Imperial College London

Spectrum of Partial Differential Equations:

from Weyl asymptotics to Lieb-Thirring inequalities

Ari Laptev

International seminar

on occasion of the 60th Anniversary

of the Institute of Mathematics Polish Academy of Science

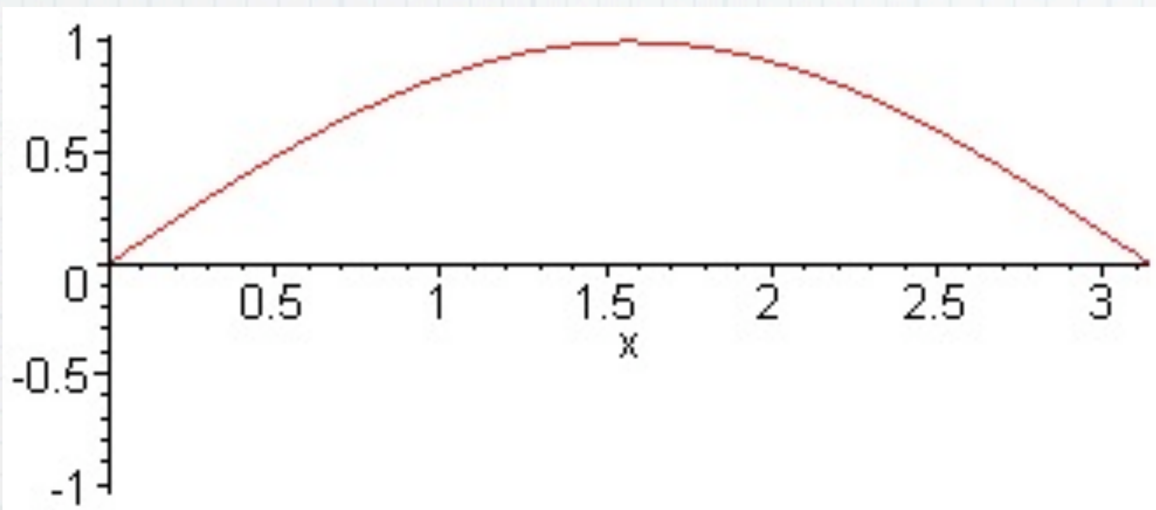
April 3, 2009

If $d = 1$, then solutions of the equation

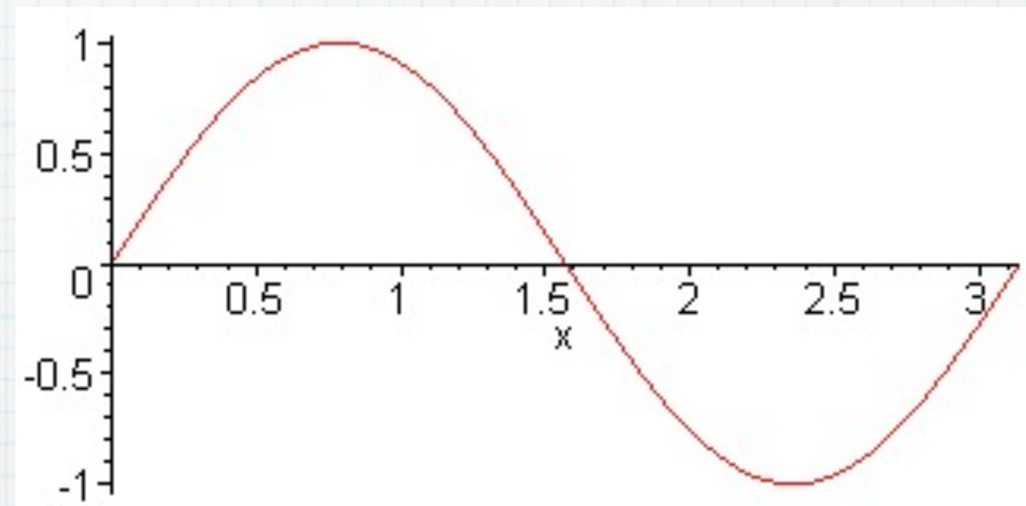
$$-u''(x) = \lambda u(x), \qquad u(0) = u(\pi) = 0$$

are

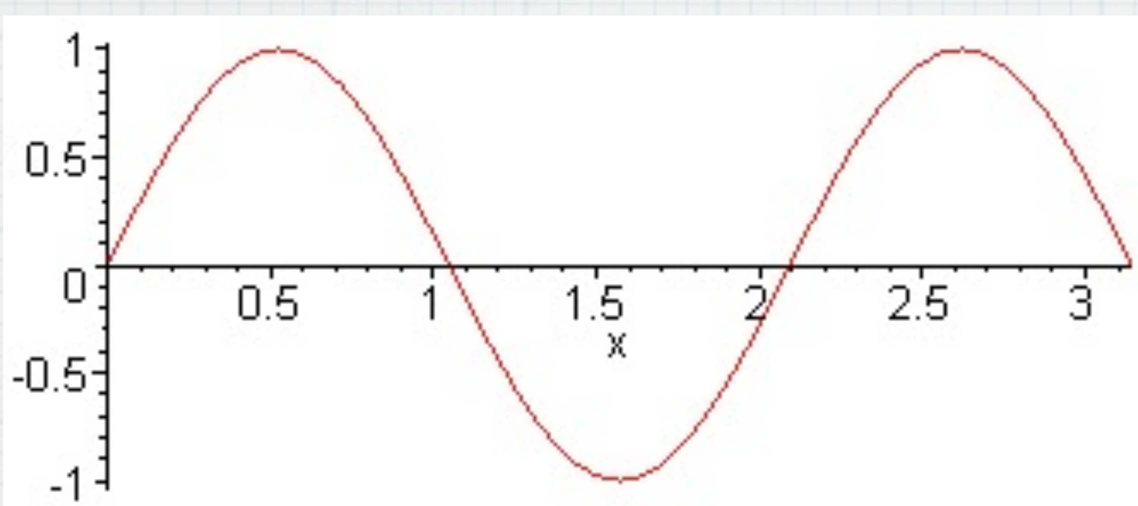
$$u(x) = \sin kx, \qquad \lambda_k = k^2, \qquad k = 1, 2, 3, \ldots$$



