 < l_2 \le \infty \\ m &: \text{right continuous increasing fnc. on } I \\ k &: \text{right continuous nondecreasing fnc. on } I \end{aligned}$$

- $\mathcal{G}_{s,m,k}$: 1-dim diffusion operator with s, m, and k

$$\mathcal{G}_{s,m,k} u = \frac{dD_s u - u dk}{dm}$$

One dimensional diffusion process

- We set

$$
\begin{aligned}
s &: \text{continuous increasing fnc. on } I = (l_1, l_2),\ -\infty \le l_1 < l_2 \le \infty \\
m &: \text{right continuous increasing fnc. on } I \\
k &: \text{right continuous nondecreasing fnc. on } I
\end{aligned}
$$

- $\mathcal{G}_{s,m,k}$: 1-dim diffusion operator with s, m, and k

$$\mathcal{G}_{s,m,k} u = \frac{dD_s u - u dk}{dm}$$

- $\mathbb{D}_{s,m,k}$: 1-dim diffusion process with $\mathcal{G}_{s,m,k}$
  [l_1 is absorbing if l_1 is regular]

One dimensional diffusion process

- $p(t,x,y)$: transition probability w.r.t. dm for $\mathbb{D}_{s,m,k}$

If l_1 is (s,m,k)-regular,

$$p(t,x,y) = \int_{[0,\infty)} e^{-\lambda t}\psi_o(x,\lambda)\psi_o(y,\lambda)\,d\sigma(\lambda), \qquad t>0,\ x,y\in I, \tag{1}$$

where $d\sigma(\lambda)$ is a Borel measure on $[0,\infty)$ satisfying

$$\int_{[0,\infty)} e^{-\lambda t}\,d\sigma(\lambda) < \infty, \qquad t>0, \tag{2}$$

and $\psi_o(x,\lambda)$, $x\in I$, $\lambda\geq 0$, is the solution of the following integral equation

$$\psi_o(x,\lambda) = s(x) - s(l_1) + \int_{(l_1,x]}\{s(x)-s(y)\}\psi_o(y,\lambda)\{-\lambda\,dm(y)+dk(y)\}$$

One dimensional diffusion process

Proposition 2.1

Assume that l_1 is (s, m, k)-entrance and

$$\int_{(l_1, c_o]} \{s(c_o) - s(x)\}^2 \, dm(x) < \infty. \tag{3}$$

Then $p(t, x, y)$ is represented as (1) with $d\sigma(\lambda)$ satisfying (2) and $\psi_o(x, \lambda)$ is the solution of the integral equation

$$\psi_o(x, \lambda) = 1 + \int_{(l_1, x]} \{s(x) - s(y)\} \psi_o(y, \lambda) \{-\lambda \, dm(y) + dk(y)\}.$$

Harmonic transform

- We set

$$\mathcal{H}_{s,m,k,\beta} = \{h > 0;\ \mathcal{G}_{s,m,k} h = \beta h\}, \quad \text{for } \beta \geq 0$$

For $h \in \mathcal{H}_{s,m,k,\beta}$,

$$ds_h(x) = h(x)^{-2} ds(x), \quad dm_h(x) = h(x)^2 dm(x)$$

Harmonic transform

- We set

$$\mathcal{H}_{s,m,k,\beta} = \{h > 0;\ \mathcal{G}_{s,m,k} h = \beta h\}, \quad \text{for } \beta \geq 0$$

For $h \in \mathcal{H}_{s,m,k,\beta}$,

$$ds_h(x) = h(x)^{-2} ds(x), \quad dm_h(x) = h(x)^2 dm(x)$$

- We obtain

$$\mathcal{G}_{s_h,m_h,0} : \text{h transform of } \mathcal{G}_{s,m,k} \quad \left[p_h(t,x,y) = e^{-\beta t} \frac{p(t,x,y)}{h(x)h(y)} \right]$$

Harmonic transform

- We set

$$\mathcal{H}_{s,m,k,\beta} = \{h > 0;\ \mathcal{G}_{s,m,k} h = \beta h\}, \quad \text{for } \beta \geq 0$$

For $h \in \mathcal{H}_{s,m,k,\beta}$,

$$ds_h(x) = h(x)^{-2} ds(x), \quad dm_h(x) = h(x)^2 dm(x)$$

- We obtain

$$\mathcal{G}_{s_h,m_h,0} : \text{h transform of } \mathcal{G}_{s,m,k} \quad \left[p_h(t,x,y) = e^{-\beta t} \frac{p(t,x,y)}{h(x)h(y)} \right]$$

- $\mathbb{D}_{s_h,m_h,0}$: 1-dim diffusion process with $\mathcal{G}_{s_h,m_h,0}$
  [l_1 is absorbing if l_1 is regular]

Harmonic transform

- $\mathbb{D}^*_{s_h,m_h,0}$: 1-dim diffusion process with $\mathcal{G}_{s_h,m_h,0}$
  [l_1 is regular and reflecting boundary]

Harmonic transform

- $\mathbb{D}^*_{s_h,m_h,0}$: 1-dim diffusion process with $\mathcal{G}_{s_h,m_h,0}$
  [l_1 is regular and reflecting boundary]
- $l^{(h*)}(t,\xi)$: local time for $\mathbb{D}^*_{s_h,m_h,0}$, that is,

$$\int_0^t f(X(u))\,du = \int_I l^{(h*)}(t,\xi)\,dm_h(\xi), \quad t > 0,$$

for bounded continuous functions f on I.

