Conservation property of

symmetric jump-diffusion processes

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1. Introduction

▷ $(\mathcal{E}, \mathcal{F})$: regular Dirichlet form on $L^2(X; m)$

$\Leftrightarrow \mathbb{M} = (\{X_t\}_{t\geq 0}, \{P_x\}_{x\in X})$: m-symm. Hunt proc. on X

▷ X: locally compact separable metric space

▷ m: positive Radon measure on X with full support

○ $\mathbb{M}$ is conservative (or stochastically complete)

$\overset{\text{def}}{\Longleftrightarrow}$ $P_x(X_t \in X) = 1$, $\forall$q.e. $x \in X$, $\forall t > 0$.

Diffusion case. **Grigor'yan, Takeda, Sturm, Davies,...**

Volume growth **and** **Coefficient growth**

Jump case.

$$\mathcal{E}(u,u) = \iint_{X\times X\setminus d} (u(x)-u(y))^2\, J(\mathrm{d}x\mathrm{d}y)$$

$\triangleright$ $J(\mathrm{d}x\mathrm{d}y)$: **symmetric positive Radon measure on** $X\times X\setminus d$

Assume: $J(\mathrm{d}x\mathrm{d}y) = J(x,\mathrm{d}y)m(\mathrm{d}x) (= J(y,\mathrm{d}x)m(\mathrm{d}y))$

$\diamond$ $\displaystyle\int_{X\setminus\{x\}} \left(1\wedge d(x,y)^2\right) J(x,\mathrm{d}y)$: **"coefficient"**

(i) Volume growth: Masamune-Uemura ('11)

Grigor'yan-Huang-Masamune ('12)

Masamune-Uemura-Wang ('12);

- $\displaystyle\sup_{x\in X}\int_{X\setminus\{x\}}(1\wedge d(x,y)^2)\,J(x,\mathrm{d}y)<\infty;$
- $\exists x_0\in X,\ \exists c>0$ **s.t**

$$m(B(x_0,r))\le e^{c\,r\log r},\ \forall r>0.$$

◇ **Diffusion case:** $m(B(x_0,r))\le e^{cr^2}$

(ii) Coefficient growth: S.-Uemura ('12).

▷ $X = \mathbb{R}^d$, $m(\mathrm{d}x) = \mathrm{d}x$: Lebesgue measure.

$\exists M_1, M_2 > 0$ s.t.

- $\displaystyle\int_{|x-y|<\gamma(x)} |x-y|^2 J(x, \mathrm{d}y) \leq M_1(1+|x|^2)$;
- $\displaystyle\int_{|x-y|\geq\gamma(x)} J(x, \mathrm{d}y) \leq M_2$;
- Condition on the "drift" ($\Leftarrow$ continuity of coefficient).

▷ $\gamma(x) \asymp |x| \quad (|x| \to \infty)$.

Purpose in this talk.

To reveal the following:

- **How the coefficient growth affects the volume growth**
- **To allow general X and m without continuity condition**

2. Result.

▷ $(\mathcal{E}, \mathcal{F})$: **regular Dirichlet form on** $L^2(X; m)$ **s.t.**

$$\mathcal{E}(u,u) = \iint_{X \times X \setminus d} (u(x) - u(y))^2 \, J(x, \mathrm{d}y) m(\mathrm{d}x)$$

Assumption 1 ("big jumps").

$\exists F(x,y)>0$: **positive function on** $X\times X\setminus d$ **s.t.**

(i) $F(x,y)=F(y,x)$;

(ii) $\displaystyle\sup_{x\in X}\int_{d(x,y)\geq F(x,y)} J(x,\mathrm{d}y)<\infty$.

$$\mathcal{E}^{(1)}(u,u)=\iint_{d(x,y)<F(x,y)}(u(x)-u(y))^2\,J(x,\mathrm{d}y)m(\mathrm{d}x)$$

$$\Longrightarrow\quad \boxed{\mathcal{E}_1\asymp\mathcal{E}_1^{(1)}}\qquad \left(\mathcal{E}_1(\cdot,\cdot)=\mathcal{E}(\cdot,\cdot)+\|\cdot\|^2_{L^2(X;m)}\right)$$

Lemma. $(\mathcal{E},\mathcal{F})$ **is conservative iff so is** $(\mathcal{E}^{(1)},\mathcal{F})$.

Takeda ('89) ("adapted distance")

▷ $\mathcal{A} := \left\{ \rho \in \mathcal{F}_{\mathrm{loc}} \cap C(X) : \begin{array}{l} \lim_{x \to \Delta} \rho(x) = \infty, \\ K_\rho(r) \text{ is compact}, \ \forall r > 0. \end{array} \right\}$

▷ $K_\rho(r) := \{x \in X : \rho(x) \le r\}$

▷ $\mathbb{M} = (\{X_t\}_{t \ge 0}, \{P_x\}_{x \in X})$: m-symm. jump process on X generated by $(\mathcal{E}^{(1)}, \mathcal{F})$

For a fixed $\rho \in \mathcal{A}$,

$$\boxed{\mathbb{M} \text{ is conservative} \iff Y_t := \rho\left(X_t^{(1)}\right) < \infty, \ \forall t > 0.}$$

Assumption 2 ("small jumps").