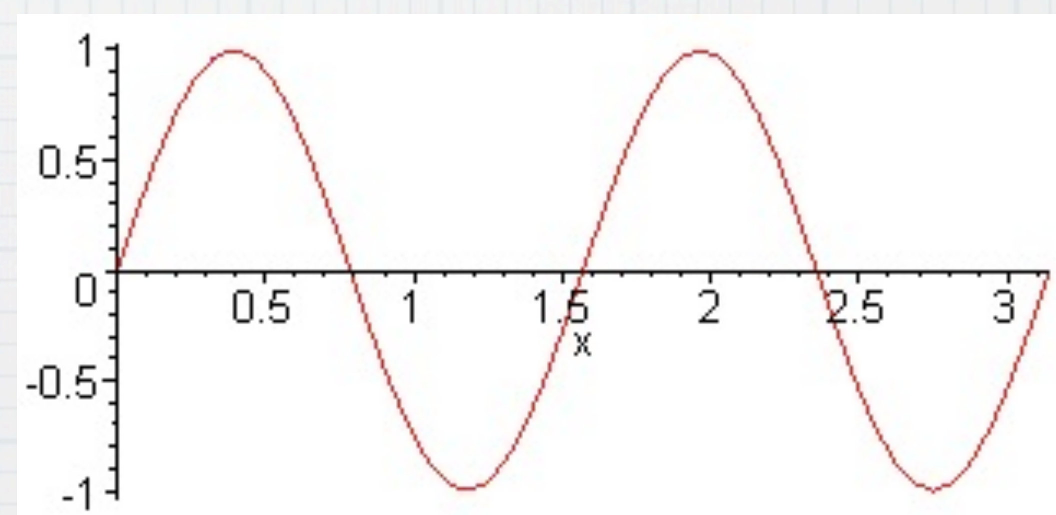
$\sin x, \quad \lambda_1 = 1$



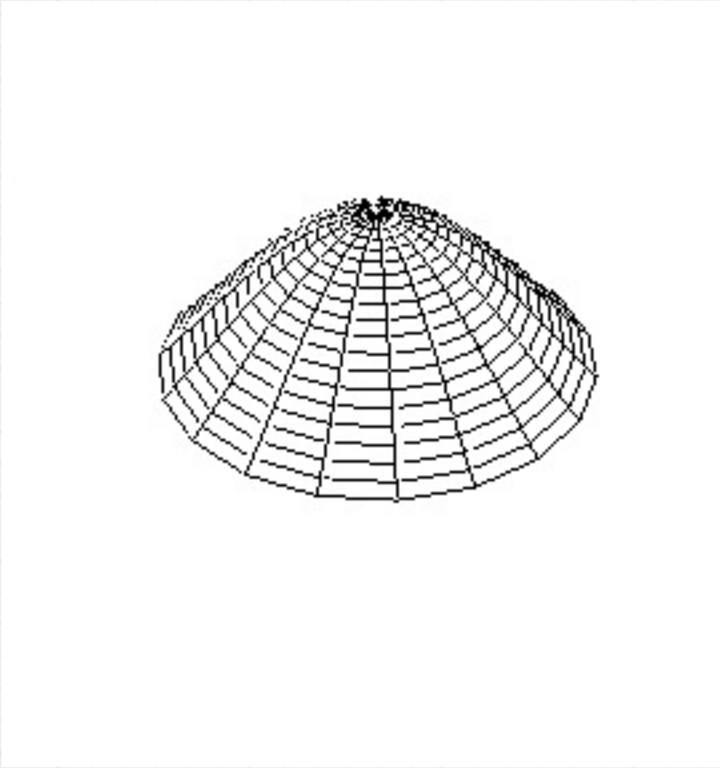
$\sin 2x, \quad \lambda_2 = 4$

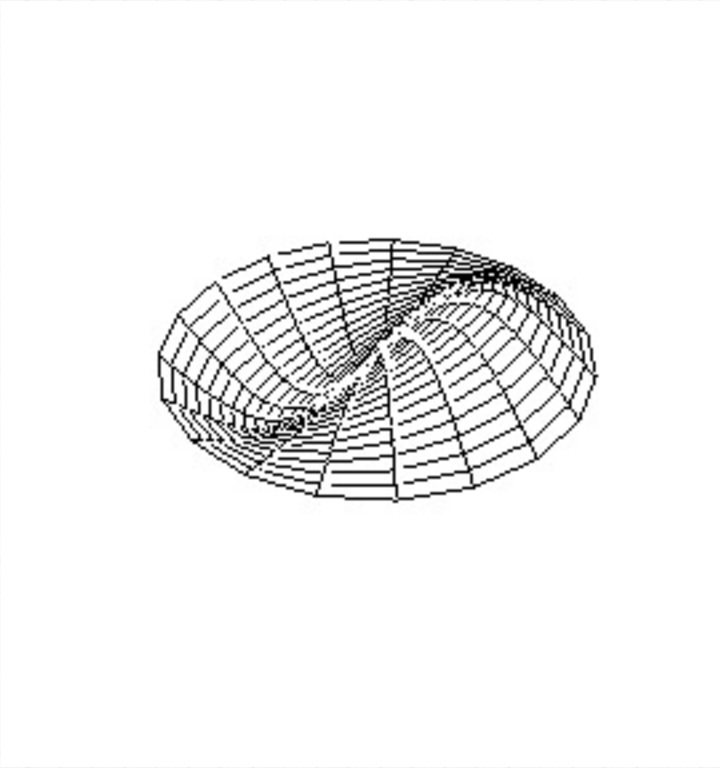


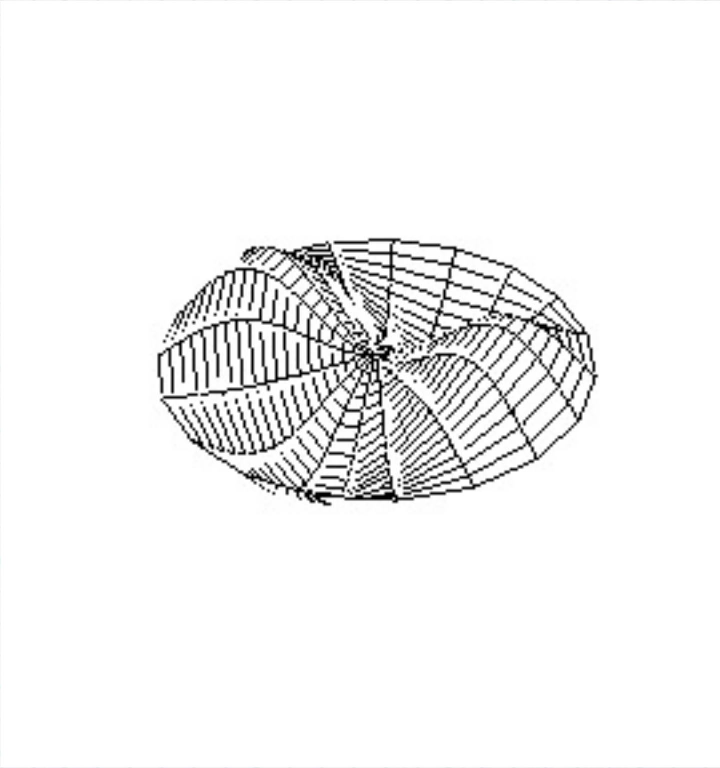
$\sin 3x, \quad \lambda_3 = 9$

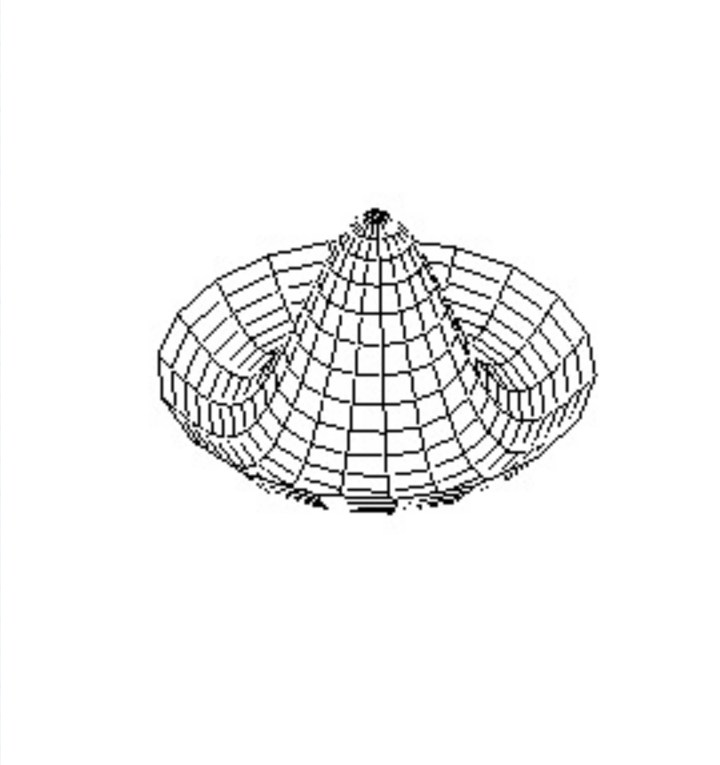


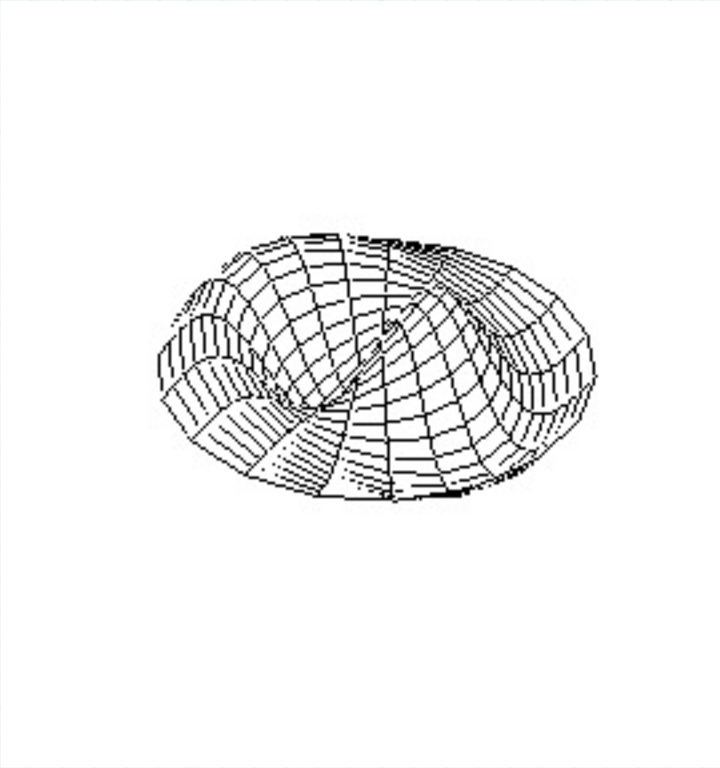
$\sin 4x, \quad \lambda_4 = 16$

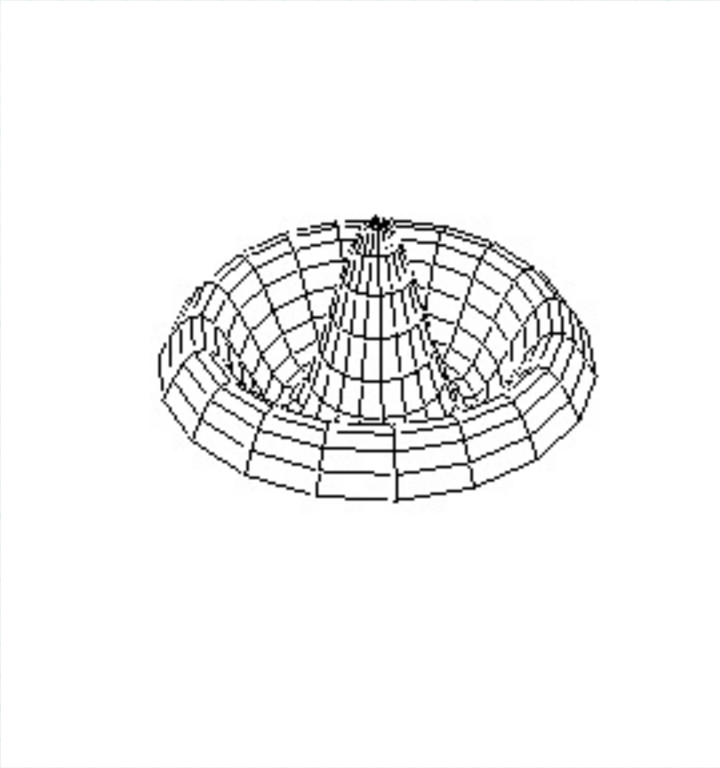




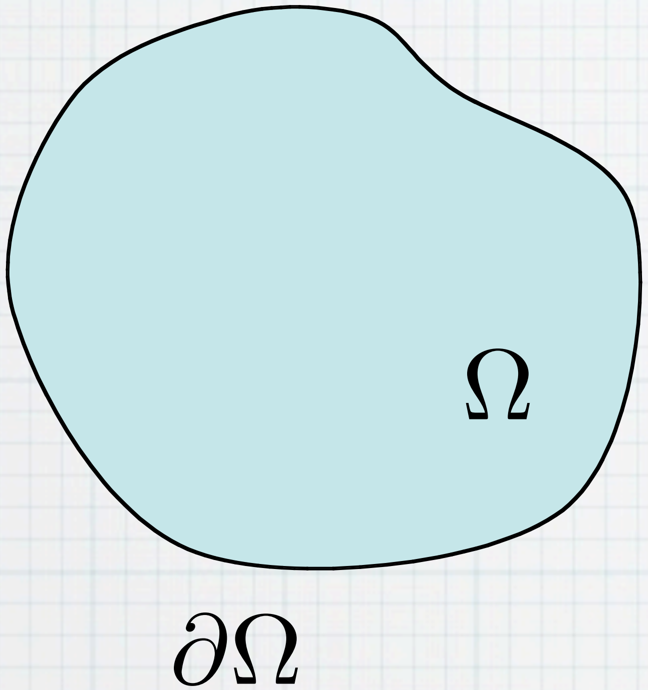








Dirichlet boundary value problem.



Consider a bounded domain $\Omega \subset \mathbb{R}^d$ with piecewise smooth boundary $\partial\Omega$.

Dirichlet boundary value problem for the Laplace operator in $L^2(\Omega)$

$$-\Delta u(x) = \lambda u(x), \qquad x \in \Omega,$$

$$u\Big|_{\partial\Omega} = 0.$$

The Dirichlet Laplacian has a discrete spectrum of infinitely many positive eigenvalues with no finite accumulation point (F. Pockels - 1892)

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots,$$

$\lambda_k \to \infty$ as $k \to \infty$.

Imperial College London

Hermann Weyl
