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Maps between classifying spaces of the unitary groups

Doctoral dissertation summary

The dissertation is devoted to a classical problem of Homotopy Theory, namely homotopy classification of maps between classifying spaces of compact Lie groups. The main result is a construction of new exotic maps between classifying spaces of the unitary groups $BU_n \to BU_m$. If $n > 18$ and $m \leqslant t(n) := \frac{1}{2}n(n-1)(n+2)$ we obtain certain classification of such maps.

Let G be a compact connected Lie group. A (homotopy class) of a map $f\colon BG \to BU_m$ will be called a homotopy representation of the group G. For a prime p, a p-toral group is a group whose identity component is a torus and group of connected components is a finite p-group. According to the Dwyer-Zabrodsky-Notbohm Theorem [11, 26] a homotopy representation f restricted to any p-toral subgroup $P \subseteq G$ is induced by a group homomorphism $\rho^f_P\colon P \to U_m$, i.e. $f|_{BP} \sim B\rho^f_P$. In particular, if $P = \mathrm{T}_G \subseteq G$ is a maximal torus we obtain a representation $\rho^f_{\mathrm{T}_G}\colon \mathrm{T}_G \to U_m$ which is invariant (up to an isomorphism) under the action of the Weyl group $W_G := N_G(\mathrm{T}_G)/\,\mathrm{T}_G$ on T_G. Thus the homotopy representation f defines a W_G-invariant element in $R(\mathrm{T}_G)$ – the representation ring of the torus. Since the restriction homomorphism $\mathrm{res}^G_{\mathrm{T}_G}\colon R(G) \to R(T)^{W_G}$ is an isomorphism [5], we can associate to every homotopy representation f its character $\rho^f \in R(G)$. The construction obviously generalizes the construction of a character of a linear representation.

The main question is: what characters $\mu \in R(G)$ are homotopy characters? The Dwyer-Zabrodsky-Notbohm Theorem implies that for any p-toral subgroup P restriction $\mu|_P \in R^+(P)$ must be a genuine representation of P. Characters of G having such property we will call p-characters, and those which are p-characters for every prime p will be called $\mathcal{P}$-characters. Thus the first step in classification of maps $BU_n \to BU_m$ is a characterisation of $\mathcal{P}$-characters of U_n. This purely algebraic question is considered in Chapters 1-4.

We begin Chapter 1 with recalling basic definitions in representation theory and then define p-characters and prove their elementary properties. In Section 1.5 we define a slant-product in representation ring of a product of two groups which, in subsequent sections, is used for reduction from larger to smaller subgroups. In some special cases the operation is called a reduction of a character.

In Chapter 2 we apply reduction of characters to formulate criteria when a virtual character of a unitary group is a p-character. For that we need a careful description of the maximal p-toral subgroup of U_n as an iterated wreath product of one-dimensional torus and cyclic groups (Sec. 2.2) and study its representations (Sec. 2.3). In Sec. 2.4 the main characterization theorems of p-characters of the unitary groups [**??**], [**??**] are proved. Detailed study of the case of U_p is carried on in Sec. 2.5 resulting in a simple characterization of p-characters of U_p **??**. In Sec. 2.6 we describe a group endomorphism of the maximal p-toral subgroup $N^n_p \subseteq U_n$ which defines the Adams operation $\Psi^k\colon R(N^n_p) \to R(N^n_p)$. In general, effect of the k-th Adams operation on a character we call its k-twisting.

In Chapter 3 we construct families of $\mathcal{P}$-characters of U_n. In Sect. 3.1 we describe some representations of U_n which are used for construction of p-characters of U_n. In Section 3.2 we list candidates for $\mathcal{P}$-characters and check which of them actually are $\mathcal{P}$-characters. In last

Section 3.3 we describe examples showing that a decomposition of p-characters into sum of the indecomposable p-characters is not unique.

In Chapter 4 we show that for unitary group U_n such that $n > 18$, $\mathcal{P}$-characters of dimension $\leqslant t(n)$ are exactly the ones constructed in Chapter 3. A proof relies on a careful analysis of dimensions of the symmetrized weights of the torus which can occur in decompositions of $\mathcal{P}$-characters restricted to the maximal torus. Proof of the main algebraic result is presented in Section 4.4.

The last Chapter 5 is devoted to topological application of the algebraic result which occupies Chapters 1-4. The main result says that all $\mathcal{P}$-characters listed in Section 4.4 are indeed homotopy characters. The key idea is a splitting property of characters recalled in Sec. 5.2. We say that a (homotopy) character $\nu \in R(G)$ has a splitting property if any character χ such that $\chi + \mu$ is a homotopy character is also a homotopy character. Results of [19], and a recent paper [21] imply that the trivial character and characters of the Adams operations have splitting property. We are lucky since the results, combined with other results of [15], suffice to prove that all our $\mathcal{P}$-characters are indeed homotopy characters of U_n.

All the groups considered in this dissertation are compact Lie; all subgroups considered are closed. We denote by Grp the category of compact Lie groups and their homomorphisms.

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