Convergence of Stochastic Processes and Collapsing of Manifolds

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Singular Perturbation

Let $\mathcal{L}_i$ be diffusion operators. Consider

- $$\mathcal{L}^\epsilon := \mathcal{L}_1 + \frac{1}{\epsilon}\mathcal{L}_2,$$

  where $\epsilon \in (0, 1)$ is a small parameter.

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  where $\epsilon \in (0,1)$ is a small parameter.
- Expand the solution, to the equation below, in ϵ,

  $$\frac{\partial u_t^\epsilon}{\partial t} = (\mathcal{L}_1 + \frac{1}{\epsilon}\mathcal{L}_2)(u_t^\epsilon).$$

  $$u_t^\epsilon = u_t^0 + \epsilon u_t^1 + \epsilon^2 u_t^2 + \dots.$$

  We seek an equation for u_t^0, and possibly for u_t^1 ...

History

- Orbits of celestial bodies are governed by a Hamiltonian system on the cotangent bundle : $\dot{u}_t = X_H(u_t)$.
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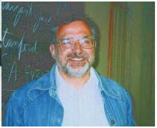
- Reduction in complexity:
  Suppose that the true dynamical system differs from this Hamiltonian system by order ϵ. After a long time of order $\frac{1}{\epsilon}$, how does the orbit deviate from that given by the Hamiltonian system, ? V.I. Arnold, ...

History: Averaging and Homogeneisation

Averaging and homogenisation of parabolic PDEs trace back to: R. Khasminskii (1963), M. Freidlin (1964),


Papanicolaou-Varadhan (1973), Papanicolaou-Stroock-Varadhan (1977).





Book by A. Bensoussan, J.-L. Lions, G. Papanicolaou. 700 pages, expect to find everything!

Development

In elasticity theory, e.g. A. Desimon, S. Müller, R.V. Kohn;
For discrete systems, e.g. A. Gloria and F. Otto.


J.-M. Bismut "Hypoelliptic Laplacian and orbital integrals", "Loops Spaces and hypoelliptic Laplacian" and cohomologies. Look for unspoken Brownian motions.



Hamilton-Jacobi equations, transport equations : E. Kosygina-F. Rezakhanlou-S.R.S. Varadhan-P.-L. Lions-P.E. Souganidis; A. Bensoussan-J. L. Lions- G. Papanicolaou.
Multi scale analysis: A.J. Majda, W. E., E. Vanden-Eijnden, A. Stuart, M. Hairer, J. Mattingly and G. Pavliotis.

Example

Let $\mathcal{L}_2 = \sum_{i,j=1}^{d} a_{i,j}^2(x,y)\frac{\partial^2}{\partial y_i \partial y_j}$, $\mathcal{L}_1 = \sum_{i,j=1}^{d} a_{i,j}^1(x,y)\frac{\partial^2}{\partial x_i \partial x_j}$,

$$\frac{\partial}{\partial t}(u_t^0 + \epsilon u_t^1 + \epsilon^2 u_t^2 + \dots) = (\mathcal{L}_1 + \frac{1}{\epsilon}\mathcal{L}_2)(u_t^0 + \epsilon u_t^1 + \epsilon^2 u_t^2 + \dots).$$

$$0 = \mathcal{L}_2 u_t^0, \qquad \frac{\partial u_t^0}{\partial t} = \mathcal{L}_1 u_t^0 + \mathcal{L}_2 u_t^1.$$

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$$\begin{aligned}\frac{\partial}{\partial t} u_t^0(x) &= \int \left(\sum a_{i,j}^1(x,y)\frac{\partial^2}{\partial x_i \partial x_j}\mu^x(dy)\right) u_t^0(x) + \int \mathcal{L}_2 u_t^1 \mu^x(dy) \\ &= \int \left(\sum a_{i,j}^1(x,y)\mu^x(dy)\right)\frac{\partial^2}{\partial x_i \partial x_j} u_t^0(x) \\ &= \bar{\mathcal{L}}_1 u_t^0(x).\end{aligned}$$

