Galois structures

Tomasz Brzeziński

Notes by
Paweł Witkowski

January 2008

Contents

Chapter 1

Comonads and Galois comodules of corings

The aim of the remaining lectures is to study Galois structures which arise in differential non-commutative geometry, in particular to show, how Galois conditions encode geometric notions such as principal (and associated vector) bundles. The Galois condition which arises in this context is very closely related to (co)monadicity described earlier. To make better connection with the preceding sections we start with the category theory considerations.

We use the following notational conventions. The identity morphism for an object X is denoted by X (though occasionally we write id for clarity). We do not write composition symbol $\circ$ when composing functors. Given a natural transformation δ between functors F and G, δ_X denotes corresponding morphism $F(X) \to G(X)$. For any other functors H, K (composable with F or G, respectively) $H\delta$ means the natural transformation $HF \to HG$ given on objects X as $H(\delta_X)$, while δK means the transformation $FK \to GK$ given on objects as $\delta_{K(X)}$.

1.1 Comonads

Definition 1.1. A **comonad** on a category $\mathbf{A}$ is a triple $G = (G, \delta, \sigma)$, where $G\colon \mathbf{A} \to \mathbf{A}$ is a functor $\delta\colon G \to GG$, $\sigma\colon G \to \mathrm{id}_{\mathbf{A}}$ are natural transformations such that the following diagrams

$$\begin{tikzcd} G \arrow[r, "\delta"] \arrow[d, "\delta"'] & GG \arrow[d, "G\delta"] \\ GG \arrow[r, "\delta G"'] & GGG \end{tikzcd}, \qquad \begin{tikzcd} G \arrow[r, "\delta"] \arrow[d, "\delta"'] \arrow[dr, equal] & GG \arrow[d, "G\sigma"] \\ GG \arrow[r, "\sigma G"'] & G \end{tikzcd}$$

commute. The transformation δ is called a **comultiplication**, and σ is called a **counit**.

Comonads form a category. A morphism between comonads $G \to G'$ is a natural transformation $\varphi\colon G \to G'$ rendering commutative the following diagrams

$$\begin{tikzcd} G \arrow[rr, "\varphi"] \arrow[d, "\delta"'] & & G' \arrow[d, "\delta'"] \\ GG \arrow[r, "G\varphi"'] & GG' \arrow[r, "\varphi G'"'] & G'G' \end{tikzcd}, \qquad \begin{tikzcd} G \arrow[rr, "\varphi"] \arrow[dr, "\sigma"'] & & G' \arrow[dl, "\sigma'"] \\ & \mathrm{id}_{\mathbf{A}} & \end{tikzcd}.$$

Definition 1.2. A **coalgebra over a comonad** $G = (G, \delta, \sigma)$ is a pair (A, ρ^A) consisting of an object A of $\mathbf{A}$ and a morphism $\rho^A\colon A \to G(A)$, such that the following diagrams commute

$$\begin{tikzcd} A \arrow[r, "\rho^A"] \arrow[d, "\rho^A"'] & G(A) \arrow[d, "\delta_A"] \\ G(A) \arrow[r, "G(\rho^A)"'] & GG(A)\ , \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\rho^A"] \arrow[dr, equal] & G(A) \arrow[d, "\sigma_A"] \\ & A\ . \end{tikzcd}$$

A morphism of coalgebras $(A, \rho^A) \to (B, \rho^B)$ is a morphism $f\colon A \to B$ in $\mathbf{A}$ compatible with the structure maps ρ^A, ρ^B in the sense of the commutativity of the following diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\rho^A"'] & B \arrow[d, "\rho^B"] \\ G(A) \arrow[r, "G(f)"'] & G(B)\ . \end{tikzcd}$$

The category of coalgebras of G is often referred to as the **Eilenberg-Moore category** and is denoted by $\mathbf{A}_G$.

Dually to comonads one considers *monads* F on a category $\mathbf{A}$ and their Eilenberg-Moore category of algebras $\mathbf{A}^F$.

The introduction of the Eilenberg-Moore category allows one to realise a close relationship between adjoint functors and comonads. Any adjoint pair of functors $L\colon \mathbf{A} \to \mathbf{B}$, $R\colon \mathbf{B} \to \mathbf{A}$ (L is the left adjoint of R) gives rise to a comonad (G, δ, σ) on $\mathbf{B}$, where $G = LR$, $\delta = L\eta R$ (that is $\delta_B = L(\eta_{R(B)})$), $\sigma = \psi$ and η is the unit of adjunction (L, R), and ψ the counit of adjunction.

Given a comonad (G, δ, σ) on $\mathbf{A}$, there is an adjunction

$L\colon \mathbf{A}_G \to \mathbf{A}\ ,$ the forgetful functor,

$R\colon \mathbf{A} \to \mathbf{A}_G\ ,$ the free coalgebra functor defined by $R(A) = (G(A), \delta_A)$

Similarly, if (L, R) is an adjoint pair of functors, then $F = RL$ is a monad on $\mathbf{M}_A$. Conversely, given a comonad F on $\mathbf{A}$, the free algebra functor $\mathbf{A} \to \mathbf{A}^F$ is the left adjoint of the forgetful functor $\mathbf{A}^F \to \mathbf{A}$.

1.2 Comonadic triangles and the descent theory

The correspondence between pairs of adjoint functors and comonads leads to the following fundamental question: What is the relationship between a category on which a pair of adjoint functors is defined and a category of coalgebras of a given comonad. The situation is summarised in

Definition 1.3. Take categories $\mathbf{A}$, $\mathbf{B}$, a comonad G on $\mathbf{A}$ and adjoint functors $L\colon \mathbf{B} \to \mathbf{A}$, $R\colon \mathbf{A} \to \mathbf{B}$. A triangle of categories and functors

$$\begin{tikzcd} \mathbf{B} \arrow[rr, "K"] \arrow[dr, shift left, "L"] & & \mathbf{A}_G \arrow[dl, "U_G"] \\ & \mathbf{A} \arrow[ul, shift left, "R"] & \end{tikzcd}$$

where U_G is the forgetful functor is called a G**-comonadic triangle** provided $U_G K = L$. The functor K is referred to as a **comparison functor**.

We would like to study, when the comparison functor K is an equivalence. First, we need to find an equivalent description of comparison functors.

Proposition 1.4. *Fix categories* $\mathbf{A}$, $\mathbf{B}$, *a comonad* G *on* $\mathbf{A}$, *and adjoint functors* $L\colon \mathbf{B} \to \mathbf{A}$, $R\colon \mathbf{A} \to \mathbf{B}$. *There is a one-to-one correspondence between comparison functors* K *in comonadic triangles made of* G, L *and* R, *and comonad morphisms* $\varphi\colon LR \to G$.

Proof. Given φ define a natural transformation

$$\beta\colon L \xrightarrow{L\eta} LRL \xrightarrow{\varphi L} GL, \quad \beta_B\colon L(B) \to G(L(B)), \quad \beta_B = \varphi_{L(B)} \circ L(\eta_B),$$

where η is the unit of adjunction (L, R). Then the functor $K\colon \mathbf{B} \to \mathbf{A}_G$ is given by $B \mapsto (L(B), \beta_B)$. Conversely, given $K\colon B \to (K(B), \rho^{K(B)})$ define

$$\beta\colon L \to GL, \text{ by } \beta_B = \rho^{K(B)}.$$

Then

$$\varphi\colon LR \xrightarrow{\beta R} GLR \xrightarrow{G\psi} G,$$

where ψ is the counit of adjunction (L, R), is the required morphism of comonads. $\square$

Proposition 1.5. *In the set-up of Proposition 1.4, If* $\mathbf{B}$ *has equalisers, then* K *has a right adjoint* $D\colon \mathbf{A}_G \to \mathbf{B}$ *defined by the equaliser*

$$\begin{tikzcd} D(A,\rho^A) \arrow[r, "\mathrm{eq}_A"] & R(A) \arrow[r, shift left, "\alpha_A"] \arrow[r, shift right, "R(\rho^A)"'] & RG(A), \end{tikzcd}$$

where

$$\alpha\colon R \xrightarrow{\eta R} RLR \xrightarrow{R\varphi} RG.$$

Proof. The unit of the adjoint pair (K, D) is given by $\widehat{\eta}_B$ in the diagram:

$$\begin{tikzcd} DK(B) \arrow[r] & RL(B) \arrow[r, shift left, "\alpha_{L(B)}"] \arrow[r, shift right, "R_{\beta_B}"'] & RGL(B) \\ & B\ . \arrow[u, "\eta_B"'] \arrow[ul, dashed, "\widehat{\eta}_B"] & \end{tikzcd}$$

The existence of such $\widehat{\eta}_B$ follows by the universal property of equalisers. The counit of the adjoint pair (K, D) is given by $\widehat{\psi}$ in the diagram

$$\begin{tikzcd} KD(A,\rho^A) \arrow[r, "L\alpha_A"] \arrow[dr, dashed, "\widehat{\psi}_{(A,\rho^A)}"'] & LR(A) \arrow[r, shift left, "L\alpha_A"] \arrow[r, shift right, "LR(\rho^A)"'] \arrow[d, "\psi_A"] & LRG(A) \\ & A\ . & \end{tikzcd}$$

Note that $\widehat{\psi}_{(A,\rho^A)}$ is a composite, the universal property of an equaliser is not used here. $\square$

Recall that a contractible equaliser of two morphisms $g, h\colon B \to G$ is a morphism $f\colon A \to B$ fitting into the following diagram

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] \arrow[l, bend left, "i"] & G \arrow[l, bend left, "j"] \end{tikzcd},$$

with two maps i, j such that

$$i \circ f = A, \quad j \circ g = B, \quad j \circ h = f \circ i, \quad g \circ f = h \circ f.$$

For objects A in $\mathbf{A}$ consider a contractible equaliser

$$\begin{tikzcd} R(A) \arrow[r, "\alpha_A"] & RG(A) \arrow[r, shift left, "\alpha_{G(A)}"] \arrow[r, shift right, "R(\delta_A)"'] \arrow[l, bend left, "R(\sigma_A)"] & RG^2(A). \arrow[l, bend left, "RG(\sigma_A)"] \end{tikzcd}$$

In view of the universal property of equalisers this implies that

$$\alpha_A = \mathrm{eq}_{G(A)}, \quad R(A) = D(G(A), \delta_A),$$

hence

$$\begin{aligned} \widehat{\psi}_{(G(A),\delta_A)} &= \psi_{G(A)} \circ L(\mathrm{eq}_{G(A)}) \\ &= \psi_{G(A)} \circ L(\alpha_A) \\ &= \psi_{G(A)} \circ LR(\varphi_A) \circ L(\eta_{R(A)}) \\ &= \varphi_A, \end{aligned}$$

where the last equality follows by one of the triangular equalities for the unit and counit of an adjunction. Since a functor which has a right adjoint is full and faithful if and only if the counit of adjunction is a (natural) isomorphism, this simple calculation of $\widehat{\psi}$ immediately establishes the following

Proposition 1.6. *If D is full and faithful, then φ is an isomorphism of comonads.*

The problem of finding when K is an equivalence is equivalent to studying the comonadicity of L. Thus the Beck monadicity theorem yields

Theorem 1.7. *Consider a comonadic triangle in Definition 1.3. If $\mathbf{B}$ has equalisers, then K is an equivalence if and only if φ is an isomorphism, L preserves equalisers that define D, and L reflects isomorphisms.*

Comonadic triangles encode (and generalise) the typical setup of *descent theory.* Let T be a monad on a category $\mathbf{B}$, and let $L\colon \mathbf{B} \to \mathbf{A}$, $R\colon \mathbf{A} \to \mathbf{B}$ be a pair of adjoint functors. Setting $G = LR$ one obtains the following comonadic triangle

$$\begin{tikzcd} \mathbf{B} \arrow[rr, "K"] \arrow[dr, shift left, "F^T"] & & (\mathbf{B}^T)_G \arrow[dl, "U_G"] \\ & \mathbf{B}^T \arrow[ul, shift left, "U^T"] & \end{tikzcd}.$$

Here K is the standard comparison functor corresponding to $\varphi = \mathrm{id}$. $(\mathbf{B}^T)_G$ is known as the category of **descent data**. We say that this triangle is of **descent type** whenever K is full and faithful, and we say that it defines an **effective descent** when K is an equivalence. The standard descent theory studies effective descent in specific situations (such as, e.g. arise in algebraic geometry).

1.3 Comonads on a category of modules. Corings.

Let A, B be associative and unital algebras over a commutative ring k, with multiplication operations denoted by μ_A, μ_B and units 1_A, 1_B (understood both as elements or linear maps $k \to A$, $k \to B$), respectively. Denote by $\mathbf{M}_A$, $\mathbf{M}_B$ the categories of right modules over A and B. Categories of modules are additive and have colimits (they are abelian categories), and we would like to study functors which preserve these structures. Such functors are fully characterised by the Eilenberg-Watts theorem.

Theorem 1.8 (Eilenberg-Watts). *Let $F\colon \mathbf{M}_A \to \mathbf{M}_B$ be an additive functor that preserves colimits. Then $F(A)$ is an (A, B)-bimodule and*

$$F \simeq - \otimes_A F(A), \qquad F(M) \simeq M \otimes_A F(A)\,.$$

We would like to study comonads (G, δ, σ) on $\mathbf{M}_A$ such that G preserves colimits. By Theorem 1.8, $G \simeq - \otimes_A G(A)$. Let $\mathcal{C} := G(A)$, so $\mathcal{C}$ is an A-bimodule. Next we explore consequences of the fact that δ, σ are natural transformations. For any $M \in \mathbf{M}_A$, $m \in M$, consider a morphism in $\mathbf{M}_A$

$$l_m\colon A \to M, \quad a \mapsto ma\,.$$

The naturality of δ implies

$$\begin{tikzcd}
\mathcal{C} \simeq A \otimes_A \mathcal{C} \arrow[r, "l_m \otimes_A \mathcal{C}"] \arrow[d, "\delta_A"] & M \otimes_A \mathcal{C} \arrow[d, "\delta_M"] \\
\mathcal{C} \otimes_A \mathcal{C} \simeq A \otimes_A \mathcal{C} \otimes_A \mathcal{C} \arrow[r, "l_m \otimes_A \mathcal{C} \otimes_A \mathcal{C}"] & M \otimes_A \mathcal{C} \otimes_A \mathcal{C}\,.
\end{tikzcd}$$

The evaluation of this diagram at $1_A \otimes_A c$, $c \in \mathcal{C}$ gives

$$\delta_M(m \otimes_A c) = m \otimes_A \delta_A(c).$$

Hence $\delta_M = M \otimes_A \delta_A$. This means, in particular, that δ_A is a left A-linear, hence an A-bilinear map (it is right A-linear as a morphism in $\mathbf{M}_A$). Similarly, $\sigma_M = M \otimes_A \sigma_A$. Let

$$\Delta_{\mathcal{C}} := \delta_A\colon \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}\,, \qquad \varepsilon_{\mathcal{C}} := \sigma_A\colon \mathcal{C} \to A\ .$$

Then diagrams for coassociativity of δ and counitarity of σ are equivalent to the following commutative diagrams

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\Delta_{\mathcal{C}}"] \arrow[d, "\Delta_{\mathcal{C}}"'] & \mathcal{C} \otimes_A \mathcal{C} \arrow[d, "\Delta_{\mathcal{C}} \otimes_A \mathcal{C}"] \\
\mathcal{C} \otimes_A \mathcal{C} \arrow[r, "\mathcal{C} \otimes_A \Delta_{\mathcal{C}}"'] & \mathcal{C} \otimes_A \mathcal{C} \otimes_A \mathcal{C}\ ,
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \arrow[r, "\Delta_{\mathcal{C}}"] \arrow[d, "\Delta_{\mathcal{C}}"'] \arrow[dr, "\simeq"] & \mathcal{C} \otimes_A \mathcal{C} \arrow[d, "\mathcal{C} \otimes_A \varepsilon_{\mathcal{C}}"] \\
\mathcal{C} \otimes_A \arrow[r, "\varepsilon_{\mathcal{C}} \otimes_A \mathcal{C}"'] & A \otimes_A \mathcal{C} \simeq \mathcal{C} \otimes_A A\ .
\end{tikzcd}
\tag{1.1}$$

Definition 1.9. An A-bimodule $\mathcal{C}$ together with A-bilinear maps $\Delta_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}$, $\varepsilon_{\mathcal{C}} : \mathcal{C} \to A$ satisfying (1.1) is called an A-**coring** (pronounced: *co-ring*). $\Delta_{\mathcal{C}}$ is called the **comultiplication** and $\varepsilon_{\mathcal{C}}$ is called the **counit** of $\mathcal{C}$.

A morphism of A-corings $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}}) \to (\mathcal{D}, \Delta_{\mathcal{D}}, \varepsilon_{\mathcal{D}})$ is an A-bimodule map $f\colon \mathcal{C} \to \mathcal{D}$ such that that makes the following diagrams commute

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, "\Delta_{\mathcal{C}}"'] & \mathcal{D} \arrow[d, "\Delta_{\mathcal{D}}"] \\ \mathcal{C} \otimes_A \mathcal{C} \arrow[r, "f \otimes_A f"'] & \mathcal{D} \otimes_A \mathcal{D}\,, \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[dr, "\varepsilon_{\mathcal{C}}"'] & \mathcal{D} \arrow[d, "\varepsilon_{\mathcal{D}}"] \\ & A\,. \end{tikzcd}$$

Using arguments similar to those establishing the correspondence between corings and (tensor functor) comonads, one easily finds that f arises as (and gives rise to) a morphism of comonads (evaluated at A). We have thus established bijective correspondences:

$$\begin{array}{c} A\text{-corings} \\ \updownarrow \\ \text{Additive comonads on } \mathbf{M}_A \text{ that preserve colimits} \\ \updownarrow \\ \text{Additive comonads on } \mathbf{M}_A \text{ that have a right adjoint} \\ \updownarrow \\ \text{Additive monads on } \mathbf{M}_A \text{ that have a left adjoint}\,. \end{array}$$

This last correspondence follows by the fact that the right adjoint of a comonad is a monad and vice versa. The correspondence between corings and comonads is explicitly given by

$$(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}}) \leftrightarrow (- \otimes_A \mathcal{C}, - \otimes_A \Delta_{\mathcal{C}}, - \otimes_A \varepsilon_{\mathcal{C}}).$$

Definition 1.10. Let $\mathcal{C}$ be an A-coring, M be a right A-module and let $\rho^M\colon M \to M \otimes_A \mathcal{C}$ be a right A-module map. A pair (M, ρ^M) is called a **right $\mathcal{C}$-comodule** if and only if the following diagrams commute

$$\begin{tikzcd} M \arrow[r, "\rho^M"] \arrow[d, "\rho^M"'] & M \otimes_A \mathcal{C} \arrow[d, "\rho^M \otimes_A \mathcal{C}"] \\ M \otimes_A \mathcal{C} \arrow[r, "M \otimes_A \Delta_{\mathcal{C}}"'] & M \otimes_A \mathcal{C} \otimes_A \mathcal{C}\,, \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\rho^M"] \arrow[dr, "\simeq"] & M \otimes_A \mathcal{C} \arrow[d, "M \otimes_A \varepsilon_{\mathcal{C}}"] \\ & M \otimes_A A\,. \end{tikzcd}$$

The map ρ^M is called a **coaction**.

