Optimal Consumption Control Problem Associated with Jump-Diffusion Processes

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Abstract. We study an optimal consumption control problem in a jump-diffusion model under the uncertainty. We show a verification result to the existence of a solution of the Hamilton-Jacobi-Bellman equation associated with the stochastic optimization problem, and then give an optimal consumption policy in terms of the solution. An application to the one-sector Ramsey theory in the economic growth is given in the appendix.

1 Introduction

We assume an asset process $z(t)$ evolves according to the 1-dimensional SDE of jump-diffusion type

$$dz(t) = \{f(z(t)) - \tilde{\mu}z(t) - c(t)\}dt - \sigma z(t)dB(t)$$
$$+z(t-)\int_{|\zeta|<1}(e^{\zeta}-1)\tilde{N}(dtd\zeta) + z(t-)\int_{|\zeta|\geq 1}(e^{\zeta}-1)N(dsd\zeta), \quad z(0) = z \geq 0, \quad (1)$$

on a complete probability space $(\Omega, \mathcal{F}, P)$. Here $\{B(t)\}$ denotes the standard Brownian motion, and $N(dtd\zeta)$ denotes a Poisson random measure on $[0, +\infty) \times \mathbf{R}$. The the mean measure of $N(dtdz)$ is $dt\mu(d\zeta)$, and we put $\tilde{N}(dtd\zeta) = N(dtd\zeta) - dt\mu(d\zeta)$ denotes the compensated random measure. That is, $\mu(d\zeta)$ is a measure satisfying $\int_{\mathbf{R}\setminus\{0\}} 1 \wedge |\zeta|^2 \mu(d\zeta) < +\infty$. The measure μ can be a singular measure such as a sum of point masses.

We assume

$$\text{supp } \mu \subset [0, +\infty) \quad (\mathbf{A.1})$$

$$\int_{|\zeta|\geq 1}(e^{\zeta}-1)\mu(d\zeta) < +\infty. \quad (\mathbf{A.2})$$

We assume $\tilde{\mu} \in \mathbf{R}$, $\sigma \geq 0$, and that the *growth function* $f(z)$ satisfies

$$f(z): \textit{ Lipschitz continuous, increasing and concave}, \ f(0) = 0. \quad (2)$$

Under **(A.2)** $z(t)$ can be written by

$$dz(t) = \{f(z(t)) - \mu z(t) - c(t)\}dt - \sigma z(t)dB(t) + z(t-)\int(e^{\zeta}-1)\tilde{N}(dtd\zeta), \quad z(0) = z \geq 0, \quad (5)$$

where $\mu = \tilde{\mu} + \int_{|\zeta|<1}(e^{\zeta} - 1 - \zeta)\mu(d\zeta) - \tilde{r}$. Note that $e^{\zeta} - 1 \geq -1$ and that, writing $e^{\zeta}z = z + (e^{\zeta}-1)z$, the fourth term in RHS of (5) is of the Lévy-Khintchine form associated with the jump $(e^{\zeta}-1)z$.

Our motivation is to maximize the expected utility

$$J(\mathbf{c}) = E[\int_0^{\tau_z} e^{-\beta t}U(c(t))dt] \quad (3)$$

over the class $\mathcal{C}$, with the above-mentioned condition that (1) has a non-negative solution $z(t)$ a.s. for $z(0) = z \geq 0$. Here $\beta > 0$, $\tau_z = \inf\{t \geq 0; z(t) \leq 0\}$, and the set $\mathcal{C}$ denotes the set of non-negative consumption policies $\mathbf{c} = \{c(t)\}$ such that it is a non-decreasing *adapted* càdlàg process satisfying

$$\int_0^t c(s)ds < \infty, \quad \forall t \geq 0, \quad \text{a.s.}. \tag{4}$$

The (potential) function $U(.)$ is regarded as a *utility function* following so-called Gossen's law which depends on the consumption rate $c(t)$, so that the hasty invester would like to maximize his or her utility, and β denotes the dumping rate of the utility as time goes by. $U(c)$ is assumed to have the following properties:

$$U \in C([0,\infty)) \cap C^2((0,\infty)), \quad U(c): \text{strictly concave and increasing on } [0,\infty), \quad (\mathbf{A.3})$$
$$U'(c): \text{ strictly decreasing}, \quad U'(\infty) = U(0+) = 0, \quad U'(0+) = U(\infty) = \infty.$$