Example: stochastic dynamics

- Let the matrix $(\sigma_1^i, \ldots, \sigma_m^i)$ be a square root of the matrix $(a_{k,j}^i)$. Let (b_t^i, w_t^k) be independent Brownian motions.

$$dx_t^\epsilon = \sum_{k=1}^m \sigma_k^1(x_t^\epsilon, y_t^\epsilon) db_t^k, \qquad dy_t^\epsilon = \frac{1}{\sqrt{\epsilon}} \sum_{k=1}^m \sigma_k^2(x_t^\epsilon, y_t^\epsilon) dw_t^k.$$

  Let $u_t^\epsilon = (x_t^\epsilon, y_t^\epsilon) \in \mathbf{R}^d \times \mathbf{R}^d$. By Itô's formula, the Markov generator for u_t^ϵ is $\frac{1}{\epsilon}\mathcal{L}_2 + \mathcal{L}_1$.

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- The special feature of u_t^ϵ is that it consists of two components, living in the **product space $\mathbf{R}^d \times \mathbf{R}^d$, one of which is clearly the fast variable**.
- In general there will be more interactions, intertwinnings such as rotations. We may consider u_t^ϵ lives in a space upstairs; x_t^ϵ or y_t^ϵ the projection. The total space, where u_t^ϵ lives, is locally a product space.

Examples

We note three classes of spaces, where singular perturbation problem occurs naturally.

1. Symplectic manifolds, e.g. with a completely integrable family of Hamiltonian. [L. 08]
2. The frame bundles of a Riemannian manifold M [L.12]. Why is it interesting? $\dot{u}_t = H_{u_t}(e_0), u_0(e_0) = v_0$ gives the geodesic flow.

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Something in common: there is an almost symplectic structure for (2) and a contact structure for (3). J. Gray 59.

The Hopf Fibration 

Let $SU(2) = \left\{ \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \;\middle|\; z, w \in \mathcal{C}, |z|^2 + |w|^2 = 1 \right\}.$

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1. There is a right action by $U(1)$ on $SU(2)$, defined below:

$$[z, w] \xRightarrow{e^{i\theta}} [e^{i\theta} z, e^{i\theta} w] = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}.$$

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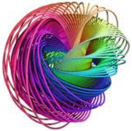
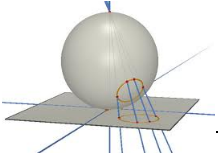
$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[r, "\text{diffeo}"] \arrow[d, "\pi"'] & U_i \times S^1 \arrow[dl, "Proj"] \\ U_i \subset M & \end{tikzcd}$$

There is, hence, a unique manifold structure on M s.t. π is smooth, surjective, a submersion (Tp is surjective), and a fibration with fibre S^1.

The Hopf fibration $\pi : S^3 \to S^2$

Hopf constructed a map from S^3 to S^2 to showed that $H_3(S^2) = Z$. We identify $SU(2)$ with S^3, the set of unit quaternion with non-abelian group multiplication.

The Hopf map is:
$\pi : [z, w] \mapsto (2Re(z\bar{w}), 2Im(z\bar{w}), |z|^2 - |w|^2)$. Indeed,

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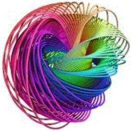
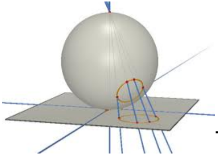
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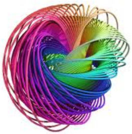

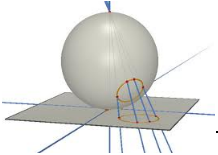

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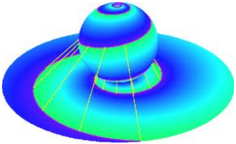
1. $SU(2)/U(1) \sim CP^1$, the complex projective space. It consists of equivalent classes in $\mathcal{C}^2$, $[z, w] \sim [\lambda z, \lambda w]$, $\lambda \neq 0$.
2. $CP^1 \sim S^2$. Let $\phi : \mathbf{R}^2 \to S^2 - \{$ North Pole$\}$ be the stereographic projection. Take $[z : w] \in CP^1$ with $w \neq 0$, define $[z : w] \to \frac{z}{w} \in C \sim \mathbf{R}^2 \xrightarrow{\phi} S^2$.