(M, ρ^M) is a $\mathcal{C}$-comodule if and only if (M, ρ^M) is a coalgebra for the corresponding comonad $G = (- \otimes_A \mathcal{C}, - \otimes_A \Delta_{\mathcal{C}}, - \otimes_A \varepsilon_{\mathcal{C}})$. A morphism in $(\mathbf{M}_A)_G$ is a right A-module map $f\colon M \to N$ rendering the following diagram commutative

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\rho^M"'] & N \arrow[d, "\rho^N"] \\ M \otimes_A \mathcal{C} \arrow[r, "f \otimes_A \mathcal{C}"'] & N \otimes_A \mathcal{C}\,. \end{tikzcd}$$

The category of right $\mathcal{C}$-comodules (i.e. the category of coalgebras of $(-\otimes_A \mathcal{C}, -\otimes_A \Delta_{\mathcal{C}}, -\otimes_A \varepsilon_{\mathcal{C}})$) is denoted by $\mathbf{M}^{\mathcal{C}}$. Morphisms between comodules (M, ρ^M) and (N, ρ^N) are denoted by $\mathrm{Hom}^{\mathcal{C}}(M, N)$.

Left comodules are defined symmetrically as coalgebras of the comonad $(\mathcal{C} \otimes_A -, \Delta_{\mathcal{C}} \otimes_A -, \varepsilon_{\mathcal{C}} \otimes_A -)$ on the category of left A-modules.

As an example of a coring we can study corings associated to a module.

Example 1.11. Take algebras A, B and look at functors $\mathbf{M}_B \to \mathbf{M}_A$ preserving colimits. By the Eilenberg-Watts theorem (Theorem 1.8) such functors have the tensor form, i.e. there is a (B, A)-bimodule M such that the functor is of the form

$$- \otimes_B M \colon \mathbf{M}_B \to \mathbf{M}_A.$$

For $N \in \mathbf{M}_A$, morphisms $\mathrm{Hom}_A(M, N)$ form a right B-module by

$$(f \cdot b)(m) = f(bm), \qquad f \in \mathrm{Hom}_A(M, N),\ m \in M,\ b \in B.$$

Since the functor $\mathrm{Hom}_A(M, -) \colon \mathbf{M}_A \to \mathbf{M}_B$ is the right adjoint to $- \otimes_B M$, there is a comonad

$$G = \mathrm{Hom}_A(M, -) \otimes_B M \colon \mathbf{M}_A \to \mathbf{M}_A\ ,$$

with comultiplication

$$\begin{gathered}\delta_N \colon\ \mathrm{Hom}_A(M, N) \otimes_B M \to \mathrm{Hom}_A(M, \mathrm{Hom}_A(M, N) \otimes_B M) \otimes_B M,\\ f \otimes_B m \mapsto [m' \mapsto f \otimes_B m'] \otimes_B m\ ,\end{gathered}$$

and counit

$$\sigma_N \colon\ \mathrm{Hom}_A(M, N) \otimes_B M \to N\,, \qquad f \otimes_B m \mapsto f(m)\,.$$

G preserves colimits if M is finitely generated and projective as a right A-module, i.e. there exists $e = \sum_i e_i \otimes_A \xi_i$, $e_i \in M$, $\xi_i \in M^* := \mathrm{Hom}_A(M, A)$, $i = 1, 2, \ldots, n$ such that, for all $m \in M$,

$$m = \sum_i e_i \xi_i(m).$$

In this case

$$\mathrm{Hom}_A(M, N) \simeq N \otimes_A M^*, \qquad G = - \otimes_A M^* \otimes_B M.$$

Hence $\mathcal{C} = M^* \otimes_B M$ is an A-coring with comultiplication and counit

$$\Delta_{\mathcal{C}}(\xi \otimes_B m) = \xi \otimes_B e \otimes_B m, \qquad \varepsilon_{\mathcal{C}}(\xi \otimes_B m) = \xi(m).$$

$G(A) = M^* \otimes_B M$ is called a **(finite) comatrix coring**.

1.4 Galois comodules for corings

We start with an A-coring $\mathcal{C}$ and take a category of right A-modules $\mathbf{A} := \mathbf{M}_A$. Denote by $G = - \otimes_A \mathcal{C}$ the corresponding comonad on $\mathbf{A}$. The category $\mathbf{A}_G$ of G-coalgebras is thus the same as the category of $\mathcal{C}$-comodules $\mathbf{M}^{\mathcal{C}}$. Take a comodule $(M, \rho^M) \in \mathbf{M}^{\mathcal{C}}$ and set $B = \mathrm{End}^{\mathcal{C}}(M) = \mathrm{Hom}^{\mathcal{C}}(M, M)$. This is an algebra with respect to composition of morphisms and M is a left B-module by evaluation ($b \cdot m = b(m)$ for $b \in B$, $m \in M$). Furthermore, the definition of comodule morphisms imply that ρ^M is a left B-linear map. Set $\mathbf{B} := \mathbf{M}_B$. Since ρ^M is a left B-linear map, there is a functor

$$K \colon \mathbf{M}_B \to \mathbf{M}^{\mathcal{C}}, \qquad V \mapsto (V \otimes_B M, V \otimes_B \rho^M). \tag{1.2}$$

Note that for the forgetful functor $U^{\mathcal{C}}\colon \mathbf{M}^{\mathcal{C}} \to \mathbf{M}_A$, $U^{\mathcal{C}}K(V) = V \otimes_B M$. Thus there is a comonadic triangle

$$\begin{tikzcd}
\mathbf{M}_B \arrow[rr, "K"] \arrow[dr, shift left, "-\otimes_B M"] & & \mathbf{M}^{\mathcal{C}} \arrow[dl, "U^{\mathcal{C}}"] \\
& \mathbf{M}_A \arrow[ul, shift left, "\mathrm{Hom}_A(M,-)"] &
\end{tikzcd}$$

By Proposition 1.4 there is a comonad morphism

$$\varphi\colon \mathrm{Hom}_A(M,-) \otimes_B M \to - \otimes_A \mathcal{C}.$$

Recall that, for all $N \in \mathbf{M}_A$, the counit ψ of the tensor-hom adjunction $(-\otimes_B M, \mathrm{Hom}_A(M,-))$ in provided by the evaluation map

$$\psi_N\colon \mathrm{Hom}_A(M,N) \otimes_B M \to N, \qquad f \otimes_B m \mapsto f(m).$$

Therefore,

$$\begin{aligned}
\varphi_N &= G(\psi_N) \circ \rho^{KR(N)}, \\
f \otimes_B m \mapsto\ & (\psi_N \otimes_A \mathcal{C})(\rho^{K(\mathrm{Hom}_A(M,N))}(f \otimes_B m)) \\
&= (\psi_N \otimes_A \mathcal{C})(f \otimes_B \rho^M(m)) = (f \otimes_A \mathcal{C})(\rho^M(m))\,.
\end{aligned}$$

Writing

$$\rho^M(m) = \sum m_{(0)} \otimes_A m_{(1)}\ ,$$

(summation index suppressed) we obtain

$$\varphi_N(f \otimes_B m) = \sum f(m_{(0)}) \otimes_A m_{(1)}\ . \tag{1.3}$$

Definition 1.12. A right $\mathcal{C}$-comodule (M, ρ^M) is called a **Galois comodule** if and only if the natural transformation φ determined by all the maps φ_N (1.3) is a natural isomorphism.

If M is finitely generated and projective as a right A-module, then the comonad $\mathrm{Hom}_A(M,-)\otimes_B M$ comes from the comatrix coring $- \otimes_A M^* \otimes_B M$. The fact that φ is a comonad morphism is equivalent to the fact that φ_A is a coring morphism. Write

$$\mathrm{can}_M := \varphi_A\colon M^* \otimes_B M \to \mathcal{C}\,, \qquad \xi \otimes_B m \mapsto \sum \xi(m_{(0)})m_{(1)}\,.$$

The map can_M is called the **canonical map**.

Definition 1.13. A Galois comodule (M, ρ^M) such that M is finitely generated projective as a right A-module is called a **finite Galois comodule**.

The Galois property of a finite Galois comodule is entirely encoded in the properties of the canonical map. More precisely,

Lemma 1.14. *A right $\mathcal{C}$-comodule (M, ρ^M) with M finitely generated projective as a right A-module is a Galois comodule if and only if the canonical map can_M is an isomorphism of A-corings.*

Since the category of modules has equalisers, the comparison functor K in (1.2) has a right adjoint D; see Proposition 1.5. Recall that D is defined by the diagram

$$\begin{tikzcd} D(N,\rho^N) \arrow[r, "\mathrm{eq}_N"] & R(N) \arrow[r, shift left, "\alpha_N"] \arrow[r, shift right, "R(\rho^N)"'] \arrow[d, equal] & RG(N) \arrow[d, equal] \\ & \mathrm{Hom}_A(M,N) & \mathrm{Hom}_A(M, N\otimes_A \mathcal{C}) \,. \end{tikzcd}$$

The equalised maps can be explicitly computed as

$$R(\rho^N)(f) = \mathrm{Hom}_A(M,\rho^N)(f) = \rho^N \circ f,$$

and

$$\alpha_N(f) = R(\varphi_N)(\eta_{R(N)}(f)) = \mathrm{Hom}_A(M,\varphi_N)(f\otimes_B -) = (f\otimes_A \mathcal{C})\circ \rho^N.$$

Therefore, D can be identified with the comodule homomorphism functor, i.e. for all right $\mathcal{C}$-comodules (N,ρ^N),

$$D(N,\rho^N) = \mathrm{Hom}^{\mathcal{C}}(M,N).$$

In order to state the conditions under which the comparison functor K, and thus also the constructed functor D, is an equivalence we need to recall the notions of flatness and faithful flatness. Consider a sequence of right B-module maps

$$V \longrightarrow V' \longrightarrow V''. \tag{1.4}$$

For any left B-module M there is then also the following sequence

$$V\otimes_B M \longrightarrow V'\otimes_B M \longrightarrow V''\otimes_B M. \tag{1.5}$$

The module M is said to be **flat** if the exactness of any sequence (1.4) implies exactness of the corresponding sequence (1.5). The module M is said to be **faithfully flat** if its flat and, for any sequence of modules (1.4), the exactness of (1.5) implies exactness of (1.4).

Combining the discussion of comodules in this and preceding sections with Beck's monadicity theorem (see Proposition 1.6 and Theorem 1.7) one derives the main characterisation of Galois comodules in terms of equivalences of categories.

Theorem 1.15 (The finite Galois comodule structure theorem)**.** *Let (M,ρ^M) be a comodule over a coring $\mathcal{C}$ such that M is finitely generated projective as a right module over A. Then the following conditions are equivalent:*

1. *the functor $\mathrm{Hom}^{\mathcal{C}}(M,-) : \mathbf{M}^{\mathcal{C}} \to \mathbf{M}_B$ is fully faithful and $\mathcal{C}$ is flat as a right A-module;*

2. *M is flat as a left B-module and (M,ρ^M) is Galois comodule.*

Furthermore the following conditions are equivalent:

1. *$\mathrm{Hom}^{\mathcal{C}}(M,-)$ is an equivalence of categories and $\mathcal{C}$ is flat as a right A-module;*

2. *M is faithfully flat as a left B-module and (M,ρ^M) is Galois comodule.*

1.5 The Maszczyk Galois condition

The notion of a Galois comodule presented in Section 1.4 is considered to be standard in the theory of corings; see [**?**]. Recently, motivated by an approach to non-commutative algebraic geometry through monoidal categories, Maszczyk introduced a different Galois condition in [**?**]. We describe this condition here and compare it with the one studied in Section 1.4.

Start with a morphism of A-corings $\gamma\colon \mathcal{D} \to \mathcal{C}$. Then $\mathcal{D}$ is a $\mathcal{C}$-bicomodule (i.e. it has both left and right $\mathcal{C}$-coaction such that the left coaction is a morphism of right $\mathcal{C}$-comodule) via

$$\begin{aligned} &(\mathcal{D} \otimes_A \gamma) \circ \Delta_{\mathcal{D}} \quad \text{(right } \mathcal{C}\text{-coaction)} \ , \\ &(\gamma \otimes_A \mathcal{D}) \circ \Delta_{\mathcal{D}} \quad \text{(left } \mathcal{C}\text{-coaction)} \ . \end{aligned}$$

Consider the k-module of $\mathcal{C}$-bicomodule maps $B := {}^{\mathcal{C}}\mathrm{Hom}^{\mathcal{C}}(\mathcal{D}, \mathcal{C})$. Then B is an algebra with the product of $b, b' \in B$ given by

$$bb' = (\mathcal{C} \otimes_A \varepsilon_{\mathcal{C}}) \circ (b \otimes_A b') \circ \Delta_{\mathcal{D}},$$

i.e. explicitly

$$bb' : d \mapsto d_{(1)} \otimes_A d_{(2)} \mapsto b(d_{(1)}) \otimes_A b'(d_{(2)}) \mapsto b(d_{(1)})\varepsilon_{\mathcal{C}}(b'(d_{(2)})) \ .$$

Furthermore, $\mathcal{D}$ is a B-bimodule. Define

$$\overline{\mathcal{D}} := \mathcal{D}/[\mathcal{D}, B], \qquad p\colon \mathcal{D} \to \overline{\mathcal{D}} \ .$$

Then $\overline{\mathcal{D}}$ is an A-coring with the structure induced by p from that of A-coring $\mathcal{D}$, and there is a commutative triangle of coring maps:

$$\begin{tikzcd} \mathcal{D} \arrow[rr, "\gamma"] \arrow[dr, "p"'] & & \mathcal{C} \\ & \overline{\mathcal{D}} \ . \arrow[ur, "\overline{\gamma}"'] & \end{tikzcd}$$

$\mathcal{C}$ is said to be a **Galois coring in the sense of Maszczyk** if the map $\overline{\gamma}$ is an isomorphism. The above triangle of coring maps induces two functors

$$F\colon \mathbf{M}^{\mathcal{D}} \to \mathbf{M}^{\overline{\mathcal{D}}}, \qquad (M, \rho^M) \mapsto (M, (M \otimes_A p) \circ \rho^M),$$

and

$$G\colon \mathbf{M}^{\mathcal{D}} \to \mathbf{M}^{\overline{\mathcal{C}}}, \qquad (M, \rho^M) \mapsto (M, (M \otimes_A \gamma) \circ \rho^M).$$

Any right $\mathcal{D}$-comodule (M, ρ^M) defines two comonadic triangles

$$\begin{tikzcd} \mathbf{M}_B \arrow[rr, "\overline{K}=-\otimes_B F(M)"] \arrow[dr, shift left, "-\otimes_B M"] & & \mathbf{M}^{\overline{\mathcal{D}}} \arrow[dl, "U^{\overline{\mathcal{D}}}"] \\ & \mathbf{M}_A, \arrow[ul, shift left, "\mathrm{Hom}_A(M,-)"] & \end{tikzcd}$$

with the corresponding (to $\overline{K}$) morphism of comonads

$$\overline{\varphi}(f \otimes_B m) = (f \otimes_A p)(\rho^M(m)), \qquad f \in \mathrm{Hom}_A(M, N) \ , \ m \in M \ ,$$

and

$$\begin{tikzcd}
\mathbf{M}_B \arrow[rr, "K=-\otimes_B G(M)"] \arrow[dr, shift left, "-\otimes_B M"] & & \mathbf{M}^{\mathcal{C}} \arrow[dl, "U^{\mathcal{D}}"] \\
& \mathbf{M}_A, \arrow[ul, shift left, "\mathrm{Hom}_A(M,-)"] &
\end{tikzcd}$$

with the corresponding (to K) morphism of comonads

$$\varphi(f \otimes_B m) = (f \otimes_A \gamma)(\rho^M(m)), \qquad f \in \mathrm{Hom}_A(M, N)\,,\ m \in M\,.$$

Proposition 1.16 (G. Böhm)**.** *Assume that $\overline{\varphi}$ is an isomorphism (i.e. that $F(M)$ is a Galois $\overline{\mathcal{D}}$-comodule). Then φ is an isomorphism (i.e. $G(M)$ is a Galois $\mathcal{C}$-comodule) if and only if $\overline{\gamma}$ is an isomorphism of corings (i.e. $\mathcal{C}$ is Galois in the sense of Maszczyk).*

References. Barr and Wells [?]; Beck [?]; Brzeziński and Wisbauer [?]; Dubuc [?]; Eilenberg and Moore [?]; Gómez-Torrecillas [?]; Grothendieck [?]; MacLane [?]; Maszczyk [?]; Mesablishvili [?], Sweedler [?]; Watts [?].

Chapter 2

Hopf-Galois extensions of non-commutative algebras

In this lecture we introduce the key notions in the Galois theory of Hopf algebras or in the algebraic approach to non-commutative principal bundles. We also show how Hopf-Galois extensions fit into the theory of Galois comodules of corings described in Chapter 1.

From now on, k denotes a field, and all algebras etc. are over k. The tensor product over k is denoted by $\otimes$.

2.1 Coalgebras and Sweedler's notation

Definition 2.1. A **coalgebra** is a vector space C with k-linear maps $\Delta_C\colon C \to C \otimes C$, $\varepsilon_C\colon C \to k$ such that the following diagrams commute

$$\begin{tikzcd} C \arrow[r, "\Delta_C"] \arrow[d, "\Delta_C"'] & C \otimes C \arrow[d, "C\otimes\Delta_C"] \\ C \otimes C \arrow[r, "\Delta_C\otimes C"'] & C \otimes C \otimes C, \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\Delta_C"] \arrow[d, "\Delta"] \arrow[dr, equal] & C \otimes C \arrow[d, "\varepsilon_C\otimes C"] \\ C \otimes C \arrow[r, "C\otimes\varepsilon_C"'] & C \end{tikzcd}$$

Δ_C is called a **comultiplication** and ε_C is called a **counit**.

In other words, a k-coalgebra is the same as a k-coring (when a vector space is viewed as a symmetric k-bimodule). Following this identification of k-coalgebras as k-corings one defines C-comodules as comodule of the k-coring C. (The reader should notice that we use the term *coalgebra* here in the sense different from that in Chapter 1.)