1885-1955

My work always tried to unite the Truth with the Beautiful, but when I had to choose one or the other, I usually choose the Beautiful.

Weyl's asymptotic formula for eigenvalues of a Dirichlet Laplacian.

H.Weyl: "Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen" Math. Ann. , 71 (1911) pp. 441–479.

Theorem.

$$\lambda_k = \frac{4\pi^2 k^{2/d}}{C_d |\Omega|^{2/d}} + o(k^{2/d}),$$

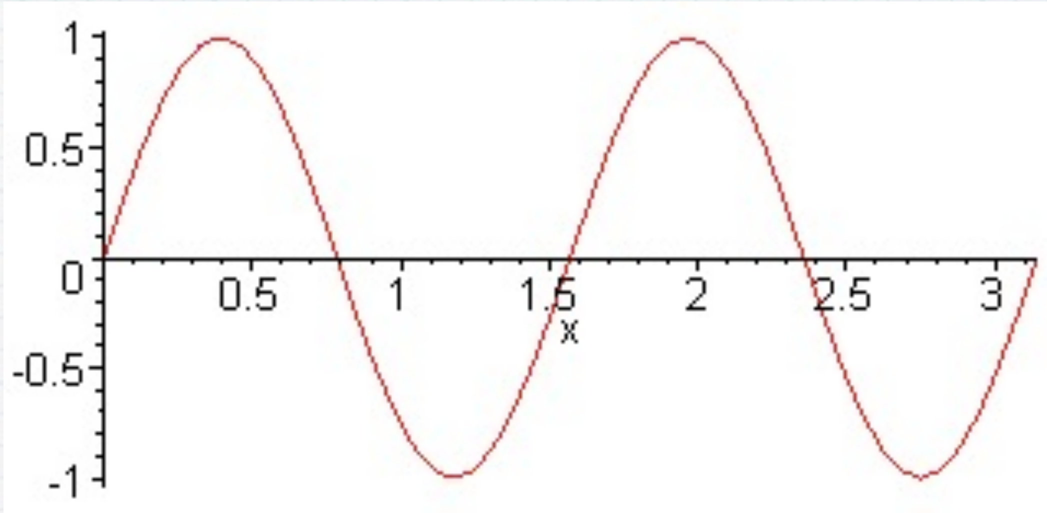
where $|\Omega|$ and $C_d = \pi^{d/2}/\Gamma(d/2+1)$ are respectively the Lebesgue measure of Ω and of the unit ball in $\mathbb{R}^d$.

It is useful to rewrite Weyl's asymptotic formula in term of the counting function of the spectrum as $\lambda \to \infty$

$$N(\lambda) = \#\{k : \lambda_k < \lambda\} = (2\pi)^{-d}\,\lambda^{d/2}|\Omega| \int_{|\xi|<1} d\xi + o(\lambda^{d/2})$$

$$= (2\pi)^{-d} \int_\Omega \int_{|\xi|^2 \le \lambda} d\xi\, dx + o(\lambda^{d/2}),$$

phase volume asymptotics.



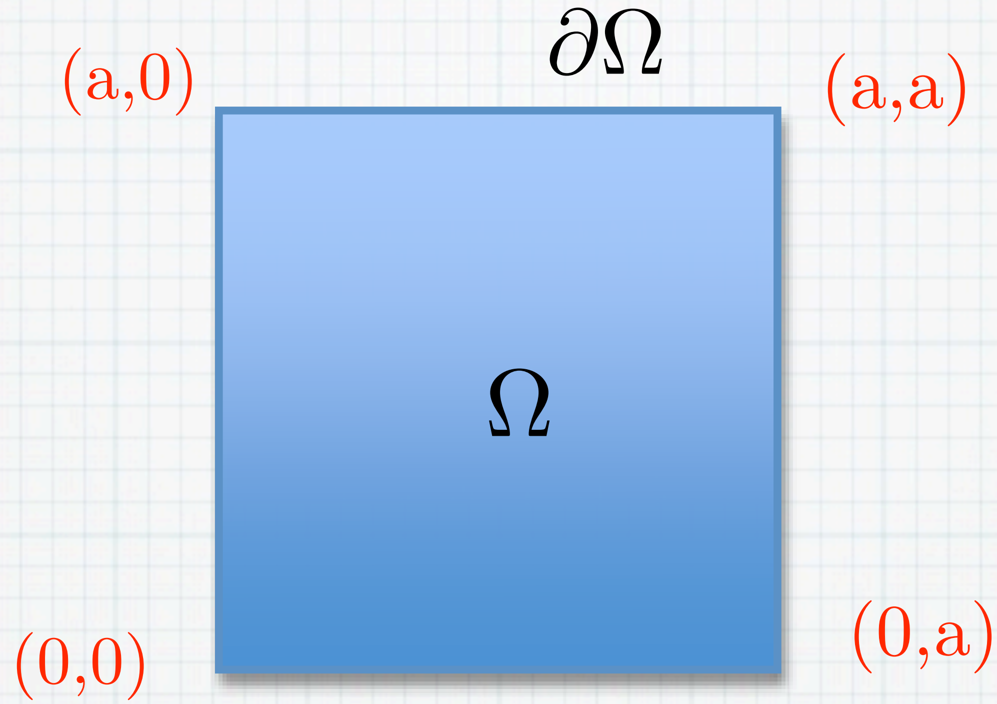
$$u_k(x) = \sin kx, \qquad \lambda_k = k^2, \qquad k = 1, 2, 3, \ldots$$