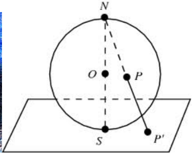
The projection $T\pi : TS^3 \to TS^2$

Consider S^3 as a subset of $\mathbf{R}^4$. Take $z = y_1 + iy_2, w = y_3 + iy_4$.



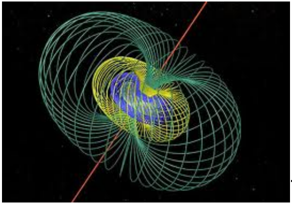
$$T_y\pi = 2\begin{pmatrix} y_3 & y_4, & y_1 & y_2 \\ -y_4 & y_3 & y_2, & -y_1 \\ y_1, & y_2, & -y_3, & -y_4 \end{pmatrix}.$$

The vertical tangent spaces are the kernels of $T_u\pi$.



It is easy to check that $V(y_1, y_2, y_3, y_4) = -y_2\partial_1 + y_1\partial_2 - y_4\partial_3 + y_3\partial_4$ is vertical. The vertical space is one dimensional. In fact at $[z, w]$ the vertical vector fields are generated by $\frac{d}{dt}[e^{it}z, e^{it}w]$.

The Riemannian Structures, $\pi : S^3 \to S^2$

- Let S^3 be given the standard Riemannian structure, that of sub-manifold of $\mathbf{R}^4$.
- There is a unique Riemmanian structure on S^2 such that π is a Riemannian submersion.
  Let $T_u S^3 = [\ker T_u \pi] \oplus HT_u S^3$ be the orthogonal decomposition.  Then $T_u \pi : HT_u \pi \to T_{\pi(u)} S^2$ is an isometry.
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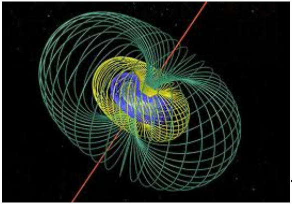
  Let $T_u S^3 = [\ker T_u \pi] \oplus HT_u S^3$ be the orthogonal decomposition.  Then $T_u \pi : HT_u \pi \to T_{\pi(u)} S^2$ is an isometry.
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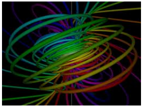
The holonomy group is S^1: any two points in $\pi^{-1}(x)$ can be connected by a horizontal curve. (Non-integrability).

The Pauli matrices

- $SU(2)$ is a simply connected Lie group. Its Lie algebra $\mathfrak{su}(2)$ consists of matrices of the form, $\begin{pmatrix} ia & \beta \\ -\bar{\beta} & -ia \end{pmatrix}$ where $a \in \mathbf{R}, \beta \in \mathcal{C}$. Define$\langle A, B \rangle = \frac{1}{2} trace(AB^*)$.

  The pauli matrices form an o.n.b.:
  $X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, X_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$.

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  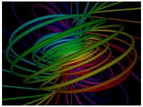

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  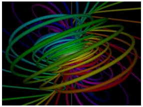

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- The structural constants are $\{-2, -2, -2\}$, see J. Milnor [Mil76] for a discussion on classifications of three dimensional Lie groups.
- Let X_i denote also the corresponding left invariant vector fields.

  $[X_2, X_3] = -2X_1, [X_3, X_1] = -2X_2, [X_1, X_2] = -2X_3$. The horizontal distributions are not integrable.

Berger's Spheres

The right invariant vector field $X_1 \sim \frac{d}{d\theta}$ is the action field.

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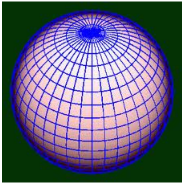
- The spaces (S^3, m_ϵ) are Berger's spheres. 
- Collapsing: The diameter of the orbits of Berger's spheres is ϵ, which shrinks to zero. The injectivity radius of $(S^3, m_\epsilon) \to 0$ as $\epsilon \to 0$. The volume of S^3 shrinks to zero.