The idea of comultiplication is somewhat counter-intuitive: out of a single element of a vector spaces, a family of elements is produced. Heyneman and Sweedler developed a short-hand notation which proves very useful in explicit computations that involve comultiplications and counits. The Sweedler notation for comultiplication is based on omitting unnecessary summation range, index and sign, and then employing the coassociativity of comultiplication (the first of diagrams in Definition 2.1) to relabel indices by consecutive numbers. Given an element $c \in C$, we write

$$\begin{aligned}\Delta_C(c) &= \sum_{i=1}^{n} c^i_{(1)} \otimes c^i_{(2)} \\ &= \sum_i c^i_{(1)} \otimes c^i_{(2)} \\ &= \sum c_{(1)} \otimes c_{(2)} \\ &= c_{(1)} \otimes c_{(2)}.\end{aligned}$$

The coassociativity of comultiplication means that the two ways to compute the result of two applications of Δ give the same result:

$$\begin{aligned}(C \otimes \Delta_C) \circ \Delta_C(c) &= (C \otimes \Delta_C)(c_{(1)} \otimes c_{(2)}) \\ &= c_{(1)} \otimes \Delta_C(c_{(2)}) = c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)} \\ (\Delta_C \otimes C) \circ \Delta_C(c) &= (\Delta_C \otimes C)(c_{(1)} \otimes c_{(2)}) \\ &= \Delta_C(c_{(1)}) \otimes c_{(2)} = c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)}\,.\end{aligned}$$

We can order all indices appearing in above expressions (and in all expressions involving multiple application of Δ_C) in the following way. Remove the brackets, put 0. in front of the index and then arrange them in increasing order. In this way we obtain

$$0.1 < 0.21 < 0.22, \qquad 0.11 < 0.12 < 0.2\,.$$

The coassociativity of Δ_C tells us that we do not need to care about exact labels but only about their increasing order. Hence we can relabel:

$$c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)} = c_{(1)} \otimes c_{(2)} \otimes c_{(3)} = c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)}.$$

Exercise 2.2. *Compute and check labelling for all three applications of Δ to an element $c \in C$.*

In terms of the Sweedler notation, the counitality of the comultiplication or the second of the diagrams in Definition 2.1 comes out as

$$c_{(1)}\varepsilon_C(c_{(2)}) = \varepsilon_C(c_{(1)})c_{(2)} = c.$$

Example 2.3. Let X be a set, $C = kX$ – the linear span of X (elements of X form a basis of the vector space kX). Define the comultiplication and counit by

$$\Delta_C(x) = x \otimes x, \qquad \varepsilon_C(x) = 1, \qquad \text{for all } x \in X.$$

Remark: for any coalgebra C an element $c \in C$ such that $\Delta_C(c) = c \otimes c$, $\varepsilon_C(c) = 1$ is called a **group-like** element.

Example 2.4. Consider the trigonometric identities

$$\begin{aligned}\sin(x + y) &= \sin(x)\cos(y) + \cos(x)\sin(y), \\ \cos(x + y) &= \cos(x)\cos(y) - \sin(x)\sin(y),\end{aligned}$$

and the values of sine and cosine at the origin $\sin(0) = 0$, $\cos(0) = 1$. We can abstract from these expressions the variables x and y and use the trignometric identities to define the comultiplication, and values at 0 to define the counit. Thus we consider a two-dimensional coalgebra C with a basis $\{\sin, \cos\}$, and comultiplication and counit

$$\Delta_C(\sin) = \sin\otimes\cos + \cos\otimes\sin, \qquad \Delta_C(\cos) = \cos\otimes\cos - \sin\otimes\sin,$$
$$\varepsilon_C(\sin) = 0, \qquad \varepsilon_C(\cos) = 1.$$

This coalgebra is often referred to as the **trigonometric coalgebra**.

Example 2.5. Let G be a monoid with unit e, $\mathcal{O}(G)$ algebra of functions $G \to k$. If G is finite we take all functions, and if G is an algebraic group then we take polynomial (or representative) functions. $\mathcal{O}(G)$ is a coalgebra with comultiplication and counit

$$\Delta_{\mathcal{O}(G)}(f)(g\otimes g') = f(gg'), \qquad \varepsilon_{\mathcal{O}(G)}(f) = f(e).$$

2.2 Bialgebras and comodule algebras

In addition to comultiplication and counit, coalgebras in Examples 2.4 and 2.5 can be equipped with the structure of an algebra in a way that is compatible with the coalgebra structure.

Definition 2.6. A **bialgebra** is a vector space H such that:
(a) H is an algebra with multiplication μ_H and unit 1_H;
(b) H is a coalgebra with comultiplication Δ_H and counit ε_H;
(c) Δ_H and ε_H are algebra maps, i.e. the following diagrams commute

$$\begin{tikzcd}
H\otimes H \arrow[rr, "\mu_H"] \arrow[d, "\Delta_H\otimes\Delta_H"'] & & H \arrow[d, "\Delta_H"] \\
H\otimes H\otimes H\otimes H \arrow[r, "H\otimes\mathrm{flip}\otimes H"] & H\otimes H\otimes H\otimes H \arrow[r, "\mu_H\otimes\mu_H"] & H\otimes H,
\end{tikzcd}$$

$$\begin{tikzcd}
H\otimes H \arrow[rr, "\mu_H"] \arrow[dr, "\varepsilon_H\otimes\varepsilon_H"'] & & H \arrow[dl, "\varepsilon_H"] \\
& k, &
\end{tikzcd}$$

and $\Delta_H(1_H) = 1_H\otimes 1_H$ and $\varepsilon_H(1_H) = 1$.

Explicitly, in terms of the Sweedler notation the first of diagrams in Definition 2.6 reads, for all $h, h' \in H$,

$$\Delta_H(hh') = h_{(1)}h'_{(1)}\otimes h_{(2)}h'_{(2)}\ .$$

Example 2.7. Let G be a monoid with unit e, and let $H = kG$ – the linear span of G. The multiplication is the monoid multiplication extended linearly, i.e. $\mu_H\colon g\otimes g' \mapsto gg'$, for all $g, g' \in G$, unit $1_H = e$, the comultiplication is given by $\Delta_H(g) = g\otimes g$, and the counit by $\varepsilon_H(g) = 1$ (see Example 2.3). With these structures kG is a bialgebra.

Example 2.8. Let G be a monoid with unit e, and let $H = \mathcal{O}(G)$ – the functions $G \to k$; see Example 2.5. H is an algebra by the pointwise multiplication $\mu_H(f \otimes f')(g) = f(g)f'(g)$, and with the unit $1_H(g) = 1$. The comultiplication is given by $\Delta_H(f)(g \otimes g') = f(gg')$, and the counit by $\varepsilon_H(f) = f(e)$ as in Example 2.5. With these operations H is a bialgebra. For example:

(i) Functions on the two element group $G = \mathbb{Z}_2$. As a vector space

$$\mathcal{O}(\mathbb{Z}_2) = k^2 \text{ with basis } e_1, e_2\,,$$

$$e_1(1) = 1, \quad e_1(-1) = 0, \qquad e_2(1) = 0, \quad e_2(-1) = 1.$$

The comultiplication derived from the rule described above comes out as

$$\Delta_H(e_1) = e_1 \otimes e_1 + e_2 \otimes e_2, \qquad \Delta_H(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1.$$

The pointwise multiplication is $e_1e_1 = e_1$, $e_1e_2 = e_2e_1 = 0$, $e_2e_2 = e_2$.

(ii) Functions on the circle group $G = S^1 = \mathrm{U}(1)$, $k = \mathbb{C}$. As an algebra $\mathcal{O}(\mathrm{U}(1))$ is isomorphic to the algebra of Laurent polynomials,

$$\mathcal{O}(\mathrm{U}(1)) \simeq \mathbb{C}[X, X^{-1}].$$

The comultiplication is given on generators by $\Delta_H(X) = X \otimes X$, $\Delta_H(X^{-1}) = X^{-1} \otimes X^{-1}$ (and is extended multiplicatively to the whole of $\mathbb{C}[X, X^{-1}]$).

Definition 2.9. Given a bialgebra H, a right H**-comodule algebra** is a quadruple $(A, \mu_A, 1_A, \rho^A)$, where

(a) $(A, \mu_A, 1_A)$ is a k-algebra with multiplication μ_A and unit 1_A;
(b) (A, ρ^A) is a right H-comodule (with coaction $\rho^A\colon A \to A \otimes H$);
(c) the coaction ρ^A is an algebra map, when $A \otimes H$ is viewed as a tensor product algebra

$$(a \otimes h)(a' \otimes h') = aa' \otimes hh', \qquad 1_{A\otimes H} = 1_A \otimes 1_H\ .$$

That is the following diagram commutes

$$\begin{tikzcd}
A \otimes A \arrow[r, "\rho^A \otimes \rho^A"] \arrow[dd, "\mu_A"'] & A \otimes H \otimes A \otimes H \arrow[d, "A \otimes \mathrm{flip} \otimes H"] \\
 & A \otimes A \otimes H \otimes H \arrow[d, "\mu_A \otimes \mu_H"] \\
A \arrow[r, "\rho^A"] & A \otimes H,
\end{tikzcd}$$

and $\rho^A(1_A) = 1_A \otimes 1_H$.

An alternative definition of a bialgebra can be given by considering the structure of the category of comodules of a coalgebra H. A coalgebra H is a bialgebra if and only if the category of right H-comodules, $\mathbf{M}^H$, is a monoidal category and the forgetful functor from $\mathbf{M}^H$ to vector spaces is strongly monoidal (i.e. the monoidal operation in $\mathbf{M}^H$ is the same as the tensor product of vector spaces). If H is a bialgebra and (M, ρ^M) and (N, ρ^N) are H-comodules, then $(M \otimes N, \rho^{M\otimes N})$ is an H-comodule with the coaction

$$M \otimes N \xrightarrow{\rho^M \otimes \rho^N} M \otimes H \otimes N \otimes H \xrightarrow{M \otimes \mathrm{flip} \otimes H} M \otimes N \otimes H \otimes H \xrightarrow{M \otimes N \otimes \mu_H} M \otimes N \otimes H \ .$$

With this interpretation a right H-comodule algebra is simply an algebra in the monoidal category of right H-comodules.

Similarly to comultiplication, in explicit expressions and calculations it is useful to use Sweedler's notation for comodules. Let $(A, \rho^A) \in \mathbf{M}^H$. For all $a \in A$, we write omitting the sum sign and summation indices

$$\rho^A(a) = a_{(0)} \otimes a_{(1)}.$$

Note that all the elements $a_{(0)}$ are in A, while all the $a_{(1)}$ are in H. The comodule property $(A \otimes \Delta_H) \circ \rho^A(a) = (\rho^A \otimes H) \circ \rho^A(a)$ can be written as

$$a_{(0)} \otimes a_{(1)(1)} \otimes a_{(1)(2)} = a_{(0)(0)} \otimes a_{(0)(1)} \otimes a_{(1)} =: a_{(0)} \otimes a_{(1)} \otimes a_{(2)} \ .$$

In general, after relabelling according to the same rules as for comultiplication, symbols with positive Sweedler indices are elements of the Hopf algebra H. The compatibility condition from Definition 2.9 can be written as

$$(aa')_{(0)} \otimes (aa')_{(1)} = a_{(0)}a'_{(0)} \otimes a_{(1)}a'_{(1)} \ .$$

Example 2.10. Since the comultiplication in a bialgebra is an algebra map, the pair (H, Δ_H) is a right comodule algebra. One often refers to (H, Δ_H) as a **(right) regular comodule**.

Example 2.11. Let G be a group, $H = kG$. Then A is an H-comodule algebra if and only if A is a G-graded algebra

$$A = \bigoplus_{g \in G} A_g, \quad A_g A_{g'} \subseteq A_{gg'}, \quad 1_A \in A_e.$$

If $a \in A_g$, then define

$$\rho^A(a) = a \otimes g.$$

Since $1_A \in A_e$, $\rho^A(1_A) = 1_A \otimes e = 1_A \otimes 1_H$.

Take $a \in A_g$, $a' \in A_{g'}$. Then $aa' \in A_{gg'}$, hence $\rho^A(aa') = aa' \otimes gg'$ as needed.

Example 2.12. Let $H = \mathcal{O}(G)$ for a monoid G. For a G-set X, take $A = \mathcal{O}(X)$ and identify $\mathcal{O}(X) \otimes \mathcal{O}(G)$ with $\mathcal{O}(X \times G)$. Then A is an H-comodule algebra with respect to

$$\rho^A(f)(x, g) = f(xg), \qquad \forall x \in X, g \in G.$$

2.3 Hopf-Galois extensions and Hopf algebras

Definition 2.13. If A is a right H-comodule algebra (of a bialgebra H), define the set of **coinvariants** (or **coaction invariants**) as

$$A^{\mathrm{co}H} := \{b \in A \mid \rho^A(b) = b \otimes 1_H\} .$$

Coinvariants $A^{\mathrm{co}H}$ are a subalgebra of A, because ρ^A is an algebra map. Furthermore

$$A^{\mathrm{co}H} = \{b \in A \mid \text{for all } a \in A,\ \rho^A(ba) = b\rho^A(a)\}\,.$$

A is an $A^{\mathrm{co}H}$-bimodule, and ρ^A is a left $A^{\mathrm{co}H}$-module map. The coaction ρ^A is also a right $A^{\mathrm{co}H}$-module map, when $A \otimes H$ has the right multiplication given by $(a \otimes h) \cdot a' = aa' \otimes h$.

Example 2.14. Take a regular comodule algebra (H, Δ_H); see Example 2.10. Then H has trivial coaction invariants, i.e.

$$H^{\mathrm{co}H} = k \cdot 1_H \ .$$

Indeed, since $\Delta_H(1_H) = 1_H \otimes 1_H$, $k1_H \subseteq H^{\mathrm{co}H}$. On the other hand if

$$h_{(1)} \otimes h_{(2)} = h \otimes 1_H \ ,$$

then apply $\varepsilon_H \otimes H$ to get $h = \varepsilon_H(h)1_H$, hence $h \in k1_H$.

Definition 2.15. A right H-comodule algebra is called a **Hopf-Galois extension** (of the coinvariants $B := A^{\mathrm{co}H}$) if the **canonical map**

$$\mathrm{can}\colon A \otimes_B A \to A \otimes H, \qquad a \otimes_B a' \mapsto a\rho^A(a')\,.$$

is bijective (an isomorphism of left A-modules and right H-comodules).

Example 2.16. Let G be a group, $H = kG$, and $A = \bigoplus_{g \in G} A_g$ be a G-graded algebra. Then A is Hopf Galois extension if and only if it is strongly graded, i.e., for all $g, g' \in G$,

$$A_g A_{g'} = A_{gg'}, \qquad B = A^{\mathrm{co}H} = A_e.$$

In this case, for all $a \in A_g$, $a' \in A_{g'}$

$$\begin{aligned} \mathrm{can}\colon a' \otimes_B a &\mapsto aa' \otimes g, \\ \mathrm{can}^{-1}\colon a' \otimes g &\mapsto \sum_i a\overline{a_i} \otimes_B a_i. \end{aligned}$$

where $a_i \in A_g$, $\overline{a_i} \in A_{g^{-1}}$ are such that $\sum_i \overline{a_i} a_i = 1_A$.

In classical differential geometry, a principal bundle with a compact structure Lie group G is defined as a manifold with a free action of G. The following example motivates the interpretation of Hopf-Galois extensions as non-commutative principal bundles.

Example 2.17. Suppose X is a free G-set (i.e. if $xg = x$, then $g = e$). Set $A = \mathcal{O}(X)$, $H = \mathcal{O}(G)$. Then

$$B := A^{\mathrm{co}H} = \{f \in \mathcal{O}(X) \mid \forall\, x \in X,\, g \in G,\ f(xg) = f(x)\} \simeq \mathcal{O}(X/G)\,,$$

$$\mathrm{can}(f \otimes_B f')(x, g) = f(x)f'(xg)\,.$$

The inverse of can is defined in a few stages. Consider a pullback

$$\begin{tikzcd} X \times_{X/G} X \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"'] & X/G, \end{tikzcd}$$

so that

$$(x, x') \in X \times_{X/G} X \Leftrightarrow \pi(x) = \pi(x').$$

Identify $A \otimes_B A$ with $\mathcal{O}(X \times_{X/G} X)$ by the isomorphism $\varphi : A \otimes_B A \to \mathcal{O}(X \times_{X/G} X)$,

$$\varphi(f \otimes_B f')(x, x') = f(x)f'(x'), \qquad \text{where} \quad \pi(x) = \pi(x')\,.$$

The map φ is well defined because the B-actions are, for all $f \in \mathcal{O}(X)$, $x \in X$, $b \in B$,

$$(f \cdot b)(x) = f(x)b(\pi(x)), \qquad (b \cdot f)(x) = b(\pi(x))f(x)\ ,$$

so that, for all $(x, x') \in X \times_{X/G} X$,

$$\varphi(f \cdot b \otimes_B f'))(x, x') = f(x)b(\pi(x))f(x') = f(x)(b \cdot f)(x') = \varphi(f \otimes_B b \cdot f')(x, x')\,.$$

Since X is a free G-set, for any x, x' such that $\pi(x) = \pi(x')$, there is a unique $g \in G$ such that $x' = xg$. Define

$$\mathrm{can}^{-1}(f \otimes h)(x, x') = f(x)h(g), \qquad f \in \mathcal{O}(X),\, h \in \mathcal{O}(G).$$

Then, for all $x \in X$, $g \in G$,

$$\begin{aligned}(\mathrm{can}^{-1} \circ \mathrm{can})(f \otimes_B f')(x, x') &= \mathrm{can}(f \otimes_B f')(x, g)\\ &= f(x)f'(xg)\\ &= f(x)f'(x')\\ &= \varphi(f \otimes_B f')(x, x'),\end{aligned}$$

and

$$\begin{aligned}(\mathrm{can} \circ \mathrm{can}^{-1})(f \otimes h)(x, g) &= \mathrm{can}^{-1}(f \otimes h)(x, xg)\\ &= f(x)h(g)\\ &= (f \otimes h)(x, g)\,.\end{aligned}$$

Therefore, the constructed map can^{-1} is the inverse of the canonical map as required.

As explained in Example 2.10 (H, Δ_H) is a right H-comodule algebra. It is thus tempting to ask the following

Question 1. When is (H, Δ_H) a Hopf-Galois extension by H?

Since the coinvariants $H^{\mathrm{co}H}$ of (H, Δ_H) coincide with the ground field k (see Example 2.14), Question 1 is equivalent to determining, when

$$\mathrm{can}_H \colon H \otimes H \to H \otimes H, \qquad h' \otimes h \mapsto h'h_{(1)} \otimes h_{(2)}\,,$$

is an isomorphism.

Lemma 2.18. *(H, Δ_H) is a Hopf-Galois extension if and only if there is a map $S \colon H \to H$ such that*

$$h_{(1)}S(h_{(2)}) = \varepsilon_H(h)1_H = S(h_{(1)})h_{(2)}\,.$$

Such a map S is called an ***antipode****.*

Proof. If such a map S exists, then the inverse of the canonical map is given by

$$\mathrm{can}^{-1}(h' \otimes h) = h' S(h_{(1)}) \otimes h_{(2)} .$$

Conversely, if can^{-1} exists, then the linear map

$$S = (H \otimes \varepsilon_H) \circ \mathrm{can}^{-1} \circ (1_H \otimes H) ,$$

has the required properties. □

Definition 2.19. A bialgebra with an antipode is called a **Hopf algebra**.

The antipode is an anti-algebra and anti-coalgebra map, and plays the role similar to the mapping which to each element of a group assigns its inverse (and hence can be heuristically understood as a generalised inverse).

Examples 2.20.