$\exists \rho \in \mathcal{A}$ **s.t. the following hold.**

(i) $|\rho(x) - \rho(y)| < 1$ **if** $d(x,y) < F(x,y)$;

(ii) $\displaystyle\int_{d(x,y)<F(x,y)} (\rho(x)-\rho(y))^2 \, J(x, \mathrm{d}y) < \infty, \; x \in X.$

$$\triangleright \; \Gamma^j(\rho)(x) := \int_{d(x,y)<F(x,y)} (\rho(x) - \rho(y))^2 \, J(x, \mathrm{d}y)$$

$$\triangleright \; M_\rho(r) := \operatorname*{ess\,sup}_{x \in K_\rho(r)} \Gamma^j(\rho)(x), \; r > 0$$

Theorem.
If $\exists\{a_n\}$: sequence s.t.

$$\liminf_{n\to\infty}\Big\{M_\rho(n+3)m(K_\rho(n+3))$$
$$\cdot\exp\Big(-na_n+a_n^2e^{a_np_n}M_\rho(n+1)T\Big)\Big\}=0 \qquad (*)$$

for some $T>0$, then $(\mathcal{E},\mathcal{F})$ is conservative.

$$\triangleright\ p_n := 2\cdot\sup_{\substack{\frac{n}{2}-1\le\rho(x)\le n+1,\\ d(x,y)<F(x,y)}}|\rho(x)-\rho(y)|$$

Approach: Adaption of the so called Davies method ('92) in a similar way to [MU], [GHM], [MUW]

3. Examples.

▷ $D \subset \mathbb{R}^d$: bounded smooth domain

▷ $\alpha \in (0, 2)$, $c(x, y) > 0$

$$\mathcal{E}(u, u) = \iint_{D \times D \setminus d} c(x, y) \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} \mathrm{d}x\mathrm{d}y$$

$$\mathcal{F} = \overline{C_0^\infty(D)}^{\sqrt{\mathcal{E}_1}}$$

• $c(x, y) \asymp 1 \Longrightarrow$ censored stable-like process

[Bogdan-Burdzy-Z.-Q. Chen ('03),. . .]

Remark. $\alpha > 1 \Longrightarrow$ non-conservative ([BBC]).

▷ $\delta_D(x) := d(x, \partial D)$, $x \in D$: **distance function**

- **For** $0 < |x-y| < (\delta_D(x) \vee \delta_D(y))/2$,

$$c(x,y) \asymp \delta_D(x)^p + \delta_D(y)^p \quad \text{for some } \; p \geq \alpha;$$

- **For** $|x-y| \geq (\delta_D(x) \vee \delta_D(y))/2$,

$$c(x,y) \asymp \delta_D(x)^q + \delta_D(y)^q \quad \text{for some } q > \alpha.$$

$\Longrightarrow$ $(\mathcal{E}, \mathcal{F})$ **is conservative.**

◦ $F(x,y) = \frac{1}{2}(\delta_D(x) \vee \delta_D(y))$

◦ $\rho(x) \asymp \sqrt{-c \log \delta_D(x)}$ as $x \to \partial D$

($\Longrightarrow \lim_{x \to \partial D} \rho(x) = \infty$ and $M := \sup_{r>0} M_\rho(r) < \infty$)

$\Longrightarrow p_n \le c/n$ and $(*)$ holds if

$$\liminf_{n\to\infty} e^{-(na_n - a_n^2 e^{ca_n/n} MT)} = 0. \qquad (**)$$

◦ $a_n = \beta n$ for some small $\beta > 0$

◇ **Diffusion case ($\alpha = 2$): M.M.H. Pang ('88).**

▷ $X = \mathbb{R}^d$, $\alpha \in (0,2)$, $\beta > 0$

▷ $m(B(x,r)) \asymp r^{\beta}$, $\forall x \in \mathbb{R}^d$, $\forall r > 0$

$$\mathcal{E}(u,u) = \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus d} c(x,y) \frac{(u(x)-u(y))^2}{|x-y|^{\alpha+\beta}} m(\mathrm{d}x) m(\mathrm{d}y)$$

$$\mathcal{F} = \overline{C_0^\infty(\mathbb{R}^d)}^{\sqrt{\mathcal{E}_1}}$$

- $c(x,y) \asymp 1 \implies$ **symmetric α-stable-like process**

[Z.-Q. Chen-Kumagai ('03)]

- For $0 < |x - y| < 1$,

$$c(x,y) \asymp (1+|x|)^p + (1+|y|)^p \quad \text{for some} \quad p \leq \alpha.$$

- For $|x - y| \geq 1$,

$$c(x,y) \asymp (1+|x|)^q + (1+|y|)^q \quad \text{for some} \quad q < \alpha.$$

- ∘ $F(x,y) = \frac{1}{2}\{(1+|x|) \vee (1+|y|)\}$
- ∘ $\rho(x) = \sqrt{\log(2+|x|)}$
- ∘ $a_n = \gamma n$ for some small $\gamma > 0$