The optimal value of $J(\mathbf{c})$ as a function of $z = z(0)$ is called the *value function* and is denoted by $v(z)$:

$$v(z) = \sup_{\mathbf{c} \in \mathcal{C}} J(\mathbf{c}).$$

- Lévy process
- Jump-diffusion processes
- History

As for the jump-diffusion type control problem, the paper [14] has studied a model (X_t) given by

$$dX_t = \mu dt + \sigma dB_t - dZ_t - dK_t, X_{0-} = x.$$

Here Z_t denotes a càdlàg (jump) process corresponding to the company's spending (paying *dividends* to the stock holders), $\sigma > 0$ and K_t corresponds to the activity to invest. The expected utility is measured up to time T in terms of Z_t by

$$E[\int_0^T e^{-\beta t} dZ_t],$$

instead of the consumption rate $c(t)$ composed in the utility function. The paper [2] studies in a similar framework the control in switching between paying dividend and going to the investment. The paper [6] studies the same model X_t, but the expected utility is measured by

$$E[U(\int_0^T e^{-\beta t} dZ_t)],$$

where $U(.)$ is a utility function. Here they measure the gross amount of dividends up to time T by $U(.)$ insted of the consumption rate $c(t)$, which makes some difference in the interpretation. In these papers the main perturbation term is the diffusion (Brownian motion).

On the other hand, Framstad [4] has studied a model (X_t) given by

$$dX_t = X_{t-}(\mu(X_t)dt + \sigma(X_t)dB_t + \int \eta(X_{t-}, \zeta)\tilde{N}(dtd\zeta)) - dH_t$$

in the Wiener-Poisson framework. Here H_t denotes a càdlàg process corresponding to the total amount of harvest up to time t, and $\sigma(.) \geq 0$. Under the setting that the expected utility is measured by

$$E[\int_{t_0}^{\infty} e^{-\beta t} dH_t],$$

he describes that the optimal harvesting policy is given by a single value x^*, which plays a role of barrier at which one reflects the process downward. The paper [6] leads also to the similar conclusion.

However, a difficulty arises in the case (1) where there exists a degeneracy in the HJB equation. Namely, the second order term will degenerate at $z = 0$, or even the coefficient σ may be identically zero. To avoid this difficulity and obtain the value fuction, we use an analytic method. Namely we first construct a weak solution, and then show the uniqueness and the existence of the solution.

2 HJB equation and Viscosity solutions

To find the value function $v(z)$, we consider the 1-dimensional Hamilton-Jacobi-Bellman (HJB for short) equation of integro-differential type on $[0, +\infty)$:

$$Lv(z) + \tilde{U}(v'(z)) = 0, z > 0, \quad v(0) = 0. \tag{6}$$

Here the symbol $\tilde{U}(x)$ is the Legendre transform of (the negative potential) $-U(-x)$, i.e.,

$$\tilde{U}(x) = \max_{c>0} \{U(c) - cx\}.$$

Here L is an integro-differential operator given by

$$Lv(z) = -\beta v(z) + \frac{1}{2}\sigma^2 z^2 v'' + (f(z) - \mu z)v'$$

$$+ \int \{v(z + \gamma(z, \zeta)) - v(z) - v'(z) \cdot \gamma(z, \zeta)\}\mu(d\zeta). \tag{7}$$

We write

$$Lv = -\beta v + L_0 v,$$

so that

$$Lv = 0 \iff \beta v = L_0 v.$$

When the value function is finite ?