1. If G is a group, then kG is a Hopf algebra with the antipode $S\colon kG \to kG$ given on G by $g \mapsto g^{-1}$.
2. Similarly, for a group G, the antipode on $\mathcal{O}(G)$ is given by

$$S\colon f \mapsto [g \mapsto f(g^{-1})] .$$

2.4 Cleft extensions

Take an algebra B and a Hopf algebra H. Let $A = B \otimes H$ and consider it as a right H-comodule with coaction

$$\rho^A\colon B \otimes H \to B \otimes H \otimes H, \qquad \rho^A = B \otimes \Delta_H .$$

Suppose furthermore that $B \otimes H$ is an algebra with multiplication and unit

$$(b \otimes h)(b' \otimes h') = bb' \otimes hh' , \qquad 1_B \otimes 1_H .$$

This makes $(B \otimes H, \mathrm{B} \otimes \Delta_H)$ into an H-comodule algebra. Clearly

$$A^{\mathrm{co}H} = (B \otimes H)^{\mathrm{co}H} = \{b \otimes 1_H \mid b \in B\} \simeq B.$$

The canonical map is

$$\begin{aligned} \mathrm{can}\colon (B \otimes H) \otimes_B (B \otimes H) \simeq B \otimes H \otimes H &\to B \otimes H \otimes H, \\ b \otimes h' \otimes h &\mapsto b \otimes h' h_{(1)} \otimes h_{(2)}, \end{aligned}$$

and hence is bijective with the inverse

$$\begin{aligned} \mathrm{can}^{-1}\colon B \otimes H \otimes H &\to B \otimes H \otimes H , \\ b \otimes h' \otimes h &\mapsto b \otimes h' S(h_{(1)}) \otimes h_{(2)}. \end{aligned}$$

Therefore, $B \otimes H$ is a Hopf-Galois extension (of B). More generally, one can study Hopf-Galois extensions built on the comodule $(B \otimes H, \mathrm{B} \otimes \Delta_H)$.

Definition 2.21. Let A be a Hopf-Galois extension of $B = A^{\mathrm{co}H}$. A is said to have a **normal basis property** if $A \simeq B \otimes H$ as a left B-module and right H-comodule.

Proposition 2.22. *Let (A, ρ^A) be a right H-comodule algebra, and let $B = A^{\mathrm{co}H}$. The following statements are equivalent:*

1. *A is a Hopf-Galois extension with a normal basis property.*

2. *There exists a map $j\colon H \to A$ such that:*

   (a) j is a right H-comodule map, i.e. the following diagram

$$\begin{tikzcd} H \arrow[r, "j"] \arrow[d, "\Delta_H"] & A \arrow[d, "\rho^A"] \\ H \otimes H \arrow[r, "j \otimes H"'] & A \otimes H \end{tikzcd}$$

   is commutative;

   (b) j is convolution invertible, i.e. there exists a linear map $\tilde{j}\colon H \to A$ such that, for all $h \in H$,

$$j(h_{(1)})\tilde{j}(h_{(2)}) = \tilde{j}(h_{(1)})j(h_{(2)}) = \varepsilon_H(h)1_A.$$

Proof. (2) $\Longrightarrow$ (1) We prove that the inverse of the canonical map can has the following form

$$\mathrm{can}^{-1}\colon a \otimes h \mapsto a\tilde{j}(h_{(1)}) \otimes_B j(h_{(2)})\,.$$

In one direction, starting with can^{-1}, we compute

$$\mathrm{can}(a\tilde{j}(h_{(1)}) \otimes j(h_{(2)})) = a\tilde{j}(h_{(1)})j(h_{(2)(1)}) \otimes h_{(2)(2)} = a\tilde{j}(h_{(1)(1)})j(h_{(1)(2)}) \otimes h_{(2)} = a \otimes h.$$

The first equality follows by the fact that the coaction ρ^A is an algebra map and by the colinearity of j (condition 2(a) in Proposition 2.22). The final equality is a consequence of condition 2(b). The proof that the composite $\mathrm{can}^{-1} \circ \mathrm{can}$ is the identity map is slightly more involved. First note that

$$\rho^A(\tilde{j}(h)) = \tilde{j}(h_{(2)}) \otimes S(h_{(1)}). \tag{2.1}$$

This is verified in a few steps. Start with the equality

$$1_A \otimes S(h_{(1)}) \otimes \tilde{j}(h_{(2)}) = \tilde{j}(h_{(1)})j(h_{(2)}) \otimes S(h_{(3)}) \otimes \tilde{j}(h_{(4)})\,,$$

which is a consequence of condition 2(b) (and the definition of a counit). Then apply $\rho^A \otimes H \otimes A$ and use the multiplicativity of ρ^A and right H-colinearity of j to obtain

$$1_A \otimes 1_H \otimes S(h_{(1)}) \otimes \tilde{j}(h_{(2)}) = \tilde{j}(h_{(1)})_{(0)}j(h_{(2)}) \otimes \tilde{j}(h_{(1)})_{(1)}h_{(3)} \otimes S(h_{(4)}) \otimes \tilde{j}(h_{(5)})\,.$$

Next multiply elements in H and use the definition of the antipode to reduce above equality to

$$1_A \otimes S(h_{(1)}) \otimes \tilde{j}(h_{(2)}) = \tilde{j}(h_{(1)})_{(0)}j(h_{(2)}) \otimes \tilde{j}(h_{(1)})_{(1)} \otimes \tilde{j}(h_{(3)})\,.$$

Finally, equality (2.1) is obtained by multiplying elements in A and then using the convolution inverse property 2(b). Equation (2.1) implies that, for all $a \in A$,

$$a_{(0)}\tilde{j}(a_{(1)}) \in B = A^{\mathrm{co}H}\,. \tag{2.2}$$

To verify this claim, simply apply ρ^A to $a_{(0)}\tilde{j}(a_{(1)})$, use the multiplicativity of ρ^A, covariance property (2.1) and the definition of the antipode to obtain

$$\begin{aligned}\rho^A(a_{(0)}\tilde{j}(a_{(1)})) &= a_{(0)}\tilde{j}(a_{(2)})_{(0)} \otimes a_{(1)}\tilde{j}(a_{(2)})_{(1)} \\ &= a_{(0)}\tilde{j}(a_{(3)}) \otimes a_{(1)}S(a_{(2)}) = a_{(0)}\tilde{j}(a_{(1)}) \otimes 1_H\,.\end{aligned}$$

Property (2.2) is used to compute $\mathrm{can}\circ\mathrm{can}^{-1}$:

$$\begin{aligned}\mathrm{can}\circ\mathrm{can}^{-1} : a' \otimes_B a &\mapsto a'a_{(0)} \otimes a_{(1)} \\ &\mapsto a' \underbrace{a_{(0)}\tilde{j}(a_{(1)})}_{\in B} \otimes_B j(a_{(2)}) \\ &= a' \otimes_B a_{(0)}\tilde{j}(a_{(1)})j(a_{(2)}) \\ &= a' \otimes_B a_{(0)}\varepsilon_H(a_{(1)}) \qquad \text{(by property 2(b) in Proposition 2.22)} \\ &= a' \otimes_B a\,.\end{aligned}$$

This completes the proof that A is a Hopf-Galois extension. We need to show that it has the normal basis property, i.e. that A is isomorphic to $B \otimes H$. Consider the map

$$\theta\colon B \otimes H \to A\,, \qquad b \otimes h \mapsto bj(h)\,.$$

This is clearly a left B-module map. It is also right H-colinear since ρ^A is left linear over the coaction invariants and j is right H-colinear by assumption (2)(a). The inverse of θ is

$$\theta^{-1}\colon A \to B \otimes H, \qquad a \mapsto a_{(0)}\tilde{j}(a_{(1)}) \otimes a_{(2)}\,.$$

The verification that θ^{-1} is the inverse of θ makes use of assumption 2(b) and is left to the reader.

(1) $\implies$ (2) Given a left B-linear, right H-colinear isomorphism $\theta\colon B \otimes H \xrightarrow{\simeq} A$, define

$$j\colon H \to A, \qquad h \mapsto \theta(1_B \otimes h)\,.$$

Since θ is right H-colinear, so is j. The convolution inverse of j is

$$\tilde{j}\colon H \xrightarrow{1_A\otimes H} A \otimes H \xrightarrow{\mathrm{can}^{-1}} A \otimes_B A \xrightarrow{A\otimes_B\theta^{-1}} A \otimes_B B \otimes H \simeq A \otimes H \xrightarrow{A\otimes\varepsilon_H} A\,.$$

□

Verification of property 2(b) is left to the reader.

Definition 2.23. A comodule algebra A such that there is a convolution invertible right H-comodule map $j\colon H \to A$ is called a **cleft extension** (of $A^{\mathrm{co}H}$). The map j is called a **cleaving map**.

Since 1_H is a grouplike element $j(1_H)\tilde{j}(1_H) = 1_A$, so $j(1_H) \neq 0$, and a cleaving map can always be normalised so that $j(1_H) = 1_A$. Proposition 2.22 establishes a one-to-one correspondence between cleft extensions and Hopf-Galois extensions with a normal basis property. Note finally that the isomorphism $\theta : A \to B \otimes H$ can be used to generate an algebra structure on $B \otimes H$. In this way one obtains an example of a *twisted tensor product* or *crossed product* algebra.

2.5 Hopf-Galois extensions as Galois comodules

In this section we would like to make a connection between Hopf-Galois extensions and Galois comodules of a coring described in Chapter 1.

Take a bialgebra H. Let A be a right H-comodule algebra, i.e. an algebra $(A, \mu_A, 1_A)$ and an H-comodule (A, ρ^A) such that $\rho^A\colon A \to A \otimes H$ is an algebra map. Then $\mathcal{C} = A \otimes H$ is an A-bimodule with the following A-actions

$$a \cdot (a' \otimes h) = aa' \otimes h \quad \text{(left } A\text{-action)}, \qquad (a' \otimes h) \cdot a = a' a_{(0)} \otimes h a_{(1)} \quad \text{(right } A\text{-action)}\,.$$

Furthermore, $\mathcal{C}$ is an A-coring with counit $\varepsilon_{\mathcal{C}} = A \otimes \varepsilon_H$ and comultiplication

$$\Delta_{\mathcal{C}}\colon A \otimes H \to (A \otimes H) \otimes_A (A \otimes H) \simeq A \otimes H \otimes H, \qquad a \otimes h \mapsto a \otimes \Delta_H(h)\,.$$

A is a right $\mathcal{C}$-comodule with the coaction

$$A \to A \otimes_A (A \otimes H) \simeq A \otimes H, \qquad a \mapsto \rho^A(a) = a_{(0)} \otimes_A (1_A \otimes a_{(1)})\,.$$

In other words, once the identification of $A \otimes_A \mathcal{C}$ with $\mathcal{C} = A \otimes H$ is taken into account, A is a $\mathcal{C}$-comodule by the same coaction by which A is an H-comodule. Next we need to compute the endomorphism ring of the right $\mathcal{C}$-comodule (A, ρ^A). $B = \mathrm{End}^{\mathcal{C}}(A)$ is a subalgebra of A, once the right A-module endomorphisms of A are identified with A by the left multiplication map, i.e.

$$B = \mathrm{End}^{\mathcal{C}}(A) \subseteq A \simeq \mathrm{End}_A(A), \qquad A \ni b \mapsto [l_b\colon a \mapsto ba] \in \mathrm{End}_A(A)\,.$$

The element $b \in A$ is an element of the subalgebra B if and only if the corresponding A-linear map l_a is right $\mathcal{C}$-colinear, i.e.

$$\begin{tikzcd}
1_A \arrow[r, mapsto, "\rho^A"] \arrow[d, mapsto, "l_b"'] & 1_A \otimes_A 1_A \otimes 1_H \arrow[d, mapsto, "l_b \otimes_A 1_A \otimes 1_H"] \\
a \arrow[r, mapsto, "\rho^A"'] & \rho^A(b) = b \otimes 1_H .
\end{tikzcd}$$

Hence $l_b \in \mathrm{End}^{\mathcal{C}}(A)$ if and only if $\rho^A(b) = b \otimes 1_H$. This means that the endomorphism algebra $B = \mathrm{End}^{\mathcal{C}}(A)$ coincides with the algebra of H-comodule invariants,

$$B = A^{\mathrm{co}H} = \{a \in A \mid \rho^A(a) = a \otimes 1_H\}\,.$$

Obviously A is a finitely generated projective right A-module and the dual module can be identified with A,

$$A^* := \mathrm{Hom}_A(A, A) \simeq A\,.$$

The dual basis for A is

$$e = (l_{1_A} \otimes_A 1_A) = 1_A \otimes_A 1_A \in A \otimes_A A\,.$$

The corresponding comatrix coring is simply the **Sweedler canonical coring** (associated to the inclusion of algebras $B \subseteq A$) $A \otimes_B A$, with the comultiplication and counit

$$\begin{aligned}
\Delta_{A \otimes_B A}&\colon a \otimes_B a' \mapsto (a \otimes_B 1_A) \otimes_A (1_A \otimes_B a')\,, \\
\varepsilon_{A \otimes_B A}&\colon a \otimes_B a' \mapsto aa'\,.
\end{aligned}$$

The canonical map for the right $\mathcal{C}$-comodule (A, ρ^A) as defined in Section 1.4 comes out as

$$\mathrm{can}_A \colon A \otimes_B \to A \otimes H, \qquad a \otimes_B a' \mapsto l_a(a'_{(0)}) \otimes_A (1 \otimes a'_{(1)}) = a\rho^A(a')\,,$$

and hence it coincides with the canonical map for the right H-comodule algebra (A, ρ^A) as defined in Definition 2.15. Consequently, a right H-comodule algebra A is a Hopf-Galois extension (of $B = A^{\mathrm{co}H}$) if and only if (A, ρ^A) is a (finite) Galois comodule of $\mathcal{C} = A \otimes H$.

Right comodules of $\mathcal{C} = A \otimes H$ are right A-modules M with a map $\rho^M \colon M \to M \otimes_A A \otimes H \simeq M \otimes H$, which is a right coaction. The coaction property means that (M, ρ^M) is a right H-comodule. The right A-module property of ρ^M yields the compatibility condition

$$\rho^M(ma) = m_{(0)}a_{(0)} \otimes m_{(1)}a_{(1)}\,.$$

Right A-modules and H-comodules M with this compatibility condition are called **relative Hopf modules** and their category is denoted by $\mathbf{M}_A^H$. Thus $\mathbf{M}_A^H$ is isomorphic to the category of right $\mathcal{C} = A \otimes H$-comodules.

The following result is often referred to as an easy part of the Schneider Theorem I.

Theorem 2.24 (Schneider)**.** *Let A be a right H-comodule algebra, $B = A^{\mathrm{co}H}$. The following statements are equivalent:*

1. *A is a Hopf-Galois extension such that A is faithfully flat as a left B-module.*
2. *The functor $- \otimes_B A \colon \mathbf{M}_B \to \mathbf{M}_A^H$ is an equivalence.*

Proof. Take $\mathcal{C} = A \otimes H$, identify right $\mathcal{C}$-comodules with relative Hopf modules $\mathbf{M}_A^H$ and apply the finite Galois comodule theorem, Theorem 1.15. $\square$

References. Blattner and Montgomery [**?**]; Brzeziński [**?**]; Brzeziński and Wisbauer [**?**]; Doi and Takeuchi [**?**]; Montgomery [**?**]; Schauenburg [**?**]; Schneider [**?**]; Sweedler [**?**]; Szlachányi [**?**].

Chapter 3

Connections in Hopf-Galois extensions

Geometric aspects of Hopf-Galois extensions are most clearly present in the theory of connections. The aim of this lecture is to outline the main points of this theory.

3.1 Connections

Connections are differential geometric objects. Thus before connections in a Hopf-Galois extension can be defined, one needs to describe what is meant by a differential structure.

Definition 3.1. A **differential graded algebra** is an $\mathbb{N} \cup \{0\}$-graded algebra

$$\Omega A = \bigoplus_{n=0}^{\infty} \Omega^n A\,,$$

with an operation

$$d\colon \Omega^n A \to \Omega^{n+1} A\,,$$

such that $d \circ d = 0$ and, for all $\omega \in \Omega^n A$ and $\omega' \in \Omega A$,

$$d(\omega\omega') = d(\omega)\omega' + (-1)^n \omega d(\omega')\,. \tag{3.1}$$

Equation (3.1) is known as the Leibniz rule.

The zero-degree part of a differential graded algebra, $\Omega^0 A$, is an algebra which is denoted by A.

Take an algebra $(A, \mu_A, 1_A)$. One associates to A a differential graded algebra ΩA as follows

$$\Omega^1 A := \ker \mu_A = \left\{ \sum_i a_i \otimes a_i' \in A \otimes A \mid \sum a_i a_i' = 0 \right\} \simeq A \otimes A/k\,,$$

$$d(a) = 1_A \otimes a - a \otimes 1_A\,,$$

$$\Omega^n A := \Omega^{n-1} A \otimes_A \Omega^1 A\,.$$

The differential d is extended to the whole of ΩA using the Leibniz rule (3.1). $\Omega^1 A$ is an A-bimodule. As an algebra $\Omega A = T_A(\Omega^1 A)$ (the tensor algebra associated to the A-bimodule $\Omega^1 A$. This $(\Omega A, d)$ is called the **universal differential envelope** of A. $(\Omega^1 A, d)$ is known as the **universal differential calculus** on A. We will only work with universal differential calculus (or envelope).

Lemma 3.2. *If (A, ρ^A) is a comodule algebra of bialgebra H, then $\Omega^1 A$ is a right H-comodule by*

$$\rho^{\Omega^1 A} \colon \Omega^1 A \to \Omega^1 A \otimes H, \qquad \sum_i a_i \otimes a'_i \mapsto \sum_i a_{i(0)} \otimes a'_{i(0)} \otimes a_{i(1)} a'_{i(1)} .$$

Furthermore, d is a right H-comodule map. We say that $(\Omega^1 A, d)$ is a ***covariant differential calculus*** *on A.*

Proof. To check that $\rho^{\Omega^1 A}$ is well-defined, we need to show that $\mathrm{Im}\rho^{\Omega^1 A} \subseteq \Omega^1 A \otimes H$. Applying $\mu_A \otimes H$ to $\rho^{\Omega^1 A}(\sum_i a_i \otimes a'_i)$ and using the multplicativity of ρ^A we obtain

$$\sum_i a_{i(0)} a'_{i(0)} \otimes a_{i(1)} a'_{i(1)} = \rho^A(\sum_i a_i a'_i) = 0, \quad \text{since } \sum_i a_i a'_i = 0.$$

Furthermore, for all $a \in A$,

$$\begin{aligned} \rho^{\Omega^1 A}(d(a)) &= \rho^{\Omega^1 A}(1_A \otimes a - a \otimes 1_A) \\ &= 1_A \otimes a_{(0)} \otimes a_{(1)} - a_{(0)} \otimes 1_A \otimes a_{(1)} = d(a_{(0)}) \otimes a_{(1)} , \end{aligned}$$

i.e. d is a right H-comodule map as required. □

Definition 3.3. Let (A, ρ^A) be a right H-comodule algebra, $B = A^{\mathrm{co}H}$. The A-subbimodule $\Omega^1_{hor} A$ of $\Omega^1 A$ generated by all $d(b)$, $b \in B$, is called a module of **horizontal one-forms**. Thus:

$$\Omega^1_{hor} A = A(\Omega^1 B)A = \left\{ \sum_i (a_i \otimes b_i a'_i - a_i b_i \otimes a'_i) \mid a_i, a'_i \in A,\ b_i \in B \right\} .$$

Equivalently, horizontal forms can be defined by the following short exact sequence

$$0 \to \Omega^1_{hor} A \to A \otimes A \to A \otimes_B A \to A ,$$

where $A \otimes A \to A \otimes_B A$ is the epimorphism defining $A \otimes_B A$.

