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Report on the PhD thesis of Jacek Krajczok:
Modular properties of locally compact quantum groups

The PhD thesis of Jacek Krajczok concerns the theory of locally compact quantum groups and investigates a variety of interesting problems connected to the modular theory of their Haar weights. The theory of locally compact quantum groups originated in the mid-20th century in non-commutative harmonic analysis, representation theory, and the search for a generalized Pontryagin duality theory for non-abelian locally compact groups. The general idea is that if G is an abelian locally compact group, then its irreducible unitary representations naturally form a locally compact abelian group $\hat{G}$, called the *Pontryagin dual* of G. $\hat{G}$ encodes the representation theory of G, and one has the Pontryagin duality theorem, which states that the double dual $\hat{\hat{G}}$ is canonically isomorphic as a topological group to G. When G is no longer abelian, its equivalence classes of irreducible unitary representations no longer form a group, and Pontryagin duality breaks down. The key idea behind locally compact quantum groups is that in order to build a satisfactory non-abelian Pontryagin duality theory, one must enlarge the category of locally compact groups to include operator algebraic objects which are (possibly non-commutative) analogues of the function spaces over a locally compact group G. More precisely, a locally compact quantum group $\mathbb{G}$ (in the sense of Kustermans and Vaes) is a quadruple $(L^\infty(\mathbb{G}), \Delta, \varphi, \psi)$, where $L^\infty(\mathbb{G})$ is a von Neumann algebra, $\Delta : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\mathbb{G})$ is a co-associative normal unital $*$-homomorphism (called the coproduct), and φ, ψ are normal semifinite weights which satisfy certain left-invariance and right-invariance conditions with respect to the coproduct. The case where $L^\infty(\mathbb{G})$ is abelian corresponds precisely to that of classical locally compact groups G, where $L^\infty(\mathbb{G}) = L^\infty(G)$, $\Delta f(s,t) = f(st)$, and the left and right Haar weights correspond to integration with respect to left and right Haar measure on G. These axioms turn out to produce a category of objects which is self-dual in the sense of a generalized Pontryagin duality theory.

The theory of locally compact quantum groups is a rather technical branch of operator algebras, requiring any researcher to have a strong background in C*-algebras, von Neumann algebras, the theory of weights, and so on. Based on the results contained in this thesis, it is clear that Jacek Krajczok has not only learned, but also mastered the field and made strong research contributions.

Krajczok's PhD thesis is divided into seven chapters. The first two chapters are devoted to notational conventions and a preliminaries section which provides a (necessarily) rapid introduction to the main objects of study: the theory of weights on operator algebras, followed by an overview of the basic theory and examples of locally compact quantum groups. Chapters 3-6 are each devoted to separate projects that Krajczok worked on during his PhD studies (partly in collaboration with others). The final chapter 7 is an appendix containing some results on direct integrals and some technical lemmas about locally compact quantum groups.

Chapter 3 is based on Krajczok's papers [49, 50] (extending work of Desmedt [31] and Caspers [17,18]), and concerns type I locally compact quantum groups. These quantum groups are defined by analogy with type I locally compact groups, and (just like their classical counterparts) represent a rich yet tractable class in terms of their representation theory beyond the compact quantum groups. The main results here are Theorems 3.24 and 3.25, which realize various objects associated to $\mathbb{G}$ and $\hat{\mathbb{G}}$ (modular operators, modular elements, and so on) as direct integrals with respect to the Plancharel weight. These results are highly technical extensions of familiar relations seen in the worlds of compact/discrete quantum groups, but require much more delicate proofs using the theory of direct integrals combined with some intense modular theory.

Chapter 4 contains a very nice joint work with Piotr Sołtan [51], where they study the quantum analogue, $\mathcal{T}_q$, of the C*-algebra of functions on the disc, for $0 \leq q < 1$. The question considered here is a natural one: Is there a quantum group structure on $\mathcal{T}_q$ for any $q < 1$? As explained in the thesis, it is well known that in the classical case ($q = 1$), the answer is no. Krajczok and Sołtan settle this in the negative for all q. The proof presented in the thesis (which differs from the published version) makes clever use of the results on type I quantum groups obtained in Chapter 3.