In the case $d = 1$, $\Omega = (0, \pi)$, $|\Omega| = \pi$, Weyl's asymptotic formula in term of the counting function could be written in a more precise way

$$N(\lambda) = \#\{k : \lambda_k = k^2 < \lambda\} = (2\pi)^{-d} \lambda^{d/2} |\Omega| \int_{|\xi|<1} d\xi + o(\lambda^{d/2})$$

$$= (2\pi)^{-1} \sqrt{\lambda}\, \pi\, 2 + O(1) = \sqrt{\lambda} + O(1), \qquad \text{as} \quad \lambda \to \infty.$$

Imperial College London



In this case Dirichlet and Neumann boundary value problems

$$-\Delta u(x,y) = \lambda u(x,y), \qquad -\Delta v(x,y) = \mu v(x,y)$$

$$u|_{\partial\Omega} = 0 \qquad \frac{\partial v}{\partial n}\Big|_{\partial\Omega} = 0$$

have solutions

$$u_{nm}(x,y) = \sin \pi a^{-1} nx \cdot \sin \pi a^{-1} my, \qquad v_{nm}(x,y) = \cos \pi a^{-1} nx \cdot \cos \pi a^{-1} my,$$

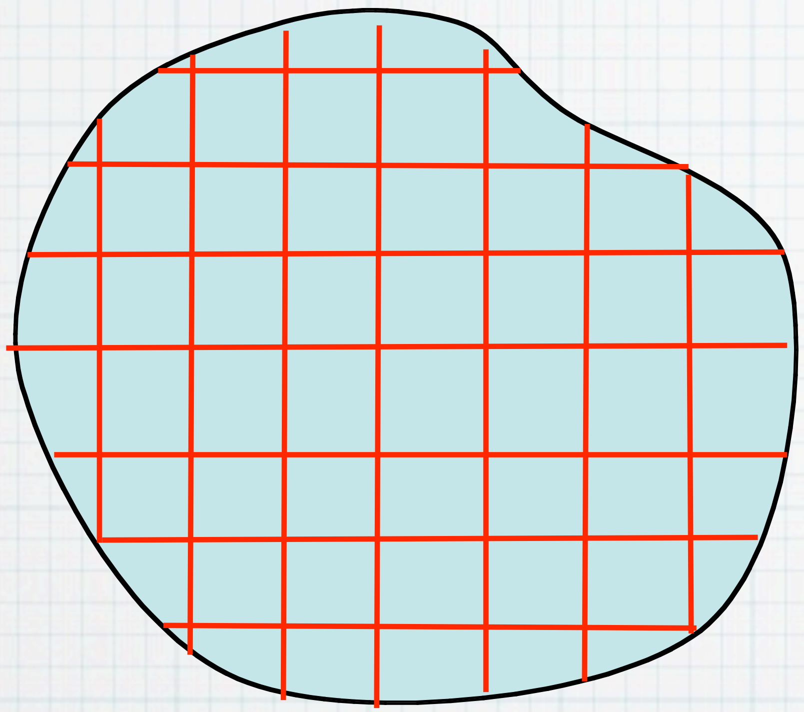
$$\lambda_{nm} = \pi^2 a^{-2}(n^2+m^2), \quad n,m = 1,2,\ldots \qquad \mu_{nm} = \pi^2 a^{-2}(n^2+m^2), \quad n,m = 0,1,2,\ldots$$

$$N^{\mathcal{D}}(\lambda) = \#\{n,m = 1,2,\cdots : \lambda_{nm} = n^2+m^2 < \lambda\} \sim (4\pi)^{-1} a^2 \lambda$$

$$N^{\mathcal{N}}(\mu) = \#\{n,m = 0,1,2,\cdots : \mu_{nm} = n^2+m^2 < \mu\} \sim (4\pi)^{-1} a^2 \mu$$

Imperial College London

$$N(\lambda) = \#\{k : \lambda_k < \lambda\} \sim (2\pi)^{-d}\,\lambda^{d/2}|\Omega| \int_{|\xi|<1} d\xi$$



Proof.

Weyl used a version of the max-min principle.

Dirichlet-Neumann bracketing:

$$N^{\mathcal{D}}(\lambda) \leq N(\lambda) \leq N^{\mathcal{N}}(\lambda).$$

For each square with side a we find that the eigenvalues are equal to

$\{\lambda^{\mathbb{D}}_{nm}(a) = \pi^2 a^{-2}(n^2 + m^2) : n, m = 1, 2, 3, \ldots\}$ for the Dirichlet problem

and

$\{\mu^{\mathbb{N}}_{nm}(a) = \pi^2 a^{-2}(n^2 + m^2) : n, m = 0, 1, 2, 3, \ldots\}$ for the Neumann problem.

Counting

$\#\{(n, m) : \lambda^{\mathbb{D}}_{nm}(a) \leq \lambda\}$

and

$\#\{(n, m) : \mu^{\mathbb{N}}_{nm}(a) \leq \lambda\}$,

summing them up and letting $a \to 0$ we proof the result.

Imperial College London

Weyl's conjecture.

In 1911 H.Weyl also conjectured that

$$N(\lambda) = (2\pi)^{-d}\, C_d\, \lambda^{d/2}|\Omega| - c_{d-1}\, \lambda^{(d-1)/2}|\partial\Omega| + o(\lambda^{(d-1)/2}),$$

where $c_{d-1} > 0$ is a standard term depending only on dimension d.

Under certain conditions on classical billiards in $T^*\Omega$ V. Ivrii proved this result in 1980.

Imperial College London

Mark Kac

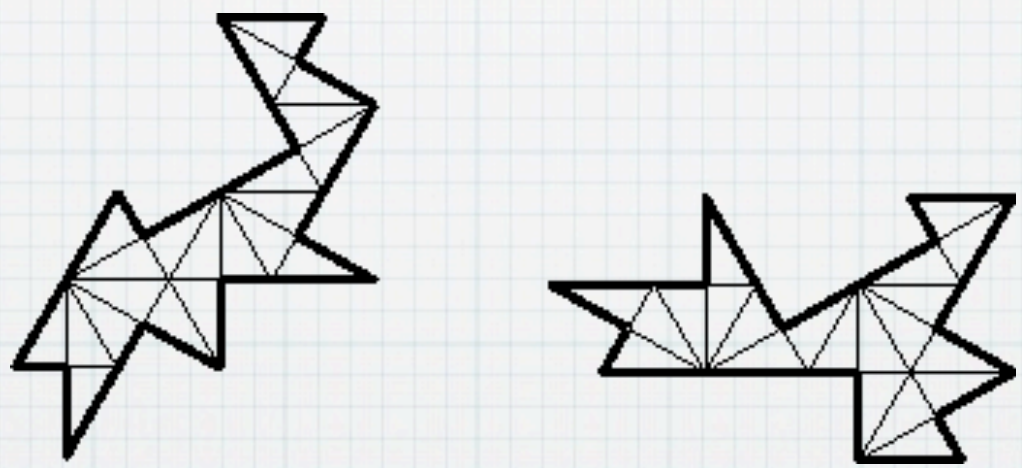
1914-1984

Isospectral domains.

In 1965 Mark Kac asked: ‘Can one hear the shape of a drum?’ (question goes back to H.Weyl)