Definition 3.4. A **connection** in a Hopf-Galois extension $B \subseteq A$ is a left A-linear map $\Pi \colon \Omega^1 A \to \Omega^1 A$, such that

(a) $\Pi \circ \Pi = \Pi$,
(b) $\ker \Pi = \Omega^1_{hor} A$,
(c) $(\Pi \otimes H) \circ \rho^{\Omega^1 A} = \rho^{\Omega^1 A} \circ \Pi$.

In other words, a connection is an H-covariant splitting of $\Omega^1 A$ into the horizontal and vertical parts.

3.2 Connection forms

In classical differential geometry connections in a principal bundle are in one-to-one correspondence with connection forms, i.e. differential forms on the total space of the bundle with values in the Lie algebra of the structure group that are covariant with respect to the adjoint action of the Lie algebra. To be able to establish a similar relationship between connections and connection forms in a Hopf-Galois extension we first need to reinterpret the definition of a Hopf-Galois extension in terms of the universal differential envelope.

Definition 3.5. Let A be a right H-comodule algebra. Set $B = A^{\mathrm{co}H}$ and $H^+ = \ker \varepsilon_H \subseteq H$. Define

$$\mathrm{ver}\colon \Omega^1 A \to A \otimes H^+, \qquad \sum_i a'_i \otimes a_i \mapsto \sum_i a'_i \rho^A(a_i)\,.$$

The map ver is called a **vertical lift**.

Note in passing that ver is well defined (its range is in $A \otimes H^+$), since $\sum_i a'_i a_i = 0$ implies

$$\sum_i a'_i a_{i(0)} \varepsilon_H(a_{i(1)}) = \sum_i a'_i a_i = 0\,.$$

Proposition 3.6. *The following statements are equivalent:*

1. *$B \subseteq A$ is Hopf-Galois extension.*
2. *The sequence*

$$0 \to \Omega^1_{hor} A \to \Omega^1 A \xrightarrow{\mathrm{ver}} A \otimes H^+ \to 0$$

is exact.

Proof. Note that

$$A \otimes A \simeq \Omega^1 A \oplus A\,, \qquad A \otimes H \simeq A \otimes H^+ \oplus A\,,$$

as left A-modules. This implies that the sequence

$$0 \to \Omega^1_{hor} A \to \Omega^1 A \xrightarrow{\mathrm{ver}} A \otimes H^+ \to 0$$

is exact if and only if the sequence

$$0 \to \Omega^1_{hor} A \to A \otimes A \xrightarrow{\overline{\mathrm{can}}} A \otimes H \to 0$$

is exact. Here $\overline{\mathrm{can}}$ is the lift of the canonical map defined by the commutative diagram

$$\begin{tikzcd} A \otimes A \arrow[rr, "\pi"] \arrow[dr, "\overline{\mathrm{can}}"'] & & A \otimes_B A \arrow[dl, "\mathrm{can}"] \\ & A \otimes H, & \end{tikzcd}$$

in which π is the defining projection of the tensor product $A \otimes_B A$. Since $\Omega^1_{hor} A = \ker \pi$ (compare Definition 3.3), the second sequence is exact if and only if the canonical map can is bijective. $\square$

The vector space H^+ is a right H-comodule by the **adjoint coaction**,

$$\mathrm{Ad}\colon H^+ \to H^+ \otimes H, \qquad h \mapsto h_{(2)} \otimes S(h_{(1)}) h_{(3)}\,.$$

Therefore $A \otimes H^+$ is a right H-comodule by combining ρ^A and Ad, i.e.

$$\begin{aligned} \rho^{A \otimes H^+}\colon A \otimes H^+ &\to A \otimes H^+ \otimes H, \\ a \otimes h &\mapsto a_{(0)} \otimes h_{(2)} \otimes a_{(1)} S(h_{(1)}) h_{(3)}\,. \end{aligned}$$

Lemma 3.7. *The vertical lift is a right H-comodule map from $(\Omega^1 A, \rho^{\Omega^1 A})$ to $(A \otimes H^+, \rho^{A \otimes H^+})$. Consequently, the sequence in Proposition 3.6 is a sequence of left A-modules and right H-comodules.*

Proof. The first statement is checked by a direct calculation that is left to the reader as an exercise. The second statement is obvious. □

Definition 3.8. A **connection form** in a Hopf-Galois extension $B \subseteq A$ is a k-linear map $\omega\colon H^+ \to \Omega^1 A$ such that

(a) $\rho^{\Omega^1 A} \circ \omega = (\omega \otimes H) \circ \mathrm{Ad}$,

(b) $\mathrm{ver} \circ \omega = 1_A \otimes H^+$.

Theorem 3.9. *Connections in a Hopf-Galois extension $B \subseteq A$ (by a Hopf algebra H) are in bijective correspondence with connection forms. The correspondence is*

$$\omega \mapsto \Pi, \quad \Pi(a'da) = a'a_{(0)}\omega(a_{(1)} - \varepsilon_H(a_{(1)})).$$

Proof. Existence of Π means that $\Omega^1_{hor}A$ is a direct summand of $\Omega^1 A$ as a left A-module and right H-comodule. This is equivalent to the existence of splitting of the left A-module and right H-comodule sequence

$$0 \to \Omega^1_{hor}A \to \Omega^1 A \xrightarrow{\mathrm{ver}} A \otimes H^+ \to 0.$$

In view of the identification

$${}_A\mathrm{Hom}^H(A \otimes H^+, \Omega^1 A) \simeq \mathrm{Hom}^H(H^+, \Omega^1 A),$$

any splitting yields an ω with required properties. □

3.3 Strong connections

Recall that given an algebra B and a left B-module Γ, a **connection** in Γ is a k-linear map

$$\nabla\colon \Gamma \to \Omega^1 B \otimes_B \Gamma,$$

such that, for all $b \in B$, $x \in \Gamma$,

$$\nabla(bx) = d(b) \otimes_B x + b\nabla(x).$$

A connection in Γ exists if and only if Γ is a projective B-module (remember that $\Omega^1 B$ is the universal differential calculus) if and only if there exists a left B-module splitting (section) of the multiplication map $B \otimes \Gamma \to \Gamma$.

A general connection in a Hopf-Galois extension $B \subset A$ does not induce a connection in the left B-module A. Only connections which are related to a more restrictive notion of horizontal forms yield connections in modules.

Definition 3.10. Given a connection Π in $B \subset A$, the right H-comodule map

$$D\colon A \to \Omega^1_{hor}A, \qquad D := d - \Pi \circ d\,,$$

is called an **covariant derivative** corresponding to Π. The connection Π is called a **strong connection** if $D(A) \subseteq (\Omega^1 B)A$.

Lemma 3.11. *Let D be a covariant derivative corresponding to a strong connection in a Hopf-Galois extension $B \subseteq A$. Then D is a connection in the left B-module A.*

Lemma 3.11 is a special case of Theorem 3.15 so is left without a proof (for the time being).

Definition 3.12. A connection form ω such that its associated connection is a strong connection is called a **strong connection form**. Thus a strong connection form is a k-linear map $\omega\colon H^+ \to \Omega^1 A$ characterised by the following properties:

(a) $\rho^{\Omega^1 A} \circ \omega = (\omega \otimes H) \circ \mathrm{Ad}$,

(b) $\mathrm{ver} \circ \omega = 1_A \otimes H^+$,

(c) $d(a) - \sum a_{(0)}\omega(a_{(1)} - \varepsilon_H(a_{(1)})) \in (\Omega^1 B)A$, for all $a \in A$.

Definition 3.13. Let (A, ρ^A) be a right H-comodule and let $(V, {}^V\rho)$ be a left H-comodule. The **cotensor** product is defined as an equaliser

$$A\Box_H V \longrightarrow A \otimes V \underset{A\otimes {}^V\rho}{\overset{\rho^A\otimes V}{\rightrightarrows}} A \otimes H \otimes V.$$

This means that

$$A\Box_H V = \{\sum_i a_i \otimes v_i \in A \otimes V \mid \sum_i \rho^A(a_i) \otimes v_i = \sum_i a_i \otimes {}^V\rho(v_i)\}.$$

The functor $A\Box_H -\colon {}^H\mathbf{M} \to \mathrm{Vect}$ is a left exact functor, and $A\Box_H H \simeq A$.

If (A, ρ^A) is a comodule algebra, $B = A^{\mathrm{co}H}$, then $A\Box_H V$ is a left B-module by

$$b(\sum_i a_i \otimes v_i) = \sum_i ba_i \otimes v_i.$$

This defines a functor $A\Box_H -\colon {}^H\mathbf{M} \to {}_B\mathbf{M}$ from the category of left H-comodules to the category of left B-modules.

Definition 3.14. Given a left H-comodule $(V, {}^V\rho)$ and a Hopf-Galois extension $B \subseteq A$, the left B-module $\Gamma := A\Box_H V$ is called a **module associated** to $B \subseteq A$.

Here Γ plays the role of module of sections of a vector bundle (with a standard fibre V) associated to the non-commutative principal bundle represented by the Hopf-Galois extension $B \subseteq A$. In the case of a cleft extension $B \subseteq A$, $A \simeq B \otimes H$, hence $\Gamma \simeq (B \otimes H)\Box_H V \simeq B \otimes V$, and thus it is a free B-module. More generally,

Theorem 3.15. *If Π is a strong connection, then*

$$\nabla\colon A\Box_H V \to \Omega^1 B \otimes_B (A\Box_H V), \quad \nabla = D \otimes V\,,$$

is a connection in the associated left B-module $\Gamma = A\Box_H V$. Consequently Γ is a projective B-module.

Proof. Since $D(A) \subseteq (\Omega^1 B)A \simeq \Omega^1 B \otimes_B A$ and D is a right H-comodule map, the map ∇ is well defined. For all $b \in B$ and $a \otimes v \in A\Box_H V$ (summation suppressed for clarity) we can compute:

$$\begin{aligned} \nabla(ba \otimes v) &= d(ba) \otimes v - \Pi(d(ba)) \otimes v \\ &= dba \otimes v + bda \otimes v - \Pi(dba) \otimes v - \Pi(bda) \otimes v \\ &= dba \otimes v + bda \otimes v - b\Pi(da) \otimes v = dba \otimes v + b\nabla(a \otimes v), \end{aligned}$$

where the second equality follows by the Leibniz rule and the third one by the left A-linearity of Π and the fact that $(db)a$ is a horizontal form, hence in the kernel of Π. We thus conclude that ∇ is a connection. The last assertion follows since every module admitting a connection (with respect to the universal differential calculus) is projective. $\square$

In general, the associated module $A\square_H V$ in Theorem 3.15 is not finitely generated as a left A-module, even if V is a finite dimensional vector space. However, if H has a bijective antipode, then $A\square_H V$ is finitely generated and projective for any finite dimensional V (and, of course, provided that A has a strong connection).

Theorem 3.16 (Dąbrowski-Grosse-Hajac)**.** *A strong connection in a Hopf-Galois extension $B \subseteq A$ by a Hopf algebra H exists if and only if A is H-equivariantly projective as a left B-module, i.e. if and only if there exists a left B-module, right H-comodule section of the multiplication map $\mu_A\colon B \otimes A \to A$ (section means $s\colon A \to B \otimes A$ such that $\mu_A \circ s = A$).*

Proof. Given a section $s : A \to B \otimes A$, define the connection by

$$\Pi(a'da) = a'da - a' \otimes a + a's(a) = a's(a) - a'a \otimes 1_A.$$

This map is clearly left A-linear and right H-colinear. It is an idempotent since, using the fact that s is a section of the multiplication map $B \otimes P \to P$, one easily finds that $-a' \otimes a + a's(a) \in \Omega^1_{hor}A$. This also implies that $\ker \Pi \subseteq \Omega^1_{hor}A$. The converse inclusion follows by the left B-linearity of s and the Leibniz rule. Write $s(a) = a^{(1)} \otimes a^{(2)} \in B \otimes A$ (summation suppressed). The splitting property means that $a^{(1)}a^{(2)} = a$, so

$$D(a) = 1_A \otimes a - s(a) = 1_A \otimes a^{(1)}a^{(2)} - a^{(1)} \otimes a^{(2)} = (da^{(1)})a^{(2)} \in (\Omega^1 B)A.$$

If Π is a strong connection, then the splitting of the product is given by

$$s(a) = a \otimes 1_A + \Pi(da).$$

The map s is obviously right H-colinear and the section of the multiplication map. Note that $s(a) = 1_A \otimes a - D(a)$, hence $s(a) \in B \otimes A$ as Π is a strong connection. An easy calculation proves that s is left B-linear. That the above assignments describe mutual inverses is immediate. $\square$

Corollary 3.17. *Let $B \subseteq A$ be a Hopf-Galois extension by H with a strong connection. Then*

1. *A is projective as a left B-module;*
2. *B is a direct summand of A as a left B-module;*
3. *A is faithfully flat as a left B-module.*

Proof. The statement (1) follows by Lemma 3.11 (or is contained in Theorem 3.16). For (2), let $s_L\colon A \to B$ be a k-linear map which is identity on B. Then the map $\mu_A \circ (B \otimes s_L) \circ s$ is a left B-linear splitting of the inclusion $B \subseteq A$. Statements (1) and (2) imply (3); see [**?**, 2.11.29]. $\square$

To give an example of a strong connection we construct such a connection in a cleft extension; see Section 2.4.

Proposition 3.18. *Let $B \subseteq A$ be a cleft extension, with a cleaving map $j\colon H \to A$ such that $j(1_H) = 1_A$. Write $\tilde{j} : H \to A$ for the convolution inverse of j; see Proposition 2.22. Then*

$$\omega\colon H^+ \to \Omega^1 A\,, \qquad h \mapsto \tilde{j}(h_{(1)}) \otimes j(h_{(2)})\,,$$

is a strong connection form.

Proof. First, for all $h \in H^+$, $\tilde{j}(h_{(1)})j(h_{(2)}) = \varepsilon_H(h) = 0$, so ω is well defined. We need to check if ω satisfies conditions (a)-(c) in Definition 3.12. This is done by the following three direct calculations.

$$\begin{aligned}
\rho^{\Omega^1 A} \circ \omega(h) &= \tilde{j}(h_{(1)})_{(0)} \otimes j(h_{(2)})_{(0)} \otimes \tilde{j}(h_{(1)})_{(1)} j(h_{(2)})_{(1)} \\
&= \tilde{j}(h_{(1)(2)}) \otimes j(h_{(2)(1)}) \otimes S(h_{(1)(1)}) h_{(2)(2)} \\
&= \tilde{j}(h_{(2)}) \otimes j(h_{(3)}) \otimes S(h_{(1)}) h_{(4)} \\
&= \tilde{j}(h_{(2)(1)}) \otimes j(h_{(2)(2)}) \otimes S(h_{(1)}) h_{(3)} \\
&= \omega(h_{(2)}) \otimes S(h_{(1)}) h_{(3)} \\
&= \omega \circ \mathrm{Ad}(h).
\end{aligned}$$

The first equality is simply the definition of $\rho^{\Omega^1 A}$, the second uses the H-colinearity of j and its consequence (2.1). Then the Sweedler indices have been rearranged and definitions of the adjoint coaction and ω used. Next,

$$\begin{aligned}
\mathrm{ver}(\omega(h)) &= \mathrm{ver}(\tilde{j}(h_{(1)}) \otimes j(h_{(2)})) \\
&= \tilde{j}(h_{(1)}) j(h_{(2)})_{(0)} \otimes j(h_{(2)})_{(1)} \\
&= \tilde{j}(h_{(1)}) j(h_{(2)(1)}) \otimes h_{(2)(2)} \\
&= \tilde{j}(h_{(1)(1)}) j(h_{(1)(2)}) \otimes h_{(2)} \\
&= \varepsilon(h_{(1)}) 1_A \otimes h_{(2)} \\
&= 1_A \otimes h.
\end{aligned}$$

The second equality is the definition of the vertical lift, then the H-colinearity of j is used and the Sweedler indices rearranged. The penultimate equality is a consequence of property (2)(b) in Proposition 2.22. Finally, using the normalisation of j (and hence also of $\tilde{j}$) one can compute, for all $a \in A$,

$$\begin{aligned}
D(a) &= d(a) - \Pi(d(a)) \\
&= d(a) - a_{(0)} \omega(a_{(1)} - \varepsilon_H(a_{(1)})) \\
&= 1 \otimes a - a \otimes 1 - a_{(0)} \tilde{j}(a_{(1)}) \otimes j(a_{(2)}) + a \tilde{j}(1_H) \otimes j(1_H) \\
&= 1 \otimes a - a_{(0)} \tilde{j}(a_{(1)}) \otimes j(a_{(2)}).
\end{aligned}$$

Since $a_{(0)}\tilde{j}(a_{(1)}) \in B$ (see (2.2) in the proof of Proposition 2.22), we obtain $D(a) \in \Omega^1 B \otimes_B A \subseteq B \otimes A$. $\square$

The normalisation of a cleaving map in Proposition 3.18 is not an essential assumption. If $j(1_H) \neq 1_A$ we can choose

$$\omega(h) = \tilde{j}(h_{(1)}) \otimes j(h_{(2)}) - \tilde{j}(1_H) \otimes j(1_H) + 1_A \otimes 1_A.$$

3.4 The existence of strong connections. Principal comodule algebras

Here we would like to determine, when a Hopf-Galois extension admits a strong connection. In all geometrically interesting situations the antipode S is a Hopf algebra is bijective, hence it is natural to restrict our considerations to this case. If a Hopf algebra H has a bijective antipode, then we make a right H-comodule algebra A iinto a left H-comodule via

$$ {}^{A}\rho\colon A \to H \otimes A, \qquad a \mapsto S^{-1}(a_{(1)}) \otimes a_{(0)}. $$

Theorem 3.19. *If a Hopf algebra H has a bijective antipode S, then strong connections in a Hopf-Galois extension $B \subseteq A$ are in bijective correspondence with k-linear maps $\ell\colon H \to A \otimes A$ such that*

(a) $\ell(1_H) = 1_A \otimes 1_A$,

(b) $\overline{\mathrm{can}} \circ \ell = 1_A \otimes H$ (or $\mu_A \circ \ell = 1_A \circ \varepsilon_H$),

(c) $(\ell \otimes H) \circ \Delta_H = (A \otimes \rho^A) \circ \ell$,

(d) $(H \otimes \ell) \circ \Delta_H = ({}^{A}\rho \otimes A) \circ \ell$.

The correspondence is given by

$$ \Pi(a'd(a)) = a' a_{(0)} \ell(a_{(1)}) - a'a \otimes 1. $$

We also refer to ℓ as a ***strong connection****.*

Proof. The idea of the proof is to show the relation between ℓ and connection forms. First we comment on two versions of condition (b). Since $(A \otimes \varepsilon_H) \circ \overline{\mathrm{can}} = \mu_A$, the first version of condition (b) immediately implies that $\mu_A \circ \ell = 1_A \circ \varepsilon_H$. The converse follows by the use of colinearity (condition (c)).