Chapter 5 contains, in the referee's humble opinion, some of Krajczok's most impressive results. The results of this chapter are based on joint work with Mateusz Wasilewski [52]. Given a compact quantum group $\mathbb{G}$, the *von Neumann algebra of class functions* is defined as the von Neumann algebra $\mathcal{C}_{\mathbb{G}}$ generated by the characters χ_α of the irreducible unitary representations of $\mathbb{G}$. This chapter concerns the structure of these algebras. When $\mathbb{G} = G$ is a classical group, $\mathcal{C}_{\mathbb{G}}$ is nothing other than the subalgebra of class functions inside the abelian von Neumann algebra $L^\infty(G)$. On the other hand, when $\mathbb{G} = \widehat{\Gamma}$ is dual of a discrete group, then $\mathcal{C}_{\mathbb{G}} = L^\infty(\mathbb{G}) = L\Gamma$, the group von Neumann algebra of Γ. In recent years there has been significant interest in the study of the inclusion $\mathcal{C}_{\mathbb{G}} \subseteq L^\infty(\mathbb{G})$. Most notably, there is the work of Freslon-Vergnioux [42], which studies the von Neumann algebra of class functions in the Kac type free orthogonal quantum groups O_N^+. For these quantum groups, it is natural to interpret $\mathcal{C}_{\mathbb{G}}$ as an analogue of the radial subalgebra in a free group factor. Indeed, it is shown in [42] that this algebra gives rise to a singular MASA, just like in the free group case. Here, Krajczok and Wasilewski take the first step in understanding the inclusion $\mathcal{C}_{\mathbb{G}} \subseteq L^\infty(\mathbb{G})$ in the non-Kac case, and in particular show that in many natural cases, this subalgebra fails to be a MASA as soon as $\mathbb{G}$ is not of Kac type. The key tools are Theorems 5.9-5.11, which give a criteria for an inclusion $N \subseteq M$ to be quasi-split in terms of the nuclearity of the map $N \to L^2(M);\ x \mapsto \nabla^{1/4} x\Omega$ (where Ω is a cyclic+separating vector for $M \subset B(H)$). Translating this criteria to the CQG setting, they obtain a criterion for

$\mathcal{C}_{\mathbb{G}} \subseteq L^{\infty}(\mathbb{G})$ to be quasi-split in terms of the finiteness of the sum $\sum_{\alpha} \sqrt{\frac{\dim \alpha}{\dim_q \alpha}}$. Using this result, the authors can prove that $\mathcal{C}_{\mathbb{G}}$ is not a MASA as soon as $L^{\infty}(\mathbb{G})$ is a type III factor (covering most non-Kac type orthogonal free quantum groups.) Krajczok and Wasilewski also develop a new approach using the stucture of the scaling group of $\mathbb{G}$ to show that as soon as the scaling group acts non-trivially on coefficients of all non-trivial irreducible representations, $\mathcal{C}_{\mathbb{G}}$ is not a MASA. This covers many known examples, including all non-Kac free orthogonal quantum groups, and quantum automorphism groups. Later on in the chapter, the authors make an interesting study of $SU_q(2)$ and a related bicrossed product construction using a restricted action of $\mathbb{Q}$ by the scaling group, to show that the associated algebra of class functions *is* a MASA. En route, they seem to construct some interesting new CQGs whose von Neumann algebras are type II_{∞}, which they claim might be the first of their kind. In the last part of the chapter, the case of U_F^+ is considered, and there they show that it only satisfies the hypotheses of their Theorem 2.7 when F is chosen not too close to the identity matrix.

Finally, in Chapter 6, Krajczok studies approximation properties of discrete quantum groups and their relations to various operator algebra approximation properties associated to the compact dual objects. Here, Krajczok takes a fresh look at the long standing open problem of whether amenabiliy for a non unimodular discrete quantum group is implied by injectivity or nuclearity of the dual von Neumann algebras/C*-algebras. The main results proved here are contained in Theorems 6.11 and 6.13, which show that the answer is yes, provided one specializes injectivity/nuclearity to take into account the structure of the dual operator algebras.

In conclusion, the PhD thesis of Jacek Krajczok is a high quality one with a broad array of important new results on interesting questions in the field of locally compact quantum groups. I enthusiastically recommend that this work be accepted as a PhD thesis. In fact, in my experience supervising and reviewing theses in this area, Jacek Krajczok's thesis easily ranks among the top 20 %. I rate the thesis of Jacek Krajczok as *excellent*.

Sincerely

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