Collapsing of (S^3, m_ϵ)

- Berger: (S^3, m_ϵ) converges to $S^2(\frac{1}{2})$ in Gromov-Hausdorff distance. The limit space is a lower dimensional manifold.



- Gromov-Cheeger, [CG86], would like to see collapsings of manifold sequences while keeping sectional curvatures uniformly bounded.
  For Berger's spheres:

$$K^\epsilon(X_1, X_2) = \epsilon^2, K^\epsilon(X_1, X_3) = \epsilon^2, K^\epsilon(X_2, X_3) = 4 - 3\epsilon.$$

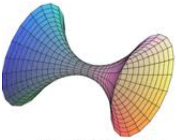
Why bounded sectional curvature?

Let us look at an example for some intuition on the requirement 'bounded sectional curvature'.
Consider Riemannian manifold (M, g_t), where $g_t \in (\wedge^2 T_M)^*$ satisfies:

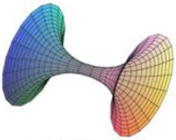
$$\dot{g}_t = -2Ric_{g_t}, \qquad g_0 \text{ smooth.}$$

R. S. Hamilton 82 proved short time existence and uniqueness.
Let $g_t, t \in (0, T)$ be a maximal flow.

- For $t < T$, the metrics are equivalent:



$\alpha = 1$ (Einstein-Rosen bridge)



$\alpha = 2$ (Ellis wormhole)

$$e^{-2Ct} g(0) \leq g(t) \leq e^{2Ct} g(0).$$

- **The norm of the Riemannian curvature blows up as** $t \uparrow T$ unless $T = \infty$ (Hamilton).

Gromov-Hausdorff Convergence

A sequence (M_n, g_n) converges strongly to (M, g) if there are diffeomorphisms $\phi_n : M_n \to M$ such that $(\phi_n)^* g_n \to g$.

- Let A, B be sets in a metric space (X, d), define

$$d_H(A, B) = \inf\{\epsilon > 0 : B \subset A_\epsilon, A \subset B_\epsilon\}.$$

  For any point x in A there is a point y in B s.t. $d(x, y) \leq \epsilon$.

- **Gromov-Hausdorff distance between metric spaces:**

$$d_{GH}((X_1, d_1), (X_2, d_2)) = \inf_{(\phi_i:(X_i,d_i)\to(X,d))} \{d_H(\phi_1(X_1), \phi_2(X_2))\}.$$

  Here ϕ_i are isometric embeddings.

Two metric spaces are isometric if their distance equals zero. The set of equivalent classes of compact metric spaces with diameter bounded above is compact.

Measured G-H convergence

If $(M_n, g_n) \to (M, g)$ how about the spectral of the Laplacian?

- K. Fukaya introduced **Measured Gromov-Hausdorff convergence**: consider the metric spaces (M_n, g_n, μ_n) where μ_n is a probability measure.
  $\lim_{n\to\infty}(M_n, g_n) = (M, g)$ in measured Gromov-Hausdorff distance if there is a family of measurable maps:
  $\psi_n : M_n \to M$ and positive numbers $\epsilon_n \to 0$ such that $|d(\psi_n(p), \psi_n(q)) - d(p, q)| < \epsilon_n$, $(\psi_n(M_n))_{\epsilon_n} = M$ and $(\psi_n)_*(\mu_n) \to \mu$ weakly.
- One a Riemannian manifold of finite volume, we take the measure to be the volume measure normalised to 1.
- Berger's sphere converges in measured Gromov-Hausdorff distance.

Fukaya's theorem on spectral convergence

Theorem (Fukaya [Fuk87])

Let $\mathcal{D}M(n, D)$ be the closure of the class of Riemannian manifolds whose sectional curvature K is bounded between -1 and 1 in the measured Gromov-Hausdorff distance. Let $\lambda_k(M)$ be the kth-eigenvalue of a manifold $M \in \mathcal{D}M(n, D)$.
Then λ_k can be extended to a continuous function on $\mathcal{D}M(n, D) - \{(point, 1)\}$. For each element $(X, \mu) \in \mathcal{D}M(n, D)$, $\lambda_k(X)$ is the kth eigenvalue of a selfadjoint operator on $L^2(X, \mu)$.