So, using the second version of (b), if $\varepsilon_H(h) = 0$, then $\mu_A \circ \ell(h) = 0$. This means that given ℓ one can define a map $\omega_\ell : H^+ \to \Omega^1 A$, by $\omega_\ell(h) = \ell(h)$. Obviously, for all $h \in H^+$, $\mathrm{ver} \circ \omega_\ell(h) = \overline{\mathrm{can}} \circ \ell(h) = 1_A \otimes h$. A straightforward calculation reveals that (c) and (d) imply that $(\omega_\ell \otimes H) \circ \mathrm{Ad} = \rho^{\Omega^1 A} \circ \omega_\ell$. Hence if ℓ exists, the corresponding ω_ℓ is a connection one-form. By Theorem 3.9 there is a connection Π_ℓ in A with the form stated. Using explicit definition of the universal differential, the corresponding covariant derivative comes out as

$$ D_\ell(a) = 1_A \otimes a - a_{(0)} \ell(a_{(1)}). \tag{3.2} $$

Now use the fact that A is a right H-comodule algebra, conditions (c) and (d) for ℓ and the fact that S^{-1} is an anti-algebra map to compute

$$ \begin{aligned} ({}^{A}\rho \otimes H)(D_\ell(a)) &= 1_H \otimes 1_A \otimes a - a_{(2)} S^{-1}(a_{(1)}) \otimes a_{(0)} \ell(a_{(3)}) \\ &= 1_H \otimes 1_A \otimes a - S^{-1}(a_{(1)} S(a_{(2)})) \otimes a_{(0)} \ell(a_{(3)}) \\ &= 1_H \otimes 1_A \otimes a - 1_H \otimes a_{(0)} \ell(a_{(1)}) = 1_H \otimes D_\ell(a). \end{aligned} $$

This implies that, for all $a \in A$, $D_\ell(a) \in B \otimes A$, i.e. the connection Π_ℓ is strong.

Conversely, given a strong connection Π with connection one-form $\omega : H^+ \to \Omega^1 A$, define $\ell_\omega : H \to A \otimes A$ by $\ell_\omega(h) = \varepsilon_H(h) 1_A \otimes 1_A - \omega(h - \varepsilon_H(h))$. Such an ℓ_ω satisfies (a) and (b) (the latter by condition (b) of Definition 3.12). Now, condition (a) of Definition 3.12 implies that

$$ (\ell_\omega \otimes H) \circ \mathrm{Ad} = \rho^{A \otimes A} \circ \ell_\omega, \tag{3.3} $$

where $\rho^{A\otimes A}$ is the diagonal coaction of H on $A \otimes A$, given by the same formula as $\rho^{\Omega^1 A}$. The covariant derivative D corresponding to Π has the same form as in equation (3.2). Since the connection Π is strong,

$$ (B \otimes \rho^A)(D(a)) = \rho^{\Omega^1 A}(D(a)), \qquad \forall a \in A. $$

In view of equation (3.2) this means that

$$(A \otimes \rho^A)(a_{(0)}\ell_\omega(a_{(1)})) = \rho^{A\otimes A}(a_{(0)}\ell_\omega(a_{(1)})), \qquad \forall a \in A. \tag{3.4}$$

Putting equations (3.3) and (3.4) together and using defining properties of the antipode one obtains

$$(A \otimes \rho^A)(a_{(0)}\ell_\omega(a_{(1)})) = a_{(0)}\ell_\omega(a_{(1)}) \otimes a_{(2)}, \qquad \forall a \in A. \tag{3.5}$$

Since A is a Hopf-Galois extension, the canonical map is bijective. This means that, for any $h \in H$, there exists $h^{[1]} \otimes_B h^{[2]} \in A \otimes_B A$ (summation assumed) such that $1 \otimes h = \mathrm{can}(h^{[1]} \otimes_B h^{[2]}) = h^{[1]}{h^{[2]}}_{(0)} \otimes {h^{[2]}}_{(1)}$. Hence equation (3.5) implies for all $h \in H$,

$$\begin{aligned}(A \otimes \rho^A)(\ell_\omega(h)) &= (A \otimes \rho^A)(h^{[1]}{h^{[2]}}_{(0)}\ell_\omega({h^{[2]}}_{(1)})) \\ &= h^{[1]}{h^{[2]}}_{(0)}\ell_\omega({h^{[2]}}_{(1)}) \otimes {h^{[2]}}_{(2)} = \ell_\omega(h_{(1)}) \otimes h_{(2)}.\end{aligned}$$

Therefore ℓ_ω satisfies property (c). Finally one easily verifies that (c) combined with equation (3.3) imply property (d). □

Theorem 3.20. *Let A be a comodule algebra of H, set $B := A^{\mathrm{co}H}$, and assume that the antipode in H is bijective. Then the following statements are equivalent.*

1. *There exists $\ell\colon H \to A \otimes A$ such that*
   - *(a) $\ell(1_H) = 1_A \otimes 1_A$,*
   - *(b) $\overline{\mathrm{can}} \circ \ell = 1_A \otimes H$ (or $\mu_A \circ \ell = 1_A \circ \varepsilon_H$),*
   - *(c) $(\ell \otimes H) \circ \Delta_H = (A \otimes \rho^A) \circ \ell$,*
   - *(d) $(H \otimes \ell) \circ \Delta_H = ({}^A\rho \otimes A) \circ \ell$.*
2. *A is a faithfully flat (as a left and right B-module) Hopf-Galois extension.*

Proof. (1) $\Longrightarrow$ (2) The inverse of the canonical map is given as the following composite

$$\mathrm{can}^{-1}\colon A \otimes H \xrightarrow{A\otimes\ell} A \otimes A \otimes A \xrightarrow{\mu_A\otimes A} A \otimes A \to A \otimes_B A\,.$$

Since ℓ is a strong connection, A is faithfully flat as a left B-module by Corollary 3.17. By symmetry, A is a left Hopf-Galois extension, ℓ is a strong connection for this left Hopf-Galois extension, hence A is faithfully flat as the right B-module (by the left-handed version of Corollary 3.17).

(2) $\Longrightarrow$ (1) Since A is faithfully flat as a right B-module, for all left H-comodules V, there is a chain of isomorphisms

$$A \otimes_B (A\Box_H V) \simeq (A \otimes_B A)\Box V \simeq (A \otimes H)\Box_H V \simeq A \otimes V.$$

The flatness of A as a B-module is crucial for the first isomorphism since, in general, the cotensor product does not commute with the tensor product. The second isomorphism is obtained by applying the canonical map. One uses this chain of isomorphisms to argue that $A\Box_H-$ is an exact functor as follows. Any exact sequence of left H-comodules $V \to W \to 0$ yields the exact sequence $A \otimes V \to A \otimes W \to 0$. By the constructed isomorphism, the sequence $A \otimes_B (A\Box_H V) \to A \otimes_B (A\Box_H W) \to 0$ is exact, hence also $A\Box_H V \to A\Box_H W \to 0$ is an exact sequence by the faithful flatness of A as a right B-module. Hence $A\Box_H-$ is right exact, and as it is always left exact, it is simply an exact functor.

For a finitely dimensional right H-comodule, (V, ρ^V), the dual vector space $V^* := \mathrm{Hom}_k(V, k)$ is a left H-comodule. Furthermore,

$$A \Box_H V^* \simeq \mathrm{Hom}^H(V, A).$$

This implies that $\mathrm{Hom}^H(-, A)$ is exact, i.e. (A, ρ^A) is an injective H-comodule. In other words there is an H-colinear map $\pi\colon A \otimes H \to A$ such that $\pi \circ \rho^A = A$. Denote by ${}_A\mathbf{M}^H$ the category with objects left A-modules M that are also right H-comodules with a left A-linear coaction ρ^M, provided $M \otimes H$ is seen as a left A-module by the *diagonal action*, $a \cdot (m \otimes h) = a_{(0)}m \otimes a_{(1)}h$. Morphisms are maps which are both left A-linear and right H-colinear. For every $(M, \rho^M) \in {}_A\mathbf{M}^H$, there is a right H-colinear retraction of the coaction $\rho^M : M \to M \otimes H$ (i.e. (M, ρ^M) is injective as an H-comodule),

$$\pi_M\colon M \otimes H \to M\,, \qquad m \otimes h \mapsto \pi(1_A \otimes hS^{-1}(m_{(1)}))m_{(0)}\,.$$

Note that the bijectivity of the antipode plays here the most crucial role. The existence of π_M implies that every short exact sequence in ${}_A\mathbf{M}^H$ splits as a sequence in $\mathbf{M}^H$. In particular, $\overline{can}\colon A \otimes A \to A \otimes H$ is an epimorphism in ${}_A\mathbf{M}^H$, where $A \otimes A$ and $A \otimes H$ are comodules with coactions

$$\begin{aligned}
\rho^{A\otimes A}(a \otimes a') &= a_{(0)} \otimes a' \otimes a_{(1)},\\
\rho^{A\otimes H}(a \otimes h) &= a_{(0)} \otimes h_{(2)} \otimes a_{(1)}S(h_{(1)}),
\end{aligned}$$

and left A-actions provided by the multiplication in A, $a \cdot (a' \otimes a'') = aa' \otimes a''$, $a \cdot (a' \otimes h) = aa' \otimes h$. Therefore, there is an H-colinear section $\alpha\colon A \otimes H \to A \otimes A$ of $\overline{\mathrm{can}}$. The map

$$s\colon A \to B \otimes A, \qquad a \mapsto a_{(0)}\alpha(1_A \otimes a_{(1)})\,,$$

is a left B-module splitting of the multiplication $B \otimes A \to A$. This shows that A is a projective left B-module. It remains to construct a section of the multiplication $B \otimes A \to A$ which is also right H-colinear.

Define a left B-module, right H-comodule map

$$\varphi\colon A \otimes H \to A, \qquad a \otimes h \mapsto a_{(0)}\pi(1_A \otimes S(a_{(1)})h).$$

A left B-module, right H-comodule splitting of the multiplication map $B \otimes A \to A$ is the composite

$$\sigma\colon A \xrightarrow{\rho^A} A \otimes H \xrightarrow{s\otimes H} B \otimes A \otimes H \xrightarrow{B\otimes\varphi} B \otimes A\,.$$

This can be checked as follows. Write

$$s(a) = \underbrace{a^{(1)}}_{\in B} \otimes \underbrace{a^{(2)}}_{\in A} \qquad \text{(summation implicit)}\,,$$

so that $a^{(1)}a^{(2)} = a$, and compute

$$\begin{aligned}
a &\xmapsto{\sigma} {a_{(0)}}^{(1)} \otimes {{a_{(0)}}^{(2)}}_{(0)}\pi(1_A \otimes S({{a_{(0)}}^{(2)}}_{(1)})a_{(1)})\\
&\xmapsto{\mu_A} {a_{(0)}}^{(1)}{{a_{(0)}}^{(2)}}_{(0)}\pi(1_A \otimes S({{a_{(0)}}^{(2)}}_{(1)})a_{(1)})\\
&= ({a_{(0)}}^{(1)}{a_{(0)}}^{(2)})_{(0)}\pi(1_A \otimes S(({a_{(0)}}^{(1)}{a_{(0)}}^{(2)})_{(1)})a_{(1)})\\
&= a_{(0)}\pi(1_A \otimes S(a_{(1)})a_{(2)})\\
&= a\pi(1_A \otimes 1_H)\\
&= a,
\end{aligned}$$

where the first equality follows by the left B-linearity of coaction ρ^A, the second one follows by the splitting property of s, the third one is the antipode axiom, and the last equality is a consequence of the fact that the composite $\pi \circ \rho^A$ is the identity on A.

Thus it has been proven that A is a Hopf-Galois extension that is an H-equivariantly projective left B-module. Theorem 3.16 now implies that there exists a strong connection and Theorem 3.19 yields the required map ℓ. $\square$

Definition 3.21. A comodule algebra of a Hopf algebra H with a bijective antipode which satisfies conditions in Theorem 3.20 is called a **principal comodule algebra**.

Principal comodule algebras are a non-commutative version of principal bundles which retains most of the features of the classical (commutative) objects.

Theorem 3.22 (The difficult part of Schneider's theorem)**.** *Let (A, ρ^A) be an H-comodule algebra that is injective as an H-comodule (i.e. there exists a right H-comodule map $\pi : A \otimes H \to A$, such that $\pi \circ \rho^A = A$). Assume that H has bijective antipode, and that lifted canonical map $\overline{\mathrm{can}}$ is injective. Then A is a principal comodule algebra.*

Proof. Follow the same steps as in the part (2) $\Longrightarrow$ (1) in Theorem 3.20, starting from the existence of π. $\square$

Theorem 3.23. *Let A be a principal comodule algebra, $B = A^{\mathrm{co}H}$. For any finitely dimensional left H-comodule V, the associated B-module $\Gamma := A\Box_H V$ is finitely generated and projective.*

Proof. By Theorem 3.15 Γ is projective as a left B-module. By arguments in proof of Theorem 3.20 $A \otimes_B \Gamma \simeq A \otimes V$. On the other hand $A \otimes V$ is finitely generated as an A-module and A is faithfully flat right B-module, hence Γ is finitely generated as a left B-module. $\square$

Put differently, Theorem 3.23 states that a principal comodule algebra defines a functor

$$A\Box_H - \colon {}^H\mathbf{M}_f \to {}_B\mathbf{P}_f \tag{3.6}$$

form finitely generated H-comodules to finitely generated projective B-modules.

On the other hand, principal comodule algebras can also be understood as monoidal functors. Start with a right H-comodule algebra (A, ρ^A) with coaction invariants B. Since the coaction ρ^A is right B-linear, there is a right B-action on $A\Box_H V$ defined by

$$\left(\sum_i a_i \otimes v_i\right) \cdot b = \sum_i a_i b \otimes v_i \,,$$

i.e. $A\Box_H V$ inherits B-bimodule structure from that in A. Both categories – of B-bimodules, ${}_B\mathbf{M}_B$, and left H-comodules, ${}^H\mathbf{M}$ – are monoidal, where the monoidal structure in ${}_B\mathbf{M}_B$ is the algebraic tensor product over B, while the monoidal structure in ${}^H\mathbf{M}$ is

$${}^{V\otimes W}\rho\colon V \otimes W \xrightarrow{{}^V\rho \otimes {}^W\rho} H \otimes V \otimes H \otimes W \xrightarrow{H \otimes \mathrm{flip} \otimes W} H \otimes H \otimes V \otimes W \xrightarrow{\mu_H \otimes V \otimes W} H \otimes W;$$

see the comments after the definition of a comodule algebra, Definition 2.9. The functor $A\Box_H - \colon {}^H\mathbf{M} \to {}_B\mathbf{M}_B$ is lax monoidal. It is monoidal if A is a Hopf-Galois extension such that A is faithfully flat as a right B-module.

Proposition 3.24 (Schauenburg-Ulbrich)**.** *If H has a bijective antipode, then there is a bijective correspondence between:*

1. *exact monoidal functors* ${}^H\mathbf{M} \to_B \mathbf{M}_B$ *(*fibre functors*),*

2. *principal comodule algebras.*

Example 3.25. Let A be a Hopf algebra with bijective antipode, and let $A \xrightarrow{\pi} H$ be a surjective map of Hopf algebras. Then A is a right H-comodule algebra with the coaction $\rho^A = (A \otimes \pi) \circ \Delta_A$, and $B = A^{\mathrm{co}H} = \{a \in A \mid a_{(1)} \otimes \pi(a_{(2)}) = a \otimes 1_H\}$. Suppose that there exists an H-bicomodule map $\iota\colon H \to A$ such that $\pi \circ \iota = H$ and $\iota(1_H) = 1_A$. Here H is understood as a left and right H-comodule via the regular coaction Δ_H and A is a left H-comodule by the induced coaction $(\pi \otimes A) \circ \Delta_A$. Then the map

$$\ell\colon H \to A \otimes A, \qquad h \mapsto S(\iota(h)_{(1)}) \otimes \iota(h)_{(2)},$$

satisfies conditions (a)–(d) in Theorem 3.20, so A is a principal H-comodule algebra (with a strong connection ℓ).

Example 3.26. As a particular application of Example 3.25, take A to be the coordinate algebra of functions on the quantum group $\mathrm{SU}_q(2)$. $A = \mathcal{O}(\mathrm{SU}_q(2))$ is generated by the 2×2 matrix of generators $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ subject to relations

$$\alpha\beta = q\beta\alpha,\;\; \alpha\gamma = q\gamma\alpha,\;\; \beta\gamma = \gamma\beta,\;\; \beta\delta = q\delta\beta,\;\; \gamma\delta = q\delta\gamma,\;\; \delta\alpha - q^{-1}\beta\gamma = 1,\;\; \alpha\delta - q\beta\gamma = 1,$$

where q is a non-zero number. When k is the field of complex numbers and q is real, then $\mathcal{O}(\mathrm{SU}_q(2))$ is a $*$-algebra with

$$\alpha^* = \delta, \qquad \beta^* = -q\gamma, \qquad \gamma^* = -q^{-1}\beta, \qquad \delta^* = \alpha.$$

The algebra $\mathcal{O}(\mathrm{SU}_q(2))$ is a Hopf algebra with coproduct given by

$$\Delta_A(\alpha) = \alpha \otimes \alpha + \beta \otimes \gamma, \qquad \Delta_A(\beta) = \alpha \otimes \beta + \beta \otimes \delta,$$
$$\Delta_A(\gamma) = \gamma \otimes \alpha + \delta \otimes \gamma, \qquad \Delta_A(\delta) = \delta \otimes \delta + \gamma \otimes \beta,$$

and extended to the whole of $\mathcal{O}(\mathrm{SU}_q(2))$ as an algebra map. The counit is

$$\varepsilon_A(\alpha) = \varepsilon_A(\delta) = 1, \qquad \varepsilon_A(\beta) = \varepsilon_A(\delta) = 0,$$

and the antipode

$$S(\alpha) = \delta, \quad S(\beta) = -q^{-1}\beta, \quad S(\gamma) = -q\gamma, \quad S(\delta) = \alpha.$$

Let $H = \mathcal{O}(U(1)) = k[w, w^{-1}]$, a commutative Hopf algebra generated by the group-like elements w, w^{-1} (cf. Example 2.8). If H is made into a $*$-algebra with $w^* = w^{-1}$, then H is the algebra of polynomials on the circle. One easily finds that, similarly to the classical case, the (diagonal) map $\pi : \mathcal{O}(\mathrm{SU}_q(2)) \to H$ defined by

$$\pi(\alpha) = w, \quad \pi(\delta) = w^{-1}, \quad \pi(\beta) = \pi(\gamma) = 0,$$

is a Hopf algebra map. The induced coaction makes A a $\mathbb{Z}$-graded algebra with the grading

$$\deg(a) = \deg(c) = 1, \qquad \deg(b) = \deg(d) = -1.$$

The coaction invariants $B = A^{coH}$ are simply the degree-zero subalgebra of A. Thus B is generated by $x = -q^{-1}\beta\gamma$, $z = -q^{-1}\alpha\beta\ z^* = \gamma\delta$. The elements x and z satisfy relations

$$zx = q^2xz, \quad xz^* = q^2z^*x, \quad zz^* = q^2x(1-q^2x), \quad z^*z = x(1-x).$$

(The coefficients are chosen so that for the $*$-algebra case x is real and z^* is the conjugate of z). An abstract algebra generated by x, z, z^* and the above relations is called a *standard* or *polar Podleś* or *quantum sphere* and is denoted by $\mathcal{O}(S_q^2)$.