By **(A.1)** the process $z(t)$ has no negative jumps. In this setting, it is known that the "trace" $v(0+) = \lim_{z \to 0+} v(z)$ exists finite for the original non-local boundary value problem on $\mathbf{R}$

$$Lv(z) + \tilde{U}(v'(z)) = 0, z > 0, \quad v(z) = 0, z \le 0. \tag{8}$$

By this reason it sufficies to consider the equation (8) replacing v with $v.1_{[0,+\infty)}$, interpretating $v(z) = 0, z \le 0$ with $v(0) = v(0+) = 0$, which is (6). The above property for L of being able to take the trace safely at the boundary is called the transmission property. See [7].

Finally we remark that we can rewrite (6) as

$$(\beta + \frac{1}{\varepsilon})v(z) = L_0 v(z) + \tilde{U}(v'(z)) + \frac{1}{\varepsilon}v(z), \ z > 0 \tag{9}$$

$$v(0) = 0,$$

for $\varepsilon > 0$ chosen later. Whereas we remark that, comparing (6) and (9), $\epsilon > 0$ is a merely apparent parameter. We shall show later that the solution $v = v_\epsilon$ is approximated by the solution $u = u_{M,\epsilon}$ of

$$(\beta + \frac{1}{\varepsilon})u(z) = L_0 u + \tilde{U}_M(u'(z)) + \frac{1}{\varepsilon}u(z), \ z > 0 \tag{10}$$

$$u(0) = 0,$$

where $\tilde{U}_M(x) \equiv \max\{U(c) - cx; 0 < c \le M\}$ and $M > 0$.

2.1 Viscosity solutions

We shall find the value function as a weak solution to an H-J-B equation.

Let

$$F(z,u,p,q,B^1(z,u,p),B_1(z,u,p)) = -\beta u + \frac{1}{2}\sigma^2 z^2 q + (f(z)-\mu z)p + B^1(z,u,p) + B_1(z,u,p) + \tilde{U}(p),$$

where

$$B^1(z,u,p) = \int_{|\zeta|>1} \{u(z+\gamma(z,\zeta)) - u(z) - p\cdot\gamma(z,\zeta)\}\mu(d\zeta),$$

and

$$B_1(z,u,p) = \int_{|\zeta|\le 1} \{u(z+\gamma(z,\zeta)) - u(z) - p\cdot\gamma(z,\zeta)\}\mu(d\zeta).$$

Definition 1 *Let a function $v \in C([0,\infty))$ satisfy $v(0)=0$.*

(1) The function v is called a viscosity subsolution *of (6) if for all $z\in(0,\infty)$ and all $(p,q)\in J^{2,+}v(z)$ there exists $\phi\in C^2((0,\infty))$ such that $p=\phi'(z), q=\phi''(z)$ and that the following relation holds:*

$$F(z,v,p,q,B^1(z,v,p),B_1(z,\phi,\phi')) \ge 0, z>0.$$

(2) The function v is called a viscosity supersolution *of (6) if for all $z\in(0,\infty)$ and all $(p,q)\in J^{2,-}v(z)$ there exists $\phi\in C^2((0,\infty))$ such that $p=\phi'(z), q=\phi''(z)$ and that the following relation holds:*

$$F(z,v,p,q,B^1(z,v,p),B_1(z,\phi,\phi')) \le 0, z>0.$$

(3) If v is both a viscosity subsolution and a viscosity supersolution, it is called a viscosity solution.

2.2 Existence

Let $M>0$ and fix it. Let $\epsilon>0$. We consider the optimization problem :

$$u_M(z) = \sup_{\mathbf{c}\in\mathcal{C}_M} E[\int_0^{\tau_z} e^{-(\beta+\frac{1}{\varepsilon})t}\{U(c(t)) + \frac{1}{\varepsilon}u_M(z(t))\}dt], \quad M>0, \tag{11}$$

where $\mathcal{C}_M$ denotes the class of all non-negative, integrable, $\mathcal{F}_t$-adapted processes $\mathbf{c}\in\mathcal{C}$ such that $0\le c(t)\le M$ for all $t\ge 0$, and the supremum is taken over all admissible control systems [3]. By (6), we have $z(t)=z(t\wedge\tau_z)\ge 0$ for each $\mathbf{c}\in\mathcal{C}_M$, because $c(t)$ is identified with $c(t)1_{\{t\le\tau_z\}}$ in (6).