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Y. Ogura [Ogu01], Y. Ogura-S. Taniguchi [OT96] studied the convergence of Brownian motions, in a suitable sense, a family of Riemannian manifolds (M_n, g_n) that converges in Kasue-Kumura's **spectral distance**, where the distance between heat kernels at time t (weighted by $e^{-(t+1/t)}$) are involved.

The Spectrum on (S^3, m_ϵ)

The k-th eigenvalue of S^d is $\lambda_k = k(d+k-1)$, $k \geq 0$ with multiplicity $\mu_k = C_k^{d+k} - C_{k-2}^{d+k-2}$.

$$S^3 : \lambda_k = k(k+2) : 0, 3, 8, \ldots$$
$$S^2 : \lambda_k = k(k+1) : 0, 4, 6, 12, \ldots$$
$$S^1 : \lambda_k = k^2 : 0, 1^2, 4, 9, \ldots.$$

$\Delta^\epsilon = \frac{1}{\epsilon}\mathcal{L}_{X_1^*}\mathcal{L}_{X_1^*} + \mathcal{L}_{X_2^*}\mathcal{L}_{X_2^*} + \mathcal{L}_{X_3^*}\mathcal{L}_{X_3^*} = \Delta^\epsilon_{S^1} + \Delta_h$. Facts:

- Δ_{S^3}, Δ_h, $\Delta^\epsilon_{S^1}$ commute. See L. Bérard-Bergery and J.-P. Bourguignon[BBB82], O'Neill [O'N67]
- $\Delta(f \circ \pi) = \Delta_{S^2} f \circ \pi$. c.f. [ELJL99]. The spectrum of Δ and Δ_h contains that of S^2.
- $\lambda_1(\Delta^\epsilon) \to \lambda_1(S^2(\frac{1}{2})) = 4 \cdot 1(1+1) = 8$,

$$\lambda_1(\Delta^\epsilon) = \min\{8 + 0, 2 + \frac{1}{\epsilon}1^2\} = 8, \text{ when } \epsilon^2 < \frac{1}{6}.$$

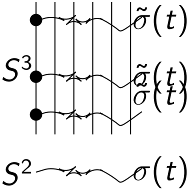
S. Tanno [Tan80][BBB82]. Fukaya's Theorem Applies, easily!

Convergences associated to collapsing of the manifolds

Horizontal Lifts

The orthogonal splitting, $T_uS^3 = H_uTS^3 \oplus \ker(T_u\pi)$, of the tangent space induces a S^1-invariant connection on S^3. Note that the kernel $\ker(T\pi)$ consists of the right invariant vector field from X_1^*. The horizontal tangent space is clearly given by the right invariant vector fields X_2^* and X_3^*: the Riemannian metric on S^1 is right invariant.

If σ is a semi-martingale on S^3, denote by $\tilde{\sigma}$ one of its



horizontal lifts. This exists c.f. [ELJL10].
On the orthonormal frames of semi-martingales,this is well known and is related to the stochastic parallel transport (K. Itô) and to the stochastic development map (J. Eells-D.

Invariant vector fields on S^3

As a Lie group there are three left invariant X_i^L and right invariant vector fields X_i^R. Since the metric on the sphere with round metric is bi-invariant, they form an o.n.b at each point. The right invariant vector fields are horizontal, and $\pi_*(X_2^R), \pi_*(X_3^3)$ is orthonormal at $\pi(u)$. However the projection do not induce vector fields on S^2. This can also be easily deduced from the fact that on S^2 there is no nowhere vanishing vector fields.
The left invariant vector fields do projects to vector fields on S^2. By the same reason the projection cannot be generate two everywhere independent vector fields. Hence the left invariant vector fields cannot lie in the horizontal distribution, and the left invariant vector field X_1^L is not in the kernel of $T\pi$.