A unital, H-bicolinear map splitting π is defined by

$$\iota\colon \mathcal{O}(U(1)) \to \mathcal{O}(\mathrm{SU}_q(2)), \qquad \iota(1) = 1, \quad \iota(w^n) = \alpha^n, \quad \iota(w^{-n}) = \delta^n.$$

The corresponding strong connection comes out as

$$\begin{aligned}
\ell(w^n) &= \sum_{k=0}^{n} \binom{n}{k}_{q^{-2}} \gamma^{*k}\alpha^{*n-k} \otimes \alpha^{n-k}\gamma^k, \\
\ell(1) &= 1 \otimes 1, \\
\ell(w^{*n}) &= \sum_{k=0}^{n} q^{2k} \binom{n}{k}_{q^{-2}} \alpha^{n-k}\gamma^k \otimes \gamma^{*k}\alpha^{*n-k},
\end{aligned}$$

where the *deformed binomial coefficients* are defined for any number ζ by

$$\binom{n}{k}_{\zeta} = \frac{(\zeta^n - 1)(\zeta^{n-1} - 1)\ldots(\zeta^{k+1} - 1)}{(\zeta^{n-k} - 1)(\zeta^{n-k-1} - 1)\ldots(\zeta - 1)}.$$

This example describes a non-commutative version of the Hopf fibration with the Dirac monopole connection.

REFERENCES. Brzeziński and Majid [?]; Connes [?]; Cuntz and Quillen [?]; Dąbrowski, Grosse and Hajac [?]; Doi [?]; Schauenburg [?]; Schauenburg and Schneider [?]; Schneider [?]; Ulbrich [?].

Chapter 4

Principal extensions and the Chern-Galois character

In the preceding chapter we have explained how a principal comodule algebra induces a functor from the category of finite dimensional comodules of a Hopf algebra to the category of finitely generated and projective modules over the coaction invariant subalgebra. When restricted to isomorphism classes this functor gives a map from the K-group of the Hopf algebra to the K-group of the invariant subalgebra. This can be followed by a map to the cyclic homology (the Chern-Connes character) and thus provides one with homological methods of studying (invariants of) Hopf-Galois extensions. The composite mapping is known as the **Chern-Galois character** and we describe its construction (in a slightly more general set-up than the Hopf-Galois theory) in this lecture.

4.1 Coalgebra-Galois extensions

One of the main examples of principal bundles in classical geometry is provided by homogenous spaces of a Lie group. The following example shows how the classical construction of a principal bundle over a homogeneous space is performed in the realm of non-commutative geometry, and how it forces one to go beyond principal comodule algebras if one wants to develop fully an example driven approach to non-commutative principal bundles.

Example 4.1. Let A be a Hopf algebra. A subalgebra $B \subseteq A$ such that

$$\Delta_A(B) \subset A \otimes B$$

is called a left A**-comodule subalgebra**.

If we think of A as of an algebra of functions on a group G, B is an algebra of functions on a homogeneous space of G.

If A is faithfully flat as a left B-module, one can construct B as a coaction invariant subalgebra (this is the non-commutative counterpart of classical identification of a homogeneous space as a quotient space). First, define

$$B^+ = \ker \varepsilon_A \cap B.$$

Then $J := B^+A$ is a right ideal in A, and a **coideal** in A, i.e., for all $x = ba \in J$

$$\begin{aligned}\Delta_A(ba) &= b_{(1)}a_{(1)} \otimes b_{(2)}a_{(2)} \\ &= ba_{(1)} \otimes a_{(2)} - b_{(1)}a_{(1)} \otimes \varepsilon_A(b_{(2)})a_{(2)} + b_{(1)}a_{(1)} \otimes b_{(2)}a_{(2)} \\ &= \underbrace{ba_{(1)} \otimes a_{(2)}}_{\in J\otimes A} + \underbrace{b_{(1)}a_{(1)} \otimes (b_{(2)} - \varepsilon_A(b_{(2)}))a_{(2)}}_{\in A\otimes J} .\end{aligned}$$

Hence $C := A/J$ is a coalgebra and a right A-module, and $\pi \colon A \to C$ is a right A-linear coalgebra epimorphism. Note that J is not an (two-sided) ideal in A, hence in general C is not a quotient algebra (or Hopf algebra) of A. However, since π is a coalgebra map, A is a right C-comodule by

$$\rho^A = (A \otimes \pi) \circ \Delta_A.$$

Define the coaction invariant subalgebra

$$A^{\mathrm{co}C} := \{b \in A \mid \text{for all } a \in A,\ \rho^A(ba) = b\rho^A(a)\}.$$

Since ρ^A is a right A-module map, the coaction invariant subalgebra can be equivalently described as

$$A^{\mathrm{co}C} = \{b \in A \mid \rho^A(b) = b\rho^A(1_A)\}.$$

Note that, for all $b \in B$,

$$\rho^A(b) = b_{(1)} \otimes \pi(b_{(2)}) = b \otimes \pi(1_A) + b_{(1)} \otimes \pi(\underbrace{b_{(2)} - \varepsilon(b_{(2)})}_{\in B^+ \subseteq J}) = b \otimes \pi(1_A),$$

so $B \subseteq A^{\mathrm{co}C}$. Faithful flatness implies also that $A^{\mathrm{co}C} \subseteq B$, that is A is an extension of B by a coalgebra C, but not necessarily by a bialgebra or a Hopf algebra, as one would naively expect guided by the classical geometric intuition. The reasons why the non-commutative geometry is reacher (or less rigid) than the classical one lie in the Poisson geometry and the reader is referred to lectures by N. Ciccoli [**Pawle, prosze sprawdz!**].

The description of quantum homogeneous spaces as invariant subalgebras in Example 4.1 justifies a generalisation of Hopf-Galois extensions in which the symmetry is given by a coalgebra rather than a Hopf algebra.

Definition 4.2. Let C be a coalgebra and let (A, ρ^A) be a C-comodule. Set

$$B = A^{\mathrm{co}C} := \{b \in A \mid \text{for all } a \in A,\ \rho^A(ba) = b\rho^A(a)\} .$$

A is called a **coalgebra-Galois extension** if the canonical left A-linear right C-colinear map

$$\mathrm{can} \colon A \otimes_B A \to A \otimes C , \qquad a \otimes_B a' \mapsto a\rho^A(a') ,$$

is bijective.

Although C in a coalgebra-Galois extension does not need to be an algebra (or have an algebra structure compatible with the coaction and the algebra structure of A), nevertheless the fact that A is an algebra gives some more information about C. In particular, since $A \otimes_B A$ is an A-coring (see Section 2.5), also $A \otimes C$ can be made an A-coring via the isomorphism can. The coproduct in $A \otimes_B A$ is transported to a coproduct in $A \otimes C$ as

$$\Delta_{A\otimes C} \colon A \otimes C \to (A \otimes C) \otimes_A (A \otimes C) \simeq A \otimes C \otimes C, \qquad \Delta_{A\otimes C} = A \otimes \Delta_C .$$

Furthermore, the right A-module structure on $A \otimes_B A$ induces a right A-module structure on $A \otimes C$,

$$(1_A \otimes c) \cdot a = \mathrm{can}(\mathrm{can}^{-1}(1_A \otimes c)a).$$

Define

$$\psi \colon C \otimes A \to A \otimes C, \qquad c \otimes a \mapsto \mathrm{can}(\mathrm{can}^{-1}(1_A \otimes c)a).$$

The map ψ is called a **canonical entwining** associated to the coalgebra-Galois extension $B \subseteq A$. The word entwining means that ψ makes the following bow-tie diagram commute

$$\begin{tikzcd}
 & & C \otimes A \otimes A \arrow[ddll, "\psi\otimes A"'] \arrow[drr, "C\otimes\mu_A"] & & & & C \otimes C \otimes A \arrow[ddrr, "C\otimes\psi"] & & \\
 & & & & C \otimes A \arrow[urr, "\Delta_C\otimes A"] \arrow[d, "\psi"] \arrow[dr, "\varepsilon_C\otimes A"] & & & & \\
A \otimes C \otimes A \arrow[ddrr, "A\otimes\psi"'] & & & C \arrow[ur, "C\otimes 1_A"] \arrow[dr, "1_A\otimes C"'] & & A & & & C \otimes A \otimes C \arrow[ddll, "\psi\otimes C"] \\
 & & & & A \otimes C \arrow[ur, "A\otimes\varepsilon_C"'] \arrow[drr, "A\otimes\Delta_C"] & & & & \\
 & & A \otimes A \otimes C \arrow[urr, "\mu_A\otimes C"'] & & & & A \otimes C \otimes C. & &
\end{tikzcd}$$

The commutativity of this bow-tie diagram for the canonical entwining can be checked by relating $A \otimes C$ to the Sweedler coring $A \otimes_B A$. In particular the right pentagon and the right triangle are a consequence of the definition of ψ in terms of right A-action on $A \otimes C$, while the left pentagon and triangle are responsible for right A-linearity of comultiplication $A \otimes \Delta_C$. An entwining is a special case of a (mixed) distributive law.

Lemma 4.3. *In a coalgebra-Galois extension $B \subseteq A$,*

$$\rho^A(aa') = a_{(0)}\psi(a_{(1)} \otimes a'), \qquad \text{for all } a, a' \in A,$$

*where ψ is the canonical entwining. This means that (A, ρ^A) is an **entwined module** (a comodule of A-coring $A \otimes C$).*

Proof. This is checked by the following calculation which uses the left linearity of can and can^{-1}, and the definition of can,

$$\begin{aligned} a_{(0)}\psi(a_{(1)} \otimes a') &= a_{(0)}\,\mathrm{can}(\mathrm{can}^{-1}(1 \otimes a_{(1)})a') \\ &= \mathrm{can}(\mathrm{can}^{-1}(a_{(0)} \otimes a_{(1)})a') = \mathrm{can}(1_B \otimes_B aa') = \rho^A(aa'). \end{aligned}$$

□

Lemma 4.3 provides one with an explicit form of the coaction in terms of the canonical entwining

$$\rho^A(a) = 1_{A(0)}\psi(1_{A(1)} \otimes a).$$

To simplify further discussions, we will assume that there is a grouplike element $e \in C$ such that

$$\rho^A(1_A) = 1_A \otimes e, \quad \text{ so } \rho^A(a) = \psi(e \otimes a).$$

This is, for example, applicable to quantum homogeneous spaces described in Example 4.1, where $e = \pi(1_A)$.

Lemma 4.4. *The coaction invariant subalgebra of A can be equivalently described as*

$$B = \{b \in A \mid \rho^A(b) = b \otimes e\}.$$

Proof. If $b \in A^{\mathrm{co}C}$, then $\rho^A(b) = b\rho^A(1_A) = b \otimes e$. If $\rho^A(b) = b \otimes e$, then, for all $a \in A$,

$$\rho^A(ba) = b_{(0)}\psi(b_{(1)} \otimes a) = b\psi(e \otimes a) = b\rho^A(a).$$

□

Example 4.5. Let H be a Hopf algebra, and let (A, ρ^A) be a Hopf-Galois extension. Then the right action in the A-coring $A \otimes H$ induced from $A \otimes_B A$ is given by $(a' \otimes h)a = a'a_{(0)} \otimes ha_{(1)}$, hence

$$\psi \colon H \otimes A \to A \otimes H, \qquad h \otimes a \mapsto a_{(0)} \otimes ha_{(1)}.$$

Note that this ψ is bijective if and only if the antipode S is bijective. Then

$$\psi^{-1}(a \otimes h) = hS^{-1}a_{(1)} \otimes a_{(0)}.$$

4.2 Principal extensions

While defining principal comodule algebras we assumed that the Hopf algebra has a bijective antipode. Example 4.5 indicates that this assumption translates to coalgebra-Galois extensions into the bijectivity of the canonical entwining ψ. If ψ is bijective, then A is a left C-comodule by

$${}^A\rho \colon A \to C \otimes A\,, \qquad a \mapsto \psi^{-1}(a \otimes e) \quad (e \in C \text{ such that } \rho^A(1_A) = 1_A \otimes e).$$

Definition 4.6. Let $B \subseteq A$ be a coalgebra-Galois extension by a coalgebra C, with a bijective canonical entwining map $\psi : C \otimes A \to A \otimes C$. Assume that $\rho^A(1_A) = 1_A \otimes e$ for a grouplike element $e \in C$. A k-linear map $\ell \colon C \to A \otimes A$ such that
(a) $\ell(e) = 1_A \otimes 1_A$,
(b) $\mu_A \circ \ell = 1_A \circ \varepsilon_C$,
(c) $(A \otimes \rho^A) \circ \ell = (\ell \otimes C) \circ \Delta_C$,
(d) $({}^A\rho \otimes A) \circ \ell = (C \otimes \ell) \circ \Delta_C$,
is called a **strong connection** in $B \subseteq A$. A coalgebra extension with a strong connection is called a **principal extension**.

Following the same reasoning as in the principal comodule algebra case one proves

Proposition 4.7. *Let $B \subseteq A$ be a principal extension. Then*

1. *A is a C-equivariantly projective left (or right) B-module (i.e. there is a B-module, C-comodule splitting of the product map $B \otimes A \to A$).*
2. *A is a faithfully flat left (or right) B-module.*
3. *B is a direct summand in A as a left B-module.*

In terms of a strong connection the left B-comodule right C-comodule splitting of the multiplication map is $s(a) = a_{(0)}\ell(a_{(1)})$.

Proposition 4.8. *Let $B \subseteq A$ be a principal extension. If $(V, {}^V\rho)$ is a finite dimensional left C-comodule, then $\Gamma := A\Box_C V$ is a finitely generated and projective left B-module.*

Proof. One can follow the same arguments as in the case of a principal comodule algebra. The module Γ has a connection $\Gamma \ni a \otimes v \mapsto 1_A \otimes a \otimes v - a_{(0)} \ell(a_{(1)}) \otimes v$, hence it is a projective B-module. Consider the sequence of isomorphisms

$$A \otimes_B (A \Box_C V) \simeq (A \otimes_B A) \Box_C V \simeq (A \otimes C) \Box_C V \simeq A \otimes V \,.$$

Since $A \otimes V$ is a finitely generated left A-module and B is a faithfully flat right B-module, $\Gamma := A \Box_C V$ is a finitely generated left B-module. $\square$

In view of Proposition 4.8, a principal extension $B \subseteq A$ can be understood as a functor

$$A \Box_C - \colon {}^C\mathbf{M}_f \to_B \mathbf{P}_f$$

from the category of finite dimensional C-comodules to the category of finitely generated projective B-modules. Passing to the Grothendieck group one obtains a map

$$\mathrm{Rep}(C) \to \mathrm{K}_0(B) \xrightarrow{\mathrm{ch}} \mathrm{HC}_{ev}(B),$$

where $\mathrm{Rep}(C)$ is the Grothendieck group of equivalence classes of finite dimensional comodules of C, ch denotes the Chern character, and $\mathrm{HC}_{ev}(B)$ is the even cyclic homology of B. This composite map is known as the **Chern-Galois character** and we will describe it presently.

4.3 Cyclic homology of an algebra and the Chern character

We begin by describing a cyclic homology of an algebra and the Chern character. For any algebra B, consider a bicomplex $\mathrm{CC}_\bullet(B)$:

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\big\downarrow{\scriptstyle \partial_3} & & \big\downarrow{\scriptstyle -\partial'_3} & & \big\downarrow{\scriptstyle \partial_3} & & \big\downarrow{\scriptstyle -\partial'_3} & & \\
B^{\otimes 3} & \xleftarrow{\tilde{\tau}_2} & B^{\otimes 3} & \xleftarrow{N_2} & B^{\otimes 3} & \xleftarrow{\tilde{\tau}_2} & B^{\otimes 3} & \xleftarrow{N_2} & \cdots \\
\big\downarrow{\scriptstyle \partial_2} & & \big\downarrow{\scriptstyle -\partial'_2} & & \big\downarrow{\scriptstyle \partial_1} & & \big\downarrow{\scriptstyle -\partial'_2} & & \\
B^{\otimes 2} & \xleftarrow{\tilde{\tau}_1} & B^{\otimes 2} & \xleftarrow{N_1} & B^{\otimes 2} & \xleftarrow{\tilde{\tau}_1} & B^{\otimes 2} & \xleftarrow{N_1} & \cdots \\
\big\downarrow{\scriptstyle \partial_1} & & \big\downarrow{\scriptstyle -\partial'_1} & & \big\downarrow{\scriptstyle \partial_1} & & \big\downarrow{\scriptstyle -\partial'_1} & & \\
B & \xleftarrow{\tilde{\tau}_0} & B & \xleftarrow{N_0} & B & \xleftarrow{\tilde{\tau}_0} & B & \xleftarrow{N_0} & \cdots,
\end{array}$$

where

$$\begin{aligned}
\partial'_n(b_0 \otimes b_1 \otimes \cdots \otimes b_n) &= \sum_{i=0}^{n-1} (-1)^i b_0 \otimes \cdots \otimes b_i b_{i+1} \otimes \cdots \otimes b_n, \\
\partial_n(b_0 \otimes b_1 \otimes \cdots \otimes b_n) &= \partial'_n(b_0 \otimes b_1 \otimes \cdots \otimes b_n) + (-1)^n b_n b_0 \otimes b_1 \otimes b_2 \otimes \cdots \otimes b_{n-1}, \\
\tau_n(b_0 \otimes \cdots \otimes b_n) &= (-1)^n b_n \otimes b_0 \otimes \cdots \otimes b_{n-1}, \\
\tilde{\tau}_n &= B^{\otimes(n+1)} - \tau_n, \\
N_n &= \sum_{i=0}^{n} (\tau_n)^i.
\end{aligned}$$

The homology of the bicomplex $\mathrm{CC}_\bullet(B)$ is known as the **cyclic homology** of B and is denoted by $\mathrm{HC}_\bullet(B)$. In case k is a field of characteristic 0, the cyclic homology can be equivalently described as the homology of the **Connes complex** of B defined as

$$\mathrm{C}^\lambda_n(B) := B^{\otimes(n+1)}/(\mathrm{id} - \tau_n),$$
$$\text{boundary:} \;\; \delta_n := \text{the quotient of } \;\partial_n.$$

It is denoted by $\mathrm{H}^\lambda(B)$. The Chern and Chern-Galois characters can be defined with respect to either of these homologies, hence – for the convenience of the reader – we will describe both these constructions in parallel. The Chern character is a map $\mathrm{ch}\colon \mathrm{K}_0(B) \to \mathrm{HC}_{ev}(B)$ defined as follows. Take a class $[P] \in \mathrm{K}_0(B)$ of a finitely generated projective B-module P. P has a finite dual basis, say $x_i \in P$, $\pi_i \in {}_B\mathrm{Hom}(P, B)$, $i = 1, \ldots, n$. Since, for all $p \in P$,

$$p = \sum_{i=1}^{n} \pi_i(p)x_i,$$

the matrix $E := (E_{ij})^n_{i,j=1} := (\pi_j(x_i))^n_{i,j=1}$ is an idempotent with image P. With the idempotent E one associates a $2n$-cycle in $\mathrm{CC}_\bullet(B)$. First define

$$\widetilde{ch}_n(E) := \sum_{i_1,i_2,\ldots,i_{n+1}} E_{i_1i_2} \otimes E_{i_2i_3} \otimes \cdots \otimes E_{i_{n+1}i_1},$$

and then $2n$-cycle

$$\bigoplus_{l=0}^{2n} (-1)^{\lfloor \frac{l}{2} \rfloor} \frac{l!}{\lfloor \frac{l}{2} \rfloor !} \widetilde{\mathrm{ch}}_l(E).$$

The class of this $2n$-cycle does not depend on the choice of P or E in $[P]$. Hence it defines an abelian group map

$$\mathrm{ch}\colon \; \mathrm{K}_0(B) \to \mathrm{HC}_{ev}(B),$$

known as the **Chern character**.