Taking κ so that

$$0<\kappa\le\mu, \tag{12}$$

we assume that there exists $A>0$ satisfying

$$f(z)-\kappa z < A \tag{13}$$

for all $z \in (0, \infty)$. Further We assume

$$\kappa + \mu < \beta. \tag{\textbf{A.3}}$$

Furthermore, we observe by **(A.3)** and (13) that the linear function

$$\varphi(z) \equiv z + B \tag{14}$$

satisfies

$$-\beta\varphi(z) + L_0\varphi(z) + \tilde{U}(\varphi'(z)) \leq -\beta B + A + \tilde{U}(1) < 0, \quad z \geq 0 \tag{15}$$

for some constant $B > 0$. Let $\mathcal{B} = \mathcal{B}_\varphi$ denote the set

$$\mathcal{B} = \{h; h \text{ is measurable on } [0, +\infty) \text{ and satisfies that there exists } C_\rho > 0 \tag{16}$$

$$\text{for any } \rho > 0 \text{ such that } |h(z) - h(\tilde{z})| \leq C_\rho|z - \tilde{z}| + \rho(\varphi(z) + \varphi(\tilde{z})), \quad z, \tilde{z} \in [0, \infty)\},$$

and denote the norm $\|h\| = \sup_{z\geq 0} |h(z)|/\varphi(z) < \infty$.

Lemma 1 *Let $M > 0$. We assume that there exists a concave function $\psi \in \mathcal{B} \cap C^2((0, +\infty))$ such that*

$$-\beta\psi(z) + L_0\psi(z) + \tilde{U}(\psi'(z)) \leq 0, \ z > 0,$$

$$\psi'(z) > 0, \ z > 0 \quad \textit{and} \quad \psi(0) = 0. \tag{17}$$

Then, under **(A.3)***, (2), for each $M > 0$ there exists a unique solution $u = u_M \in \mathcal{B}$ of (11) for some $\varepsilon > 0$ such that $\frac{1}{\epsilon} > \kappa + \mu - \beta$.*

Remark. The condition (17) refers to the existence of the C^2 function related to the viscosity supersolution of (11). The existence of such function, given L_0, depends on the form of $\tilde{U}$. Analytically, we are assuming the existence of a Lyapunov function associated to the equation. Instead of providing a sufficient condition for the existence of $\psi \in \mathcal{B} \cap C^2$ to (17), we give an example for it.

Example. Let $U(c) = \frac{1}{r}c^r$ $(0 < r < 1)$. Then $\psi(z) = \frac{R}{r}z^r$ for $R > 0$ sufficiently large satisfies the condition (17).

This $u = u_M$ is a viscosity solution :

Proposition 1 *We assume* **(A.3)***, (2) and (17). Then $u = u_M \in \mathcal{B}$ of (11) is a viscosity solution of (10), and is concave. Here*

$$(\beta + \frac{1}{\varepsilon})u(z) = L_0 u + \tilde{U}_M(u'(z)) + \frac{1}{\varepsilon}u(z), \ z > 0 \tag{10}$$

$$u(0) = 0,$$

where $\tilde{U}_M(x) \equiv \max\{U(c) - cx; 0 < c \leq M\}$ and $\epsilon > 0$ is a parameter.

We let $M \to +\infty$ and obtain the solution $v(z)$ as a limit as a weak solution.

Proposition 2 *We assume* **(A.3)***, (2) and (17). Then there exists a concave viscosity solution v of* (6)*.*

Proofs of these propositions continue in the next section.