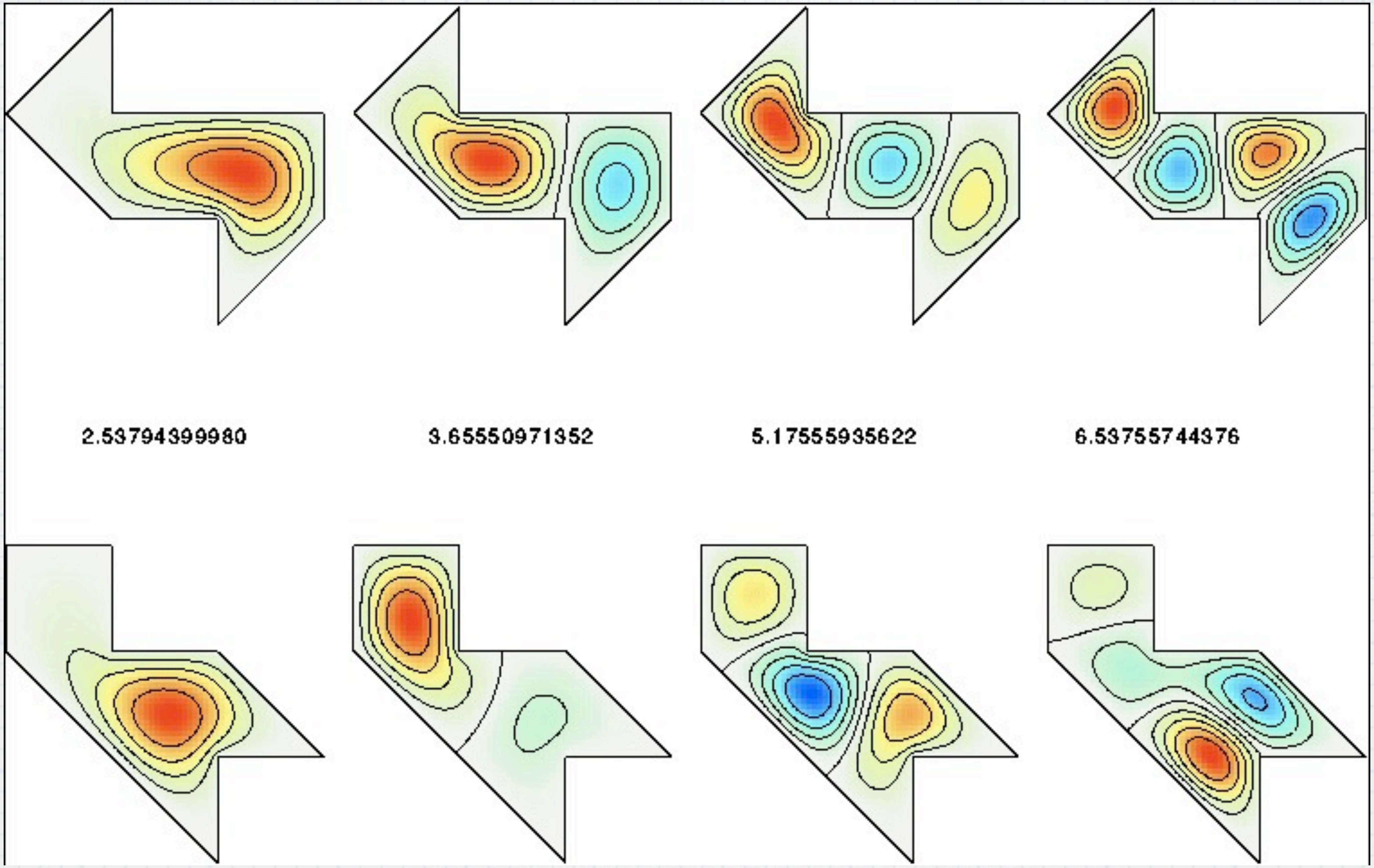
T. Sunada 1985 found two different domains in $\mathbb{R}^{16}$ which have the same ”Dirichlet” spectrum.

Gordon, Webb, and Wolpert 1992, found planar isospectral domains.



Peter Buser, John Conway, Peter Doyle, and Klaus-Dieter Semmler, 1994.


Sunada triple:
"almost
conjugate
WASH. U. MATH


2.53794399980
3.65550971352
5.17555935622
6.53755744376

Imperial College London

Weyl's inequalities. Pólya's conjecture.

George Pólya
1887 - 1985

In 1961 Pólya proved that if $\Omega \subset \mathbb{R}^2$ is a tiling domain then

$$\lambda_k \geq \frac{4\pi k}{|\Omega|}, \qquad k = 1, 2, 3, \ldots$$

or equivalently

$$N(\lambda) = \#\{k : \lambda_k \leq \lambda\} \leq (2\pi)^{-2} \pi \lambda |\Omega|$$

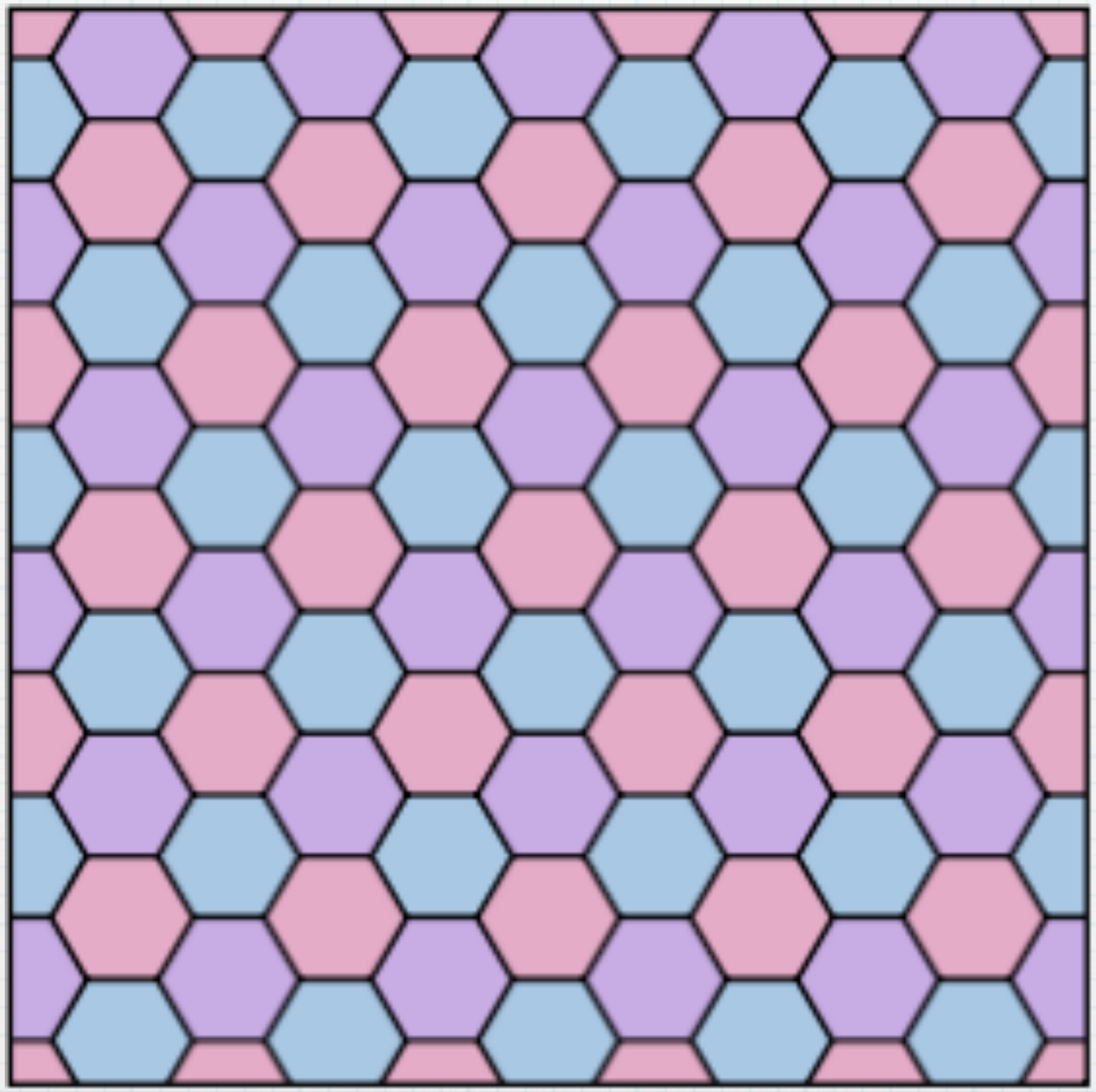
$$= (2\pi)^{-2} \int_\Omega \int_{|\xi|^2 \leq \lambda} d\xi \, dx,$$