Harmonic transform

- $\mathbb{D}^*_{s_h,m_h,0}$: 1-dim diffusion process with $\mathcal{G}_{s_h,m_h,0}$
  [l_1 is regular and reflecting boundary]
- $l^{(h*)}(t,\xi)$: local time for $\mathbb{D}^*_{s_h,m_h,0}$, that is,

$$\int_0^t f(X(u))\,du = \int_I l^{(h*)}(t,\xi)\,dm_h(\xi), \quad t > 0,$$

  for bounded continuous functions f on I.
- $\tau^{(h*)}(t)$: inverse local time ${l^{(h*)}}^{-1}(t, l_1)$ at the end point l_1

Lévy measure density

Proposition 2.2 (Itô and McKean)

Assume the following conditions.

$$l_1 \text{ is } (s, m, 0)\text{-regular and reflecting.}$$
$$s(l_2) = \infty, \quad \text{or} \quad l_2 \text{ is } (s, m, 0)\text{-regular and reflecting.}$$

Then $[\tau^*(t),\ t \geq 0]$ *is a Lévy process and there is a Lévy measure density* $n^*(\xi)$ *such that*

$$E^*_{l_1}\left[e^{-\lambda\tau^*(t)}\right] = \exp\left\{-t\int_0^\infty (1 - e^{-\lambda\xi})n^*(\xi)\,d\,\xi\right\},$$
$$n^*(\xi) = \lim_{x \to l_1} q^*(\xi, x)/\{s(x) - s(l_1)\},$$

where $E^*_{l_1}$ *stands for the expectation with respect to* $P^*_{l_1}$,

Lévy measure density

$$\int_0^t q^*(\xi, x)\, d\xi = P_x^*(\sigma_{l_1} < t), \quad x \in I,\ t > 0,$$

and σ_{l_1} is the first hitting time for l_1. In particular, if $s(l_2) = \infty$, then

$$n^*(\xi) = \lim_{x,y \to l_1} D_{s(x)} D_{s(y)} p(\xi, x, y) = \int_{[0,\infty)} e^{-\lambda\xi} d\sigma(\lambda),$$

where $p(t, x, y)$ is the the transition probability density with respect to dm for $\mathbb{D}_{s,m,0}$, and $d\sigma(\lambda)$ is the Borel measure appeared in the representation (1) satisfying (2).

Main theorem

Now we give a representation of $n^{(h*)}(\xi)$ by means of items corresponding to the diffusion process $\mathcal{D}_{s,m,k}$. l_1 is $(s_h, m_h, 0)$-regular if and only if one of the following conditions is satisfied.

$$l_1 \text{ is } (s, m, k)\text{-regular and } h(l_1) \in (0, \infty). \quad (4)$$

$$l_1 \text{ is } (s, m, k)\text{-entrance, } h(l_1) = \infty, \text{ and } |m_h(l_1)| < \infty. \quad (5)$$

$$l_1 \text{ is } (s, m, k)\text{-natural, } h(l_1) = \infty, \text{ and } |m_h(l_1)| < \infty. \quad (6)$$

Main theorem

Theorem 2.3

*Let $h \in \mathcal{H}_{s,m,k,\beta}$. Assume one of (**??**), (**??**), and (**??**). Further assume that l_1 is reflecting and $s_h(l_2) = \infty$. Then there exists Lévy measure density $n^{(h*)}(\xi)$. In particular, if (**??**) is satisfied, then*

$$
\begin{aligned}
n^{(h*)}(\xi) =& h(l_1)^2 e^{-\beta\xi} \int_{[0,\infty)} e^{-\xi\lambda}\, d\sigma(\lambda) \\
=& h(l_1)^2 e^{-\beta\xi} \lim_{x,y \to l_1} D_{s(x)} D_{s(y)} p(\xi, x, y).
\end{aligned}
$$

*If (**??**) is satisfied, then*

$$
\begin{aligned}
n^{(h*)}(\xi) =& D_s h(l_1)^2 e^{-\beta\xi} \int_{[0,\infty)} e^{-\xi\lambda}\, d\sigma(\lambda) \\
=& D_s h(l_1)^2 e^{-\beta\xi} \lim_{x,y \to l_1} p(\xi, x, y).
\end{aligned}
$$

Examples

Let us consider the following diffusion generators on $(0, \infty)$.

$$\mathcal{L}_1 = \frac{1}{2}\frac{d^2}{dx^2} + \left\{ \frac{1}{2x} + \sqrt{2\beta}\,\frac{K'_\nu\left(\sqrt{2\beta}\,x\right)}{K_\nu\left(\sqrt{2\beta}\,x\right)} \right\} \frac{d}{dx},$$

$$\mathcal{L}_2 = \frac{1}{2}\frac{d^2}{dx^2} + \left\{ -\frac{1}{2x} + 2\kappa x \frac{W'_{-\frac{\beta}{2\kappa}+\frac{\nu+1}{2},\frac{|\nu|}{2}}(\kappa x^2)}{W_{-\frac{\beta}{2\kappa}+\frac{\nu+1}{2},\frac{|\nu|}{2}}(\kappa x^2)} \right\} \frac{d}{dx},$$

where $-1 < \nu < 1$, $\kappa > 0$ and $\beta > 0$. $K_l(x)$ and $W_{k,l}(x)$ are the modified Bessel function and the Whittaker function. Lévy measure densities $n_i^*(\xi)$ of $\tau^*(t)$ are given as follows.

$$n_1^*(\xi) = C\xi^{-(|\nu|+1)}e^{-\beta\xi}, \quad n_2^*(\xi) = C\left(\frac{\kappa}{\sinh(\kappa\xi)}\right)^{|\nu|+1} e^{\{\kappa(\nu+1)-\beta\}\xi}$$

where $C = 2^{-(|\nu|+1)}\Gamma(|\nu|+1)$.