Sub-Riemannian Geometry on S^3

The horizontal distribution has the following properties:
Any point can be reached from a given one by a horizontal curve (Hörmander condition).
The horizontal lift of a geodesic on S^3, or on (S_3, m_ϵ) given below, is a horizontal geodesic (of minimal length among all horizontal curves).

SDE's on Berger's spheres

Using unit vectors on Berger's spheres, we arrive at a number of singularly perturbed SDE's:

- Brownian motion on (S^3, m_ϵ) : $dx_t^\epsilon = \frac{1}{\epsilon} X_1(x_t^\epsilon) \circ db_t^1 + \sum_{i=2}^3 X_i(x_t^\epsilon) \circ db_t^i$. What we like to do: converges of the processes, the derivative process, propose a convergence corresponding to collapsing with bounded geometry.
- Hypoelliptic SDE's with the hypo-elliptic Laplacians as Markov generator:

$$dx_t^\epsilon = X_2(x_t) \circ db_t^2 + X_3(x_t) \circ db_t^3,$$

$$dx_t^\epsilon = \frac{1}{\sqrt{\epsilon}} X_1(x_t^\epsilon) \circ db_t^1 + X_2(x_t^\epsilon) \circ db_t^2,$$

$$dx_t^\epsilon = \frac{1}{\sqrt{\epsilon}} X_1(x_t^\epsilon) \circ db_t + X_3(x_t^\epsilon) dt.$$

- Degenerate system: $dx_t^\epsilon = \frac{1}{\sqrt{\epsilon}} X_1(x_t^\epsilon) \circ db_t^1 + X_2(x_t^\epsilon) \circ dt$.

Effective Hypoelliptic Diffusions

Let $x_t = \pi(u_t^\epsilon)$ and $\tilde{x}_t^\epsilon$ its horizontal lift. Take $Y_0 \in span\{X_2, X_3\}$. We investigate rotations of the vector Y_0 by elements of (S^1, g_ϵ):

Theorem ([Li12c])

Take $u_0 \in SU(2)$. Consider the SDE on $SU(2) \times U(1)$,

$$du_t^\epsilon = (Y_0 g_t^\epsilon)^*(u_t^\epsilon)dt + \frac{1}{\sqrt{\epsilon}} X_1^*(u_t^\epsilon) \circ db_t, \qquad u_0^\epsilon = u_0$$

$$dg_t^\epsilon = \frac{1}{\sqrt{\epsilon}} g_t^\epsilon X_1 \circ db_t, \qquad g_0^\epsilon = 1$$

Then $\tilde{x}_t^\epsilon$ converges in probability to the hypoelliptic diffusion with generator $\bar{\mathcal{L}}F = \frac{1}{2}|Y_0|^2 \Delta_H$.

If $|Y_0| = 1$, x_t^ϵ converges in law to the Brownian motion on S^2.

Remark

We have mentioned that a Brownian motion on S^2 cannot be constructed by an SDE on $\mathbf{R}^2$driven than less than 3 independent Brownian motion, e.g.

$$dx_t = \sum_{i=1}^{m} V_i(x_t) \circ db_t^i$$

with $m < 3$.
In the above we constructed a Brownian motion on S^2 with one driving Brownian motion. We remark that : (1) The SDE on S^3 does not project to an SDE on S^2.
(2) With one single driving Brownian motion, we obtain a hypoelliptic Brownian motion.

Check the scaling is correct

Note that if $g \in S^1$, $Y_0 \in span\{X_2, X_3\}$. Then

$Y_0 g \in span\{X_2, X_3\}$.

We make a multi scale analysis to confirm that we have the correct scaling and that there is indeed an effective motion. Let $F : SU(2) \times U(1) \to \mathbf{R}$ be C^∞. Then

$$\mathcal{L}^\epsilon(g)F(u) = \frac{1}{\epsilon}\mathcal{L}_0 F(u) + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_{Z_1^g} F(u) + \mathcal{L}_Z F.$$