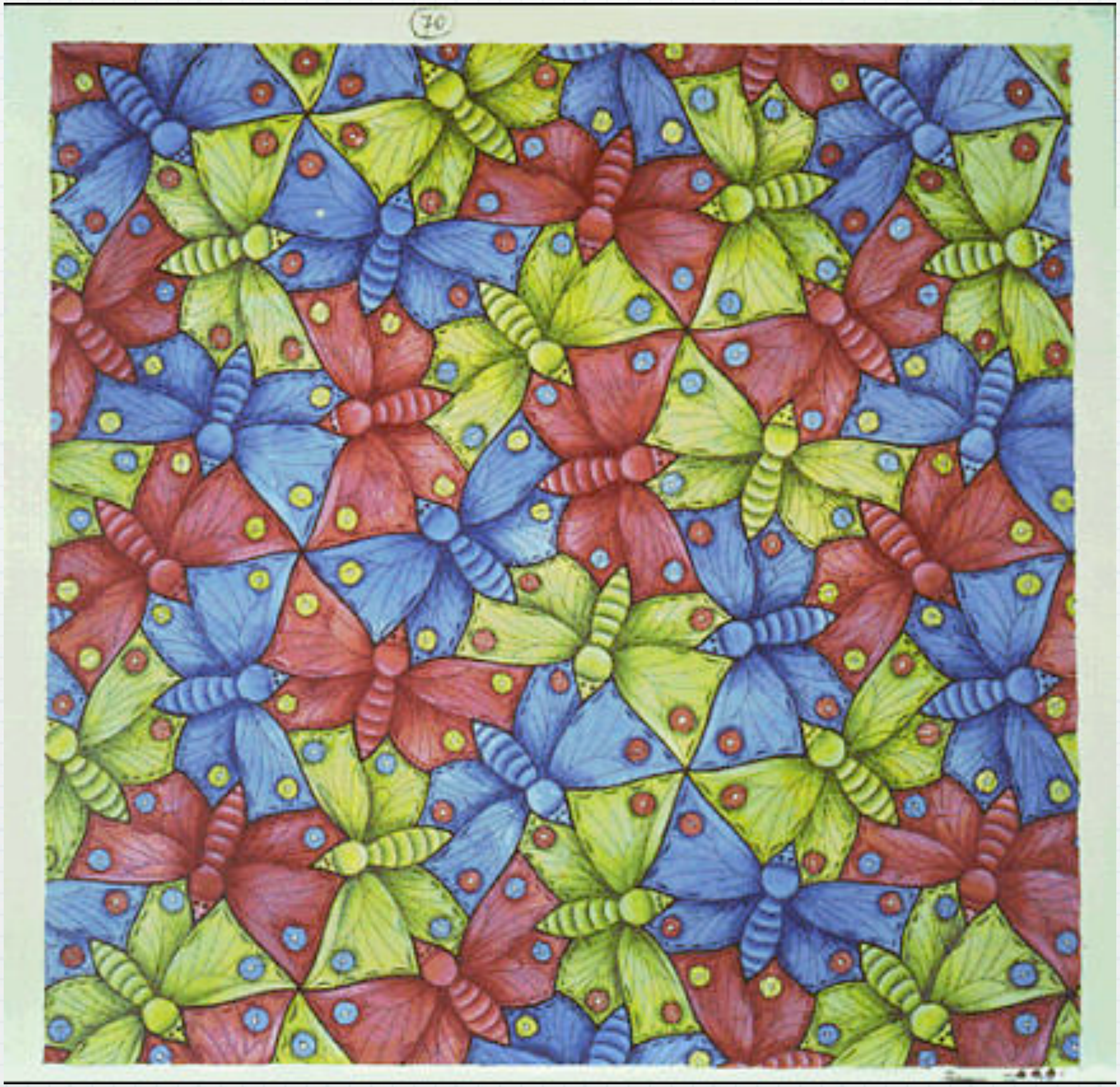
phase volume inequality.

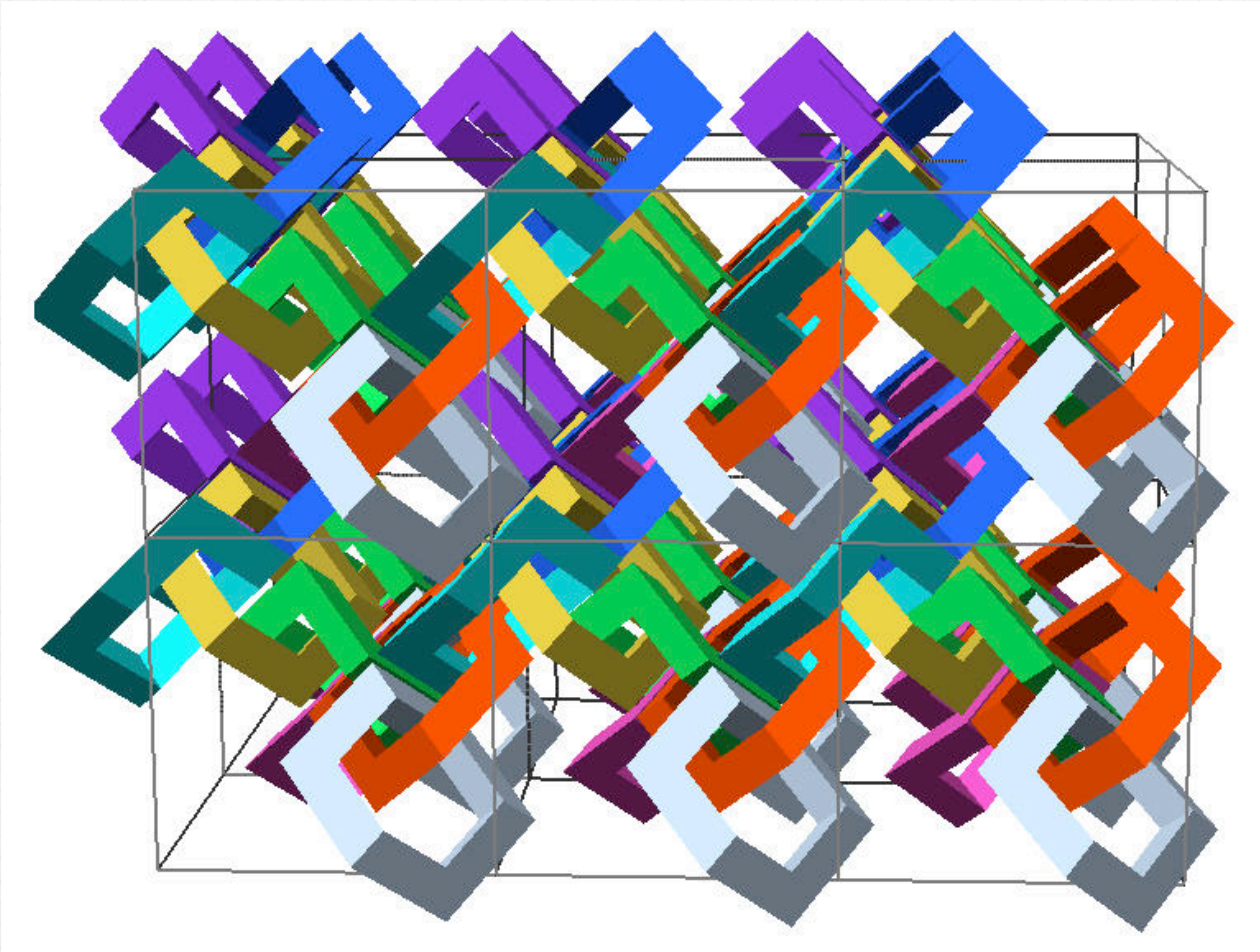
Pólya's Conjecture

Prove that the latter inequality holds for arbitrary domains.

Pólya's conjecture is still open for $\Omega = \{x \in \mathbb{R}^d : |x| < 1\}$.




70



Imperial College London

Another easier question:

Is it true that for some $\gamma \geq 0$ the following phase volume estimate holds?

$$\sum (\lambda - \lambda_k)_+^\gamma \leq (2\pi)^{-d} \lambda^{\gamma+d/2} |\Omega| \int_{\mathbb{R}^d} (1 - |\xi|^2)_+^\gamma \, d\xi,$$

where $(x)_+ = (|x| + x)/2$ is the positive part of x.

Theorem. (Berezin, Li-Yau)
If $\gamma \geq 1$ then the above Weyl inequality holds true.

Proof.
Let $\gamma = 1$ and let φ_k be the orthonormal basis in $L^2(\Omega)$ consisting of eigenfunctions of the Dirichlet Laplacian which is denoted by A. Let $\hat{\varphi}$ be the Fourier transform of φ. Then by using Parceval formula we find

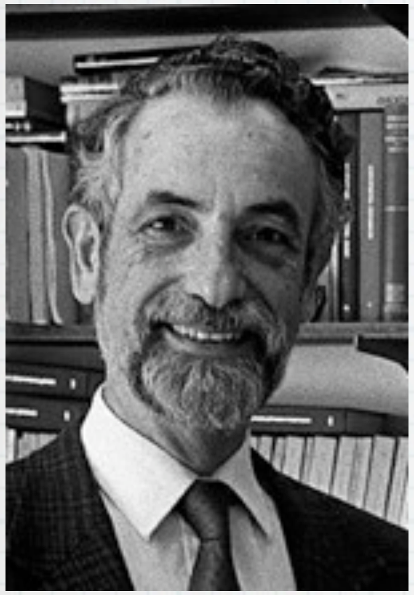
$$\sum_k (\lambda - \lambda_k)_+ = \sum_k \big(\lambda - (A\varphi_k, \varphi_k)\big)_+ = \sum_k \left((2\pi)^{-d} \int_{\mathbb{R}^d} (\lambda - |\xi|^2)\, |\hat{\varphi}_k|^2 \, d\xi \right)_+$$

$$\leq (2\pi)^{-d} \sum_k \int_{\mathbb{R}^d} (\lambda - |\xi|^2)_+ |\hat{\varphi}_k|^2 \, d\xi$$