3 Bellman principle (dynamic programing principle)

To prove that $u = u_M$ is a viscosity solution to (10), we need the Bellman principle.

Idea of proof of Prop. 2

3.1 Use of Bellman principle

In this paper we assume the *dynamic programming principle* (Bellman principle) that $u(z)$ is continuous on $(0, +\infty)$ and satisfies

$$u(z) = \sup_{\mathbf{c} \in \mathcal{C}_M} E[\int_0^{\tau_z \wedge \tau} e^{-\beta t}\{U(c(t))\}dt + e^{-\beta \tau_z \wedge \tau} u(z(\tau_z \wedge \tau))]$$

for any bounded stopping time τ.

The proof of the Bellman principle in the general setting (i.e., without continuity or measurability assumptions) is not easy. Occasionally one is advised to decompose the equality into 2 parts ([1] Cor. 3.1) :

$$(i)\ u(z) \le \sup_{\mathbf{c} \in \mathcal{C}_M} E[\limsup_{z' \to z} \int_0^{\tau_{z'} \wedge \tau} e^{-(\beta + \frac{1}{\varepsilon})t}\{U(c(t)) + \frac{1}{\varepsilon} u(z(t))\}dt + e^{-(\beta + \frac{1}{\varepsilon})\tau_{z'} \wedge \tau} u(z(\tau_{z'} \wedge \tau))],$$

and the converse

$$(ii)\ u(z) \ge \sup_{\mathbf{c} \in \mathcal{C}_M} E[\liminf_{z' \to z} \int_0^{\tau_{z'} \wedge \tau} e^{-(\beta + \frac{1}{\varepsilon})t}\{U(c(t)) + \frac{1}{\varepsilon} u(z(t))\}dt + e^{-(\beta + \frac{1}{\varepsilon})\tau_{z'} \wedge \tau} u(z(\tau_{z'} \wedge \tau))].$$

The proof for (i) is relatively easy. The proof for (ii) is difficult if it is not assumed to be continuous.

This principle is proved in the diffusin case in [10] Theorem 4.5.1, and in a 2-dimensional case for some jump-diffusion process in [8] Lemma 1.5.

3.2 From the Bellman principle to the viscosity solution

By the above we assume u is in $C([0, +\infty))$. Then we prove that u is a viscosity solution of (10) in the following way.

(a) *u is a viscosity supersolution.*

(b) *u is a viscosity subersolution.*

4 Uniqueness and Smoothness of the viscosity solution

Proposition 3 *Let $f_i, i = 1, 2$, satisfy (2) and let $v_i \in C([0, \infty))$ be the concave viscosity solution of (6) for f_i in place of f such that $0 \le v_i \le \varphi$. Suppose*

$$f_1 \le f_2. \tag{24}$$

Then, under **(A.3)**, *(2) and (17), we have*

$$v_1 \le v_2.$$

This proposition proves the uniqueness of $v(z)$.

Proposition 4 *Under* **(A.3)**, *(2), (17) and that $\sigma > 0$, we have $v \in C^2((0, \infty))$ and $v'(0+) = \infty$.*

See [9] Theorems 2.7, 3.1 for the proofs.

5 Optimal consumption policy

Now we consider the equation of the form:

$$dz^*(t) = (f(z^*(t)) - \mu z^*(t) - c^*(t))dt + \sigma z^*(t)dB(t) + z^*(t-)\int (e^{\zeta} - 1)\tilde{N}(dtd\zeta), \qquad z^*(0) = z > 0, \tag{35}$$

where

$$c^*(t) = (U')^{-1}(v'(z^*(t-)))1_{\{t \le \tau_{z^*}\}}. \tag{36}$$

Here $c^*(t)$ is the consumption rate which maximizes the Hamilton function (associated with L) operated to v. For a typical case $U(c) = \frac{1}{r}c^r$ ($r \in (0, 1)$),

$$c^*(t) = (v'(z^*(t-)))^{\frac{1}{r-1}}.1_{\{t \le \tau_{z^*}\}}.$$

Proposition 5 *Under* **(A.0)**, *(2) and (17), there exists a unique solution $z^*(t) \ge 0$ of (35).*

We assume the S.D.E. (35) has a strong solution. The optimal consumption policy is given by as follows.