Here $Z = (Y_0 g)^*$, $Z_1^g = \frac{1}{2}(Y_0 g X_1)^*$, $\mathcal{L}_0 = \frac{1}{2}\mathcal{L}_{X_1^*}\mathcal{L}_{X_1^*}$. The middle term comes from interaction between u and g. Let F be solution to $\frac{\partial F}{\partial t} = \mathcal{L}^\epsilon(g)F$. Expand F in ϵ,

$$F = F_0 + \sqrt{\epsilon}F_1 + \epsilon F_2 + o(\epsilon).$$

Multi-scale Analysis

$$\frac{\partial F}{\partial t} = \mathcal{L}^{\epsilon}(g)F, \quad F = F_0 + \sqrt{\epsilon}F_1 + \epsilon F_2 + o(\epsilon)$$

Expand the equation in $\sqrt{\epsilon}$, $\dot{F}_0 + \sqrt{\epsilon}\dot{F}_1 + \epsilon\dot{F}_2 + o(\epsilon) = (\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_{Z_1^g} + \mathcal{L}_Z)(F_0 + \sqrt{\epsilon}F_1 + \epsilon F_2 + o(\epsilon))$.

$$\mathcal{L}_0 F_0 = 0 \qquad \Longrightarrow F_0 \text{ does not depend on the } \theta\text{-vari}$$

$$\mathcal{L}_{Z_1} F_0 = -\mathcal{L}_0 F_1 \qquad \Longrightarrow F_1 = \mathcal{L}_0^{-1}(\mathcal{L}_{Z_1} F_0).$$

$$\dot{F}_0 = \mathcal{L}_Z F_0 + L_{Z_1} F_1 + \mathcal{L}_0 F_2. \qquad \int \mathcal{L}_0 F_2 d\theta = 0$$

$\mathcal{L}_0 = \frac{1}{2}\mathcal{L}_{X_1^*}\mathcal{L}_{X_1^*}$, Integrate lat equation with respect to $d\theta$, $\int \dot{F}_0 = \int \mathcal{L}_Z F_0 + \int L_{Z_1} F_1 + \int \mathcal{L}_0 F_2$. Define $\bar{F}_0 = \int F_0$.

$$\frac{d}{dt}\bar{F}_0 = \mathcal{L}_Z \bar{F}_0 + L_{Z_1}\bar{F}_1 = \mathcal{L}_Z \bar{F}_0 + L_{Z_1}\mathcal{L}_0^{-1}(L_{Z_1}\bar{F}_0),$$

We have a second order differential operator.

Idea of Proof

- Observe that $\tilde{x}_t^\epsilon$ and u_t^ϵ live in the same fibre, there is an element $a_t^\epsilon \in S^1$ such that $u_t^\epsilon = R_{a_t^\epsilon}\tilde{x}_t^\epsilon$. Then

  $$du_t^\epsilon = TR_{a_t^\epsilon}(dx_t^\epsilon) + ((a_t^\epsilon)^{-1}da_t^\epsilon)^*(u_t^\epsilon).$$

  We apply the connection 1-form ϖ to the above equation and to the SDE for u_t^ϵ. Note that the horizontal distribution is right invariant and $TR_{a_t^\epsilon}(dx_t^\epsilon)$ is horizontal. Also $\varpi_u A^*(u) = A$ for any $u \in \mathfrak{u}(1)$. This means,

  $$\frac{1}{\sqrt{\epsilon}}X_1 \circ db_t = (a_t^\epsilon)^{-1}da_t^\epsilon.$$

  Thus $a_t^\epsilon = g_t^\epsilon$.
- Deduce an equation for $\tilde{x}_t^\epsilon$:

  $$\begin{aligned} d\tilde{x}_t^\epsilon &= TR_{(g_t^\epsilon)^{-1}} \circ du_t^\epsilon + (g_t^\epsilon d(g_t^\epsilon)^{-1})^*(\tilde{x}_t^\epsilon) \\ &= TR_{(g_t^\epsilon)^{-1}}(g_t^\epsilon Y_0)^*(u_t^\epsilon)dt = (g_t^\epsilon Y_0)^*(\tilde{x}_t^\epsilon)dt. \end{aligned}$$