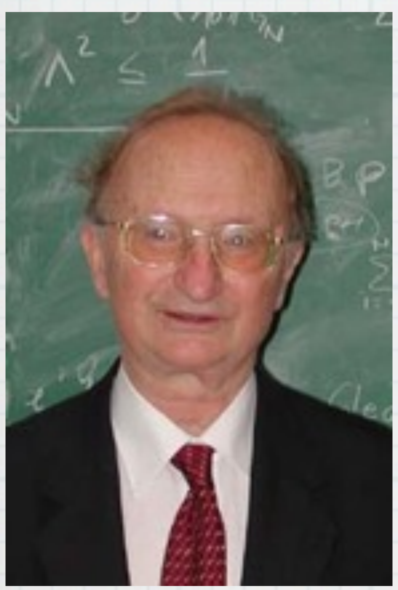
$$= (2\pi)^{-d} \sum_k \int_{\mathbb{R}^d} (\lambda - |\xi|^2)_+ \left| \int_\Omega e^{i(x,\xi)} \varphi_k(x)\, dx \right|^2 d\xi$$

$$= (2\pi)^{-d} \int_{\mathbb{R}^d} (\lambda - |\xi|^2)_+ \left(\sum_k |(e^{i(\cdot,\xi)}, \varphi_k)|^2 \right) d\xi = (2\pi)^{-d} \int_{\mathbb{R}^d} (\lambda - |\xi|^2)_+ \, d\xi \underbrace{\|e^{i(\cdot,\xi)}\|^2}_{=|\Omega|}.$$

Imperial College London



E.H. Lieb

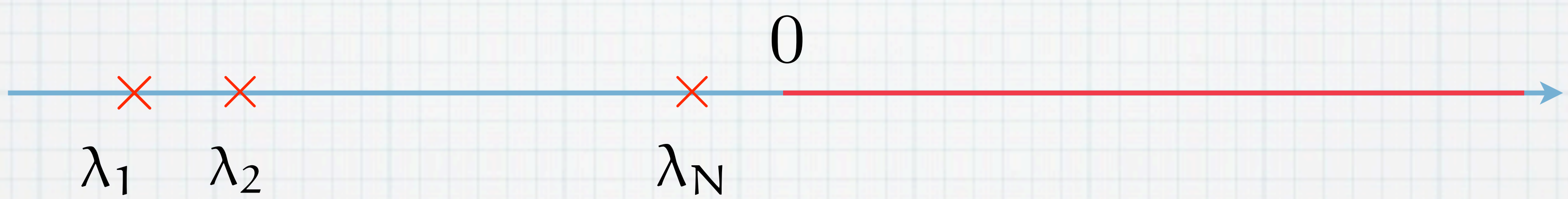


W. Thirring

Lieb-Thirring inequalities.

Let $H = -\Delta + V$ be a Schrödinger operator in $L^2(\mathbb{R}^d)$, $V \to 0$, as $x \to \infty$.

Spectrum:



$$\sum_j |\lambda_j|^\gamma = \sum_j \lambda_j^\gamma(V) \leq \frac{C_{d,\gamma}}{(2\pi)^d} \int\int \Big(|\xi|^2 + V(x)\Big)_-^\gamma \, dx d\xi = L_{\gamma,d} \int V(x)_-^{\gamma+d/2} \, dx.$$

Compare with Weyl's asymptotic formula:

$$\sum_j |\lambda_j(\alpha V)|^\gamma \sim_{\alpha\to\infty} L_{\gamma,d}^{cl} \int (\alpha V_-)^{\gamma+d/2} \, dx = (2\pi)^{-d} \iint (\xi^2 + \alpha V)_-^\gamma \, d\xi dx,$$

which implies $L_{\gamma,d}^{cl} \leq L_{\gamma,d}$.

Imperial College London

Example.

If in $H = -\Delta + V$,

$$V(x) = \begin{cases} -\lambda, & x \in \Omega, \\ +\infty, & x \notin \Omega, \end{cases} \qquad \Omega \in \mathbb{R}^d,$$

then the spectrum of H coincides with the spectrum of the Dirichlet Laplacian in Ω.

Therefore Pólya inequalities are special cases of L-Th inequalities.

Imperial College London

$$(-\Delta + V)u = \lambda u.$$

$$\sum_j |\lambda_j|^\gamma \le L_{\gamma,d} \int V(x)_-^{\gamma+d/2}\, dx.$$

Theorem.
The constant $L_{\gamma,d} < \infty$ if $d = 1$, $\gamma \geq 1/2$, $d > 2$, $\gamma > 0$ and $d \geq 3$, $\gamma \geq 0$.

E.Lieb, W.Thirring, T.Weidl, M.Cwikel, G.Rozenblum.

Theorem.
It is known that $L_{1/2,1} = 1/2$ ($L^{cl}_{1/2,1} = 1/4$) and
$L_{\gamma,d} = L^{cl}_{\gamma,d}$ if $\gamma \geq 3/2$, $d \geq 1$.
In other cases the sharp constants are unknown.

E.Lieb, W.Thirring, M.Aizenmann, D.Hundertmark, L.Thomas, AL & T.Weidl.

Buslaev-Faddeev-Zakharov trace formula, $d = 1$.

Let ψ solves the equation

$$-\frac{d^2}{dx^2}\psi + V\psi = k^2\psi, \qquad \psi(x,k) = \begin{cases} e^{ikx}, & \text{as } x \to \infty \\ a(k)e^{ikx} + b(k)e^{-ikx}, & \text{as } x \to -\infty. \end{cases}$$

Fundamental property:

if $k \in \mathbb{R}$ then $W[\psi, \bar\psi] = \psi\bar\psi' - \psi'\bar\psi = \text{const}.$

This implies $1 = |a|^2 - |b|^2 \Leftrightarrow |a| \geq 1.$

Let

$$\lambda_j = (i\kappa_j)^2, \qquad \kappa_j > 0.$$

Theorem. (BFZ trace formula.)
If $V \leq 0$, then

$$\frac{3}{2\pi}\int k^2 \ln|a|^2\, dk + \sum_j \kappa_j^3 = \frac{3}{16}\int V^2\, dx = (2\pi)^{-1}\iint (|\xi|^2 + V)_-^{3/2}\, d\xi\, dx.$$

Corollary. (L-Th inequality.)

$$\sum_j |\lambda_j|^{3/2} = \sum_j \kappa_j^3 \leq \frac{3}{16}\int V^2\, dx.$$

Imperial College London

Soliton's approach (Lieb & Thirring, Lax, Kruskal).

Let us consider the KdV equation

$$U_t = 6UU_x - U_{xxx}, \qquad U|_{t=0} = V.$$

Then

$$U_t = \Big[-\frac{d^2}{dx^2} + U, M\Big], \quad \text{where} \quad M = 4\frac{d^3}{dx^3} - 3\Big(U\frac{d}{dx} + \frac{d}{dx}U\Big).$$

- Discrete spectrum is independent of t:

$$\lambda_j\Big(-\frac{d^2}{dx^2} + U\Big) = \lambda_j\Big(-\frac{d^2}{dx^2} + V\Big).$$

- $a(k,t) = e^{i8k^3 t} a(k,0)$.
- $\int U^2(x,t)\,dx = \int V^2(x)\,dx$.

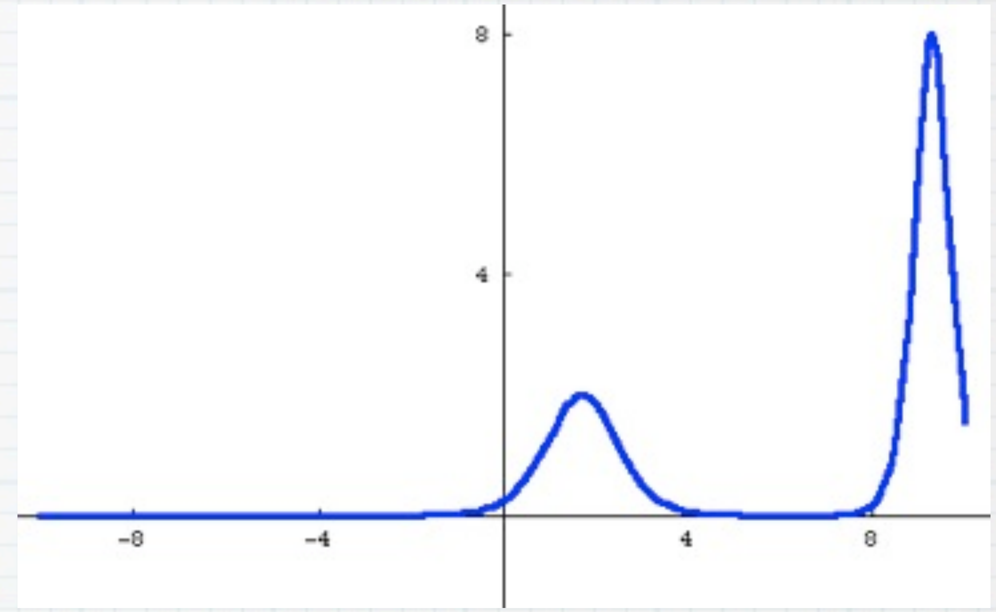
Therefore terms in the trace formula

$$\frac{3}{2\pi}\int k^2 \ln|a|^2\,dk + \sum_j |\lambda_j|^{3/2} = \frac{3}{16}\int U^2\,dx$$

are independent of time.