In the case of the Connes complex, with the idempotent E one associates a $2n$-cycle in $\mathrm{C}^\lambda_\bullet(B)$ by taking the quotient of

$$\widetilde{ch}_{2n}(E) := \sum_{i_1,i_2,\ldots,i_{2n+1}} E_{i_1i_2} \otimes E_{i_2i_3} \otimes \cdots \otimes E_{i_{2n+1}i_1}.$$

Note that similar construction for even number of factors yields $0 \in \mathrm{C}^\lambda_\bullet(B)$. The class of this $2n$-cycle does not depend on the choice of E or P in $[P]$. It is also compatible with the direct sums of P's and additive structure of $\mathrm{C}^\lambda_\bullet(B)$. Hence it defines an abelian group map

$$\mathrm{ch}\colon \; \mathrm{K}_0(B) \to \mathrm{H}^\lambda_{ev}(B),$$

also known as the **Chern character**.

If $B = C^\infty(X)$, then

$$\mathrm{ch}\colon \; \mathrm{K}^0(X) \to \mathrm{H}^{ev}_{\mathrm{dR}}(X), \qquad [E] \mapsto \mathrm{Tr}(EdE \ldots dE),$$

which is simply the Chern character in differential geometry.

4.4 The Chern-Galois character

Let $B \subseteq A$ be a principal extension by a coalgebra C. Take a strong connection ℓ and introduce a Sweedler-type notation for ℓ,

$$\ell(c) = c^{\langle 1\rangle} \otimes c^{\langle 2\rangle}. \tag{4.1}$$

Let $(V, {}^V\rho)$ be a finite dimensional left C-comodule with a basis $\{v_i\}$. This defines an $n \times n$ matrix of elements $(e_{ij})_{i,j=1}^n$ by

$${}^V\rho(v_i) = \sum_{j=1}^{n} e_{ij} \otimes v_j.$$

The trace of $(e_{ij})_{i,j=1}^n$ is known as a character of the comodule V. The coassociativity of ${}^V\rho$ implies that (e_{ij}) is a *coidempotent matrix*, i.e.,

$$\Delta_C(e_{ij}) = \sum_{l=1}^{n} e_{il} \otimes e_{lj}, \qquad \varepsilon_C(e_{ij}) = \delta_{ij}, \qquad i,j = 1, \ldots, n.$$

Lemma 4.9. *For any $c \in C$,*

$$\ell(c_{(1)})\ell(c_{(2)}) \in A \otimes B \otimes A.$$

Proof. Use the introduced notation for the strong connection (4.1) and apply $A \otimes \rho^A \otimes A$ to $\ell(c_{(1)})\ell(c_{(2)})$ to obtain

$$\begin{aligned}
{c_{(1)}}^{\langle 1\rangle} \otimes \rho^A({c_{(1)}}^{\langle 2\rangle}{c_{(2)}}^{\langle 1\rangle}) \otimes {c_{(2)}}^{\langle 2\rangle} &= {c_{(1)}}^{\langle 1\rangle} \otimes {c_{(1)}}^{\langle 2\rangle}{}_{(0)}\psi({c_{(1)}}^{\langle 2\rangle}{}_{(1)} \otimes {c_{(2)}}^{\langle 1\rangle}) \otimes {c_{(2)}}^{\langle 2\rangle} \\
&= {c_{(1)}}^{\langle 1\rangle} \otimes {c_{(1)}}^{\langle 2\rangle}\psi(c_{(2)} \otimes {c_{(3)}}^{\langle 1\rangle}) \otimes {c_{(3)}}^{\langle 2\rangle} \\
&= {c_{(1)}}^{\langle 1\rangle} \otimes {c_{(1)}}^{\langle 2\rangle}\psi({}^A\rho({c_{(2)}}^{\langle 1\rangle})) \otimes {c_{(2)}}^{\langle 2\rangle} \\
&= {c_{(1)}}^{\langle 1\rangle} \otimes {c_{(1)}}^{\langle 2\rangle}\psi(\psi^{-1}({c_{(2)}}^{\langle 1\rangle} \otimes e)) \otimes {c_{(2)}}^{\langle 2\rangle} \\
&= {c_{(1)}}^{\langle 1\rangle} \otimes {c_{(1)}}^{\langle 2\rangle}{c_{(2)}}^{\langle 1\rangle} \otimes e \otimes {c_{(2)}}^{\langle 2\rangle}.
\end{aligned}$$

The first equality follows by the entwined module property of A, Lemma 4.3, the second one is the right colinearity of ℓ (condition (c) in Definition 4.6). The third equality follows by condition (d) in Definition 4.6 (left C-colinearity of a strong connection), next one is the definition of left coaction ${}^A\rho$. Finally, employ Lemma 4.4 to conclude that the middle term in $\ell(c_{(1)})\ell(c_{(2)})$ is an element of the coaction invariant subalgebra. □

Next we describe the Chern-Galois character in Connes' complex.

Theorem 4.10. *Given a finite dimensional C-comodule V and the corresponding coidempotent matrix $\mathbf{e} = (e_{ij})_{i,j=1}^n$, define*

$$\widetilde{\mathrm{chg}}_n(\mathbf{e}) := \sum_{i_1, i_2, \ldots, i_{n+1}} {e_{i_1 i_2}}^{\langle 2\rangle}\ell(e_{i_2 i_3})\ell(e_{i_3 i_4})\ell(e_{i_3 i_4}) \ldots \ell(e_{i_{n+1} i_1}){e_{i_1 i_2}}^{\langle 1\rangle} \in B^{\otimes(n+1)}.$$

Then $\widetilde{\mathrm{chg}}_{2n}(\mathbf{e})$ is a $2n$-cycle in $\mathrm{C}^\lambda(B)$, $\widetilde{\mathrm{chg}}_{2n+1}(\mathbf{e}) = 0$. It does not depend on the choice of a basis for V and it is the same for isomorphic comodules.

Proof. Note that $\widetilde{\mathrm{chg}}_n(\mathbf{e})$ is an element of $B^{\otimes(n+1)}$ by Lemma 4.9. An easy calculation that uses $\mu_A \circ \ell = 1_A \circ \varepsilon_C$ gives

$$\partial_{2n}(\widetilde{\mathrm{chg}}_{2n}(\mathbf{e})) = -\widetilde{\mathrm{chg}}_{2n-1}(\mathbf{e}).$$

Since

$$\tau_{2n-1}(\widetilde{\mathrm{chg}}_{2n-1}(\mathbf{e})) = -\widetilde{\mathrm{chg}}_{2n-1}(\mathbf{e}),$$

$\widetilde{\mathrm{chg}}_{2n-1}(\mathbf{e}) = 0$ in $\mathrm{C}^{\lambda}(B)$. Thus $\widetilde{\mathrm{chg}}_{2n}(\mathbf{e})$ is a $2n$-cycle in $\mathrm{C}^{\lambda}(B)$.

The $\widetilde{\mathrm{chg}}_{2n}(\mathbf{e})$ do not depend on the choice of basis and a representative in the isomorphism class of comodules, since they are defined only using the character of the comodule V, $\mathrm{tr}(\mathbf{e}) = \sum_i e_{ii}$. □

Similarly in the full cyclic bicomplex

Theorem 4.11. *Given a finite dimensional C-comodule V and the corresponding coidempotent matrix $\mathbf{e} = (e_{ij})_{i,j=1}^{n}$, define*

$$\widehat{\mathrm{chg}}_{2n} := \bigoplus_{l=0}^{2n} (-1)^{\lfloor \frac{l}{2} \rfloor} \frac{l!}{\lfloor \frac{l}{2} \rfloor !} \widetilde{\mathrm{chg}}_l(\mathbf{e}).$$

Then $\widehat{\mathrm{chg}}_{2n}$ is a $2n$-cycle in $\mathrm{CC}_{\bullet}(B)$, and it does not depend on the choice of a basis for V and is the same for isomorphic comodules.

Proof. Since $\mu_A \circ \ell = 1_A \circ \varepsilon_C$, one finds

$$\begin{aligned}
N_n(\widetilde{\mathrm{chg}}_n(\mathbf{e})) &= (n+1)\widetilde{\mathrm{chg}}_n(\mathbf{e}),\\
\partial_n(\widetilde{\mathrm{chg}}_n(\mathbf{e})) &= \widetilde{\mathrm{chg}}_{n-1}(\mathbf{e}), \quad \text{if } n \text{ is even},\\
\partial'_n(\widetilde{\mathrm{chg}}_n(\mathbf{e})) &= \widetilde{\mathrm{chg}}_{n-1}(\mathbf{e}),\\
\tilde{\tau}_n(\widetilde{\mathrm{chg}}_n(\mathbf{e})) &= 2\widetilde{\mathrm{chg}}_n(\mathbf{e}) \quad \text{if } n \text{ is odd}.
\end{aligned}$$

This implies that $\widehat{\mathrm{chg}}_{2n}$ is a cycle in $CC_{\bullet}(B)$ as claimed.

A representative in the isomorphism class of comodules does not depend on the choice of basis, since it is defined only using the character of the comodule V, $\mathrm{tr}(\mathbf{e}) = \sum_i e_{ii}$. □

The cycles constructed in Theorem 4.10 or Theorem 4.11 might depend on the choice of a strong connection (at least their form explicitly depends on this choice). The full independence is achieved by going to homology.

Theorem 4.12. *The class of the Chern-Galois cycle $\widehat{\mathrm{chg}}_{2n}(\mathbf{e})$ (or $\widetilde{\mathrm{chg}}_{2n}(\mathbf{e})$ in the case of the Connes complex) defines a map of abelian groups*

$$\mathrm{chg}\colon \mathrm{Rep}(C) \to \mathrm{HC}_{ev}(B)\,, \tag{4.2}$$

*known as the **Chern-Galois character** of the principal extension $B \subseteq A$. The Chern-Galois character is independent of the choice of a strong connection.*

Proof. The independence of chg on the choice of ℓ follows by observing that there is a factorisation

$$\begin{tikzcd}
\mathrm{Rep}(C) \arrow[rr, "\mathrm{chg}"] \arrow[dr, "A\square_C -"'] & & \mathrm{HC}_{ev}(B) \\
& \mathrm{K}_0(B) \arrow[ur, "\mathrm{ch}"'] &
\end{tikzcd}$$

in which both factors are independent of ℓ.

In more detail, an idempotent for the left B-module $\Gamma = A\Box_C V$ is

$$\mathbf{E} = (E_{(i,p),(j,q)}) := \varphi(l_p(e_{ij})x_q)_{(i,p),(j,q)},$$

where φ is a left B-module retraction of $B \subseteq A$, which exists since B is a direct summand in A, (e_{ij}) is the coidempotent matrix defining the comodule V, $\{x_q\}$ is a finite basis of the subspace of A generated by the ${e_{ij}}^{\langle 1\rangle}{}_\nu$, where ν is a summation index in $\ell(c) = \sum_\nu c^{\langle 1\rangle}{}_\nu \otimes c^{\langle 2\rangle}{}_\nu$. Finally, $\ell_p = (\xi_p \otimes A) \circ \ell$, where $\{\xi_p\}$ is a dual basis to $\{x_q\}$. Then

$$\widetilde{\mathrm{chg}}_n(\mathbf{e}) = \widetilde{\mathrm{ch}}_n(\mathbf{E}).$$

This justifies the stated factorisation property. $\square$

4.5 Example: the classical Hopf fibration

We illustrate the construction of the Chern-Galois character on the classical example of the Hopf fibration. The reader is encouraged to compare this example with its non-commutative counterpart described in Example 3.26. In this example we take $k = \mathbb{C}$, and

$$\mathrm{SU}(2) = \left\{ M = \begin{pmatrix} w & -\bar{z} \\ z & \bar{w} \end{pmatrix} \mid w, z \in \mathbb{C},\ \det(M) = 1 \right\}.$$

The condition $\det(M) = 1$ means that $|w|^2 + |z|^2 = 1$, i.e. SU(2) is a 3-sphere.

The algebra of functions on SU(2), $\mathcal{O}(\mathrm{SU}(2))$ is generated by

$$a\colon M \mapsto w, \quad c\colon M \mapsto z, \quad a^*\colon M \mapsto \bar{w}, \quad c^*\colon M \mapsto \bar{z},$$

with the relation

$$(aa^* + cc^*)(M) = w\bar{w} + z\bar{z} = 1.$$

Hence

$$A := \mathcal{O}(\mathrm{SU}(2)) = \mathbb{C}[a, a^*, c, c^*]/(aa^* + cc^* = 1).$$

There is an action of the group U(1) (the unit circle $\{u \in \mathbb{C} \mid |u|^2 = 1\}$) on SU(2):

$$\begin{pmatrix} w & -\bar{z} \\ z & \bar{w} \end{pmatrix} \cdot u = \begin{pmatrix} wu & -\overline{zu} \\ zu & \overline{wu} \end{pmatrix}.$$

The algebra $\mathcal{O}(\mathrm{U}(1))$ is generated by

$$x\colon u \mapsto u, \qquad x^*\colon u \mapsto \bar{u},$$

with the relation $xx^* = x^*x = 1$. Hence

$$\mathcal{O}(\mathrm{U}(1)) = \mathbb{C}[x, x^*]/(xx^* = x^*x = 1).$$

As a Hopf algebra

$$H := \mathcal{O}(\mathrm{U}(1)) = \mathbb{C}[\mathbb{Z}], \quad x^n \mapsto n,\ x^{*n} = x^{-n} \mapsto -n.$$

A comodule of $\mathcal{O}(\mathrm{U}(1))$ can be viewed as a $\mathbb{Z}$-graded vector space. In particular, the algebra $\mathcal{O}(\mathrm{SU}(2))$ is $\mathbb{Z}$-graded, $\deg(a) = \deg(c) = 1$, $\deg(a^*) = \deg(c^*) = -1$. In fact it is strongly graded, that is $\mathcal{O}(\mathrm{SU}(2))$ is a Hopf-Galois extension by $\mathcal{O}(\mathrm{U}(1))$. Invariant subalgebra B is a degree 0 part generated by the following three polynomials

$$\xi := aa^* - cc^*, \quad \eta := ac^* + ca^*, \quad \zeta := i(ac^* - ca^*),$$

satisfying $\xi^2 + \eta^2 + \zeta^2 = 1$. This means that B is an algebra of functions on the two-sphere, $B = \mathcal{O}(S^2)$.

Since $\mathrm{SU}(2)$ is a group, $A = \mathcal{O}(\mathrm{SU}(2))$ is a Hopf algebra with comultiplication

$$\Delta_A(a) = a \otimes a^* - c^* \otimes c, \qquad \Delta_A(c) = c \otimes a + a^* \otimes c.$$

The $\mathbb{Z}$-grading comes from the Hopf algebra map

$$\pi\colon A \to H, \qquad \pi(a) = x, \quad \pi(a^*) = x^*, \quad \pi(c) = \pi(c^*) = 0.$$

The algebra $\mathcal{O}(S^2)$ is an algebra of functions on a homogenous space. The connection is determined by an H-colinear map (see Example 3.25)

$$\iota\colon H = \mathcal{O}(\mathrm{U}(1)) \to \mathcal{O}(\mathrm{SU}(2)) = A, \qquad x^n \mapsto a^n, \quad x^{*n} \mapsto a^{*n}, \quad 1 \mapsto 1.$$

The resulting strong connection form, $\ell(x) = a^* da + c^* dc$, is known as the Dirac monopole connection

To compute the Chern-Galois character (for line bundles), take smooth functions on $\mathrm{SU}(2)$ and define

$$\begin{aligned} A := \widehat{C}(\mathrm{SU}(2)) &= \{f \in C^\infty(\mathcal{U}(2)) \mid \widehat{\rho}(f) \in C^\infty(\mathrm{SU}(2)) \otimes \mathcal{O}(\mathrm{U}(1))\} \\ &= \bigoplus_{n\in\mathbb{Z}} C^\infty_n(\mathrm{SU}(2)), \end{aligned}$$

where $\widehat{\rho}(f)(x, g) = f(xg)$, and $C^\infty_n(\mathrm{SU}(2))$ is the algebra of smooth functions on S^2 and all polynomials of $\mathbb{Z}$-degree n on $\mathrm{SU}(2)$ (recall that $\mathcal{O}(\mathrm{SU}(2))$ is a strongly $\mathbb{Z}$-graded algebra). Then $\widehat{C}(\mathrm{SU}(2))$ is a Hopf-Galois extension of $B := C^\infty(S^2)$ by $H = \mathcal{O}(\mathrm{U}(1)) \simeq \mathbb{C}[\mathbb{Z}]$.

For any $n \in \mathbb{Z}$, take a one-dimensional left H-comodule $(V_n, {}^{V_n}\rho)$ with coaction

$${}^{V_n}\rho(v) := x^n \otimes v.$$

Then

$$\Gamma_{-n} = \widehat{C}(\mathrm{SU}(2)) \Box_{\mathcal{O}(\mathrm{U}(1))} V_n = C^\infty_n(\mathrm{SU}(2))$$

is a line bundle over S^2. The idempotents for Γ_{-n} coming from the strong connection induced by ι can be written explicitely. For example, for Γ_{-1},

$$\mathbf{E}_{-1} = \begin{pmatrix} aa^* & ac^* \\ cA^* & cc^* \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} 1+\xi & \eta - i\zeta \\ \eta + i\zeta & 1-\xi \end{pmatrix}.$$

Furthermore, the Chern-Galois character is given by the following diagram

$$\begin{array}{ccccc} \mathrm{Rep}(\mathrm{U}(1)) = \mathrm{Rep}(\mathcal{O}(\mathrm{U}(1))) & \longrightarrow & \mathrm{K}_0(C^\infty(S^2)) & \longrightarrow & \mathrm{H}^\lambda_{ev}(C^\infty(S^2)) \\ & & \| & & \downarrow \\ & & \mathrm{K}^0(S^2) & \longrightarrow & \mathrm{H}_{\mathrm{dR}}(S^2)\,. \end{array}$$

In particular, the first two terms of the Chern (or Chern-Galois) character come out as

$$\mathrm{Tr}(E_{-1}) = 1,$$
$$\mathrm{Tr}(E_{-1}dE_{-1}dE_{-1}) = \tfrac{1}{2}(\xi d\eta \wedge d\zeta + \eta d\zeta \wedge d\xi + \zeta d\xi \wedge d\eta).$$

Integration over the sphere S^2 gives the Chern number

$$\mathrm{ch}(\Gamma_{-1}) = \frac{1}{2\pi i}\int S^2\, \mathrm{Tr}(E_{-1}dE_{-1}dE_{-1}) = -1.$$

Similarly, for Γ_{-n} we compute

$$\mathrm{ch}(\Gamma_{-n}) = \frac{1}{2\pi i}\int S^2\, \mathrm{Tr}(E_n dE_n dE_n) = -n.$$

References. Beck [**?**]; Böhm and Brzeziński [**?**]; Brzeziński and Hajac [**?**]; Brzeziński and Majid [**?**]; Connes [**?**]; Loday [**?**].