Proof

- $\frac{d}{dt}\tilde{x}_t^\epsilon = (g_t^\epsilon Y_0)^*(\tilde{x}_t^\epsilon)$. This bounded variation term will leads to a diffusion term in the limit.
- We prove the tightness of relevant measures for the weak convergence.
- Let $F : S^3 \to \mathbf{R}$ be any smooth function. Since $Y_0 \in span\{X_2, X_3\}$,

$$F(\tilde{x}_t^\epsilon) = F(u_0) + \sum_{j=2}^{3} \int_0^t dF(\tilde{x}_s^\epsilon X_j)\langle X_j, g_s^\epsilon Y_0\rangle ds.$$

- Note the right hand side is bounded variation term. However we seek an approximate 'semi-martingale' decomposition, of the form, 'martingale +drift+ $o(\epsilon)$-terms. To the drift term we may apply the ergodic theorem.
- Solve a Poisson Equation and use Stroock-Varadhan's martingale method to identify the limits.

Further Investigations

- Generalise the results to manifolds with the following structures: almost contact structures. Homogeneous spaces, Nioptent Li Groups, Warped product manifolds.
- Dynamics associated with collapsing with bounded sectional curvature. The study of exchange limit with homogenisation.

Discussions can be found in [Li12a, Li12b].

Collapsing with bounded sectional curvature

Collapsing with bounded sectional curvature or convergence with bounded derivative flows?
Let ∇ be the Levi-Civita connection. Then

$$Dv_t^\epsilon = \frac{1}{\sqrt{\epsilon}} \nabla_{v_t^\epsilon} X_1 \circ db_t^1 + \sum_{i=2}^{3} \nabla_{v_t^\epsilon} X_i \circ db_t^i, \quad v_t^\epsilon = v_0.$$

In general, let $P_t^\epsilon f(u_0) = \mathbf{E} f(u_t^\epsilon)$. Then
$d(P_t^\epsilon) f(v_0) = \mathbf{E} df(v_t^\epsilon)$.
Then $W_t^\epsilon(v_0) = \mathbf{E}\{v_t^\epsilon | \mathcal{F}_t^{u_\cdot^\epsilon}\}$ solves $\frac{DW_t^\epsilon(v_0)}{dt} = -\frac{1}{2} Ric^{\#}(W_t(v_0))$ and $W_0 = v$. If $Ric \geq K$ then

$$|d(P_t f)(v_0)| = |\mathbf{E} df(W_t(v))| \leq e^{Kt} \mathbf{E} |df|_{u_t^\epsilon}.$$

Convergence with bounded derivatives

Since *Ric* bounded from below is equivalent to $|\nabla P_t f|^2 \le e^{2Kt}|\nabla f|^2$, we propose to study collapsing of manifolds with one of the following constraints: (1) $|\nabla P_t^\epsilon f|_\epsilon^2 \le e^{2Kt}|\nabla f|_\epsilon^2$ or (2) the derivative flows $|\mathbf{E}\{v_t^\epsilon|\mathcal{F}_t^\epsilon\}|_\epsilon \le C|v_0|_\epsilon$.

Some Estimates

In our case, the terms $\nabla_{X_2} X_1 = \epsilon X_3$, $\nabla_{X_3} X_1 = -\epsilon X_2$, ... can be computed explicitly.

Since $/\!/_t$ is an isometry, we see that $|v_t^\epsilon|$ is nicely bounded. This can also follow from the following computation:

$$Ric^\epsilon(X_2) = 4 - 2\epsilon, Ric^\epsilon(X_1, X_1) = 2\epsilon^2.$$

Proposition

Consider the SDE with (X_1, X_2^L, X_3^L). With respect to the round metric, $|v_t^\epsilon|_1$ is a constant in t. The eigenvalues of $(u_t^\epsilon)^{-1} v_t^\epsilon$ is constant in t.

Remark: Suppose that $v_0 \neq 0$. The derivative flow of the equation for $\tilde{x}_t^\epsilon$ satisfies that $(\tilde{x}_t^\epsilon)^{-1} v_t^\epsilon$ is hypoelliptic on the unit sphere $\{h \in \mathfrak{su}(2) : |h| = |v_0|\}$.

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