Proposition 6 *We make the assumptions of Theorem 3.1. Then an optimal consumption* $\mathbf{c}^* = \{c^*(t)\}$ *is given by (36).*

6 Application - Ramsey theory

Define the following quantities:

$y(t) =$ labour supply at time $t \geq 0$,
$x(t) =$ capital stock at time $t \geq 0$,
$\lambda =$ the constant rate of depreciation (written as β in the text), $\lambda \geq 0$,
$F(x, y) =$ production function producing
the commodity for the capital stock $x \geq 0$ and the labour force $y \geq 0$.

We now state the setting in the model. Suppose that the labour supply $y(t)$ at time t, and the capital stock $x(t)$ at t are governed by the following stochastic differential equations

$$dy(t) = ry(t)dt + \sigma y(t)dB(t), \qquad y(0) = y > 0, \quad r \neq 0, \sigma \geq 0, \tag{a.1}$$

$$dx(t) = (F(x(t), y(t)) - \lambda x(t) - c(t)y(t))dt \tag{a.2}$$
$$+x(t-)\int_{|\zeta|<1}(e^{\zeta}-1)\tilde{N}(dtd\zeta) + x(t-)\int_{|\zeta|\geq 1}(e^{\zeta}-1)N(dtd\zeta), \qquad x(0) = x > 0.$$

An intuitive explaination is as follows. Suppose there exists a farm (or a company, a small country, ...) who makes economical activities based on labour and capital. At time t, he or she makes production, which is expressed by $F(x(t), y(t))$. At the same time, he or she has to consume the capital at the rate $c(t)$, and the capital may depreciate as time goes by. The second and third terms in RHS of (a.2) corresponds to a random fluctuation in capital.

Under this interpretation we may even assume that the labour supply $y(t)$ is constant, and consider the optimal consumption under the randomly fluctuated capital. This assumption has a due economical meaning, since the labour supply is not easy to control (i.e., make decrease) in short time.

This type of problem is called the stochastic Ramsey problem. For reference, see [5]. We denote by $V(x, y)$ the value function associated with $x(t)$ and $y(t)$. The HJB equation associated with this problem reads as follows:

$$\beta V(x, y) = \frac{1}{2}\sigma^2 y^2 V_{yy}(x, y) + ryV_y(x, y) + (F(x, y) - \lambda x)V_x(x, y)$$
$$+ \int \{V(e^{\zeta}x, y) - V(x, y) - V_x(x, y) \cdot x(e^{\zeta}-1)\}\mu(d\zeta) + \tilde{U}(V_x(x, y)y),$$
$$V(0, y) = 0, \quad x > 0, \quad y > 0. \tag{a.3}$$

We seek for the solution $V(x, y)$ in a speacial form, namely

$$V(x, y) = v(z), \quad z = \frac{x}{y}. \tag{a.4}$$

Then (a.3) turns into the form (6)-(7) with $f(z) = F(z^+, 1), \mu = r - \tilde{r} + \lambda - \sigma^2$. Here we assume the homogeneity of F with respec to y : $F(x, y) = F(\frac{x}{y}, 1)y$. [1] We can show

[1] A well-known Cobb-Douglas model of the production function satisfies this requirement.

that $V(x,y)$ defined as above is a viscosity solution of (a.3), and that if $\sigma > 0$ then it is in $C^2((0,\infty)\times(0,\infty))$. The condition $z(t) \geq 0$ a.s. implies that the economy can sustain.

Although it is not implied in general that we can find the solution of (a.3) as in (a.4) in the setting (a.1)-(a.2), under the same token as above (i.e., the labour supply is constant), the problem will exactly correspond to what we have studied in the main text. That is, we can identify the value function in the form $v(z) = V(z,1)$.

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