Imperial College London

$$U(x,t) \sim_{t\to\infty} \sum_{j=1}^{N} U_j(x - 4\lambda_j t) + U_\infty,$$



- $\|U_\infty\|_\infty \le \varepsilon(t) \to_{t\to\infty} 0$ and U_j are solitons

$$U_j(x) = -2\lambda_j \cosh^{-2}(\sqrt{\lambda_j}x).$$

- $\left(-\frac{d^2}{dx^2} + U_j\right)\cosh^{-1}(\sqrt{\lambda_j}x) = -\lambda_j \cosh^{-1}(\sqrt{\lambda_j}x).$

Finally, since $4\int \cosh^{-4} x\,dx = 16/3$, we obtain

$$\int V^2\,dx \ge \sum_{j=1}^{N} \int U_j^2\,dx = \frac{16}{3}\sum_{j=1}^{N} \lambda_j^{3/2}.$$

Multidimensional Lieb-Thirring inequalities.

The main argument is based on 1D matrix Lieb-Thirring inequality.

Theorem. (AL & T.Weidl)

Let Q be a Hermitian $m \times m$ matrix-function and let $H = -\Delta + Q$. Then

$$\sum_j \lambda_j^{3/2}(H) \leq \frac{3}{16} \int \operatorname{Tr} Q^2(x)\, dx.$$

Lifting argument with respect to dimension.

Let for simplicity $d = 2$, $V \in C_0^\infty(\mathbb{R}^2)$, $V \leq 0$, $x = (x_1, x_2)$. Then

$$H = -\Delta + V = -\partial^2_{x_1 x_1} - \underbrace{(\partial^2_{x_2 x_2} - V)}_{\tilde{H}(x_1)}.$$

Spectrum $\sigma(\tilde{H})$ of $\tilde{H}(x_1)$ has a finite number of positive eigenvalues $\mu_l(x_1)$. Thus $\tilde{H}_+(x_1)$ has a finite rank. Let, for instance, $\gamma = 3/2$

$$\sum_j \lambda_j^{3/2}(H) \leq \sum_j \lambda_j^{3/2}(-\partial^2_{x_1 x_1} - \tilde{H}_+)$$

$$\leq \frac{3}{16} \int \operatorname{Tr} \tilde{H}_+^2(x_1)\, dx_1 \leq \underbrace{\frac{3}{16} L_{2,1}}_{L^{cl}_{3/2,2}} \iint V^{3/2+1}(x)\, dx.$$

Imperial College London

Recent Result (jointly with J.Dolbeault & M.Loss)

Let H be a Schrödinger operator in $L^2(\mathbb{R}^d)$

$$H = -\Delta - Q,$$

The main new result of this talk is the following Theorem:

Theorem. Let $Q \geq 0$ be a Hermitian $M \times M$ matrix-function defined on $\mathbb{R}$ and let λ_n be all negative eigenvalues of the operator H. Then

$$\sum |\lambda_n| \leq \frac{2}{3\sqrt{3}} \int_{\mathbb{R}} \mathrm{Tr}\left[Q^{3/2}(x)\right] dx\,.$$

Corollary. For any dimension $d \geq 1$, the negative eigenvalues of the operator H satisfy inequalities

$$\sum |\lambda_n| \leq L_{d,1} \int_{\mathbb{R}^d} \mathrm{Tr}\left[Q^{d/2+1}(x)\right] dx\,,$$

where

$$L_{d,1} \leq R \times L^{cl}_{d,1} = R \times \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (1 - |\xi|)_+ \, d\xi,$$

and $R = \frac{\pi}{\sqrt{3}} = 1.8138\ldots$.

Imperial College London

Scalar case

For simplicity we consider a scalar version of this theorem based on a 1D generalised Sobolev inequality due to Eden and Foias.

Let $\{\psi_j\}_{j=1}^n$ be in orthonormal system of function in $L^2(\mathbb{R})$ and let

$$\rho(x) = \sum_{j=1}^n \psi_j^2(x).$$

In this case the previous theorem can be reformulated as

Theorem.

$$\int_{\mathbb{R}} \rho^3(x)\, dx = \int \Big(\sum_{j=1}^n |\psi_j(x)|^2\Big)^3 dx \le \sum_{j=1}^n \int_{\mathbb{R}} |\psi'{}_j(x)|^2\, dx.$$

Imperial College London

Proof.

We first derive a simple inequality

$$\|\psi\|_{L^\infty} \le \|\psi\|_{L^2}^{1/2} \|\psi'\|_{L^2}^{1/2}.$$

Indeed

$$|\psi(x)|^2 = \frac{1}{2}\Big| \int_{-\infty}^{x} |\psi^2|'\,dt - \int_{x}^{\infty} |\psi^2|'\,dt \Big| \le \int |\psi||\psi'|\,dt \le \|\psi\|_{L^2}\|\psi'\|_{L^2}.$$

Let now $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$. Then

$$\Big|\sum_{j=1}^{n} \xi_j \psi_j(x)\Big| \le \Big(\sum_{j,k=1}^{n} \xi_j \bar{\xi}_k (\psi_j, \psi_k)\Big)^{1/4} \Big(\sum_{j,k=1}^{n} \xi_j \bar{\xi}_k (\psi_j', \psi_k')\Big)^{1/4}$$

$$\le \Big(\sum_{j=1}^{n} \xi_j^2\Big)^{1/4} \Big(\sum_{j,k=1}^{n} \xi_j \bar{\xi}_k (\psi_j', \psi_k')\Big)^{1/4}.$$

If we set $\xi_j = \psi_j(x)$ then the latter inequality becomes

$$\rho(x) = \sum_{j=1}^{n} |\psi_j(x)|^2 \le \rho^{1/4}(x) \Big(\sum_{j,k=1}^{n} \psi_j(x)\overline{\psi_k(x)} (\psi_j', \psi_k')\Big)^{1/4}.$$

Thus

$$\rho^3(x) \le \sum_{j,k=1}^{n} \psi_j(x)\overline{\psi_k(x)}(\psi_j', \psi_k').$$

Integrating both sides we arrive at

$$\int \Big(\sum_{j=1}^{n} |\psi_j(x)|^2\Big)^3 dx \le \sum_{j=1}^{n} \int |\psi_j'|^2\,dx.$$

Imperial College London

Spectrum of Schrödinger operators

Let $\{\psi_j\}_{j=1}^{\infty}$ be the orthonormal system of eigenfunctions corresponding to the negative eigenvalues of the Schrödinger operator

$$-\frac{d^2}{dx^2}\psi_j - Q\psi_j = -\lambda_j\psi_j,$$

where we assume that $Q \geq 0$. Then by using the latter result and Hölder's inequality we obtain

$$\int\Big(\sum_{j=1}^{n}|\psi_j(x)|^2\Big)^3 dx - \Big(\int Q^{3/2}\,dx\Big)^{2/3}\int\Big(\sum_{j=1}^{n}|\psi_j(x)|^2\Big)^3 dx\Big)^{1/3}$$

$$\leq \sum_j \int\Big(|\psi_j'|^2 - Q|\psi_j|^2\Big)\,dx = -\sum_j \lambda_j.$$

Denote

$$X = \Big(\int\Big(\sum_{j=1}^{n}|\psi_j(x)|^2\Big)^3 dx\Big)^{1/3},$$

then the latter inequality can be written as

$$X^3 - \Big(\int V^{3/2}\,dx\Big)^{2/3} X \leq -\sum_j \lambda_j.$$

Maximizing the left hand side we find $X = \frac{1}{\sqrt{3}}\Big(\int Q^{3/2}\,dx\Big)^{1/3}$. This implies

$$\frac{1}{3\sqrt{3}}\int Q^{3/2}\,dx - \frac{1}{\sqrt{3}}\int Q^{3/2}\,dx = -\frac{2}{3\sqrt{3}}\int Q^{3/2}\,dx \leq -\sum_j \lambda_j$$

and we finally obtain $\sum_j \lambda_j \leq \frac{2}{3\sqrt{3}}\int Q^{3/2}\,dx$.

This is the best known constant in Lieb-Thirring's inequality.

Imperial College London

Thank you
for your attention

Professor Ari Laptev, Head of the Department of Mathematics, Imperial College London

Inaugural lecture: “Spectrum of partial differential equations: from Weyl asymptotics to Lieb-Thirring inequalities"

In the chair: **Martin Liebeck**
Department of Mathematics

Vote of thanks: **Professor John Elgin**
Department of Mathematics