Lecture notes on singular integrals, projections, multipliers and rearrangements

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Preface

In March 2011, Michal Wojciechowski (IMPAN Warszawa) organized a winter school in Harmonic Analysis at the Bedlewo Conference Center of the Institute for Mathematics PAN.

The lectures at the winter school were primarily addressed at graduate students of Warsaw University who had significant knowledge in Harmonic Analysis. The programme consisted of four mini-courses covering the following subjects

- Non commutative harmonic analysis (M. Bozeko, Wroclaw)
- The Heisenberg group (J. Dziubanski, Wroclaw)
- The Banach space approximation property (A. Szankowski, Jerusalem)
- Singular integral operators (P. Müller, Linz)

With considerable delay - for which the first named author assumes responsibility - the present lecture notes grew out of the minicourse on singular integral operators. Our notes reflect the original selection of classical and recent topics presented at the winter school. These were,

- H^p atomic decomposition and absolutely summing operators
- Rearrangement operators, interpolation, extrapolation and boundedness
- Calderón-Zygmund operators on $L^p(L^q), 1 < p, q < \infty$.
- Calderón-Zygmund operators on $L^p(L^1), 1 < p < \infty$.
- Interpolatory estimates, compensated compactness, Haar projections.

In the present notes we worked out Figiel's proof of the David and Journé theorem, Kislyakov's proof of Bourgain's theorem on operators in $L^p(L^1)$, the proof of Kislyakov's embedding theorem, and our recent, explicit formulas for the Pietsch measure of absolutely summing multiplication operators. Where appropriate, we emphasized the role played by dyadic tools, such as averaging projections, rearrangements and multipliers acting on the Haar system.

The authors acknowledge financial support during the preparation of these notes provided by the Austrian Science Foundation under Project Nr. P22549, P23987.

Linz, July 2015 Paul F. X. Müller Johanna Penteker

1 Singular integral operators: Basic examples

We start these notes with a short list of important singular integral operators, together with brief remarks concerning their usage and motivation. The classic texts by L. Ahlfors [Ahl66], M. Christ [Chr90] R. Coifman and Y. Meyer [CM78], V.P. Khavin and N. K. Nikolski [KN92], P. Koosis [Koo98], E. M. Stein [Ste70] or A. Zygmund [Zyg77], provide suitable references to the material reviewed in this section. The exception to this rule is the last paragraph on Fourier multipliers where we discuss the comparatively recent Marcinkiewicz decomposition of the Banach couple $(H^p(\mathbb{T}), SL^\infty \cap H^\infty(\mathbb{T}))$, for which the reference is [JM04].

1.1 Hilbert transformation

Here we recall the Hilbert transform, its basic L^p estimates and the link to the convergence of the partial sums of Fourier series.

Let $f \in L^2(]-\pi, \pi[)$. The Hilbert Transform $H(f)$ of f is defined by

$$H(f)(\theta) = \lim_{\epsilon \to 0} \int_{\{|t|>\epsilon\}} \frac{f(\theta - t)}{2\tan\frac{t}{2}} \frac{dt}{\pi},$$

where we take $f(\theta + 2\pi) = f(\theta)$. The integral on the right-hand side exists a.e. if $f \in L^1(]-\pi, \pi[)$ and defines a construction on $L^2(]-\pi, \pi[)$,

$$\|H(f)\|_{L^2} \leq \|f\|_{L^2}.$$

(See P. Koosis [Koo98])

Motivation

We recall the use of the Hilbert transform in the study of holomorphic functions in the unit disk, and cite the book of Koosis [Koo98] as a classic introductory text. Let g be defined on the unit circle by $g(e^{i\theta}) = f(\theta)$ with Fourier expansion

$$g(e^{i\theta}) = \sum_{n=-\infty}^{\infty} A_n e^{in\theta}$$

and harmonic extension to the unit disk given by

$$u(re^{i\theta}) = \sum_{n=-\infty}^{\infty} r^{|n|} A_n e^{in\theta}, \quad r < 1.$$

The harmonic conjugate of u is then

$$\tilde{u}(re^{i\theta}) = -\sum_{n=-\infty}^{\infty} i\mathrm{sign}(n) r^{|n|} A_n e^{in\theta}, \quad r < 1,$$

since

$$u(re^{i\theta}) + i\tilde{u}(re^{i\theta}) = A_0 + \sum_{n=1}^{\infty} 2r^{|n|} A_n e^{in\theta}$$

is analytic. By Fatou's theorem the L^2 function

$$\tilde{g}(e^{i\theta}) = -\sum_{n=-\infty}^{\infty} i\mathrm{sign}(n) A_n e^{in\theta},$$

is the a.e. boundary value of $\tilde{u}(re^{i\theta})$, i.e.

$$\tilde{g}(e^{i\theta}) = \lim_{r\to 1} \tilde{u}(re^{i\theta}) \quad \text{for a.e } \theta,$$

and

$$\tilde{g}(e^{i\theta}) = \lim_{\epsilon\to 0} \int_{\{|t|>\epsilon\}} \frac{f(\theta - t)}{2\tan\frac{t}{2}} \frac{dt}{\pi} = H(f).$$

Consequently Plancherel's theorem gives L^2 estimates for the Hilbert transform

$$\|Hf\|_{L^2} \le \|f\|_{L^2}. \tag{1.1}$$

Inequality (1.1) continues to hold true in the reflexive L^p spaces.

Theorem 1.1 (M. Riesz)**.** *Let $f \in L^p(]-\pi, \pi[)$, $1 < p < \infty$. Then (taking $f(\theta + 2\pi) = f(\theta)$)*

$$H(f)(\theta) = \lim_{\epsilon\to 0} \int_{\{|t|>\epsilon\}} \frac{f(\theta - t)}{2\tan\frac{t}{2}} \frac{dt}{\pi}$$

exists a.e. and

$$\|Hf\|_{L^p} \le C_p \|f\|_{L^p},$$

where $C_p \sim p^2/(p-1)$.

By algebraic manipulations the Theorem of Riesz implies the norm-convergence of Fourier series in L^p. Let $f \in L^p(]-\pi, \pi[)$, $1 < p < \infty$ with formal Fourier series

$$\sum_{n=-\infty}^{\infty} A_n e^{in\theta}.$$

Let

$$S_N(f)(\theta) = \sum_{n=-N}^{N} A_n e^{in\theta}$$

and

$$P(f)(\theta) = \frac{1}{2}(\,\mathrm{Id} + iH)f(\theta) + \frac{1}{2}A_0.$$

By the above computation we have

$$P(f)(\theta) = \sum_{n=0}^{\infty} A_n e^{in\theta}.$$

The partial sums $S_N(f)$ coincide with

$$e^{-iN\theta}P(e^{iN(\cdot)}f(\cdot)) - e^{i(N+1)\theta}P(e^{-i(N+1)(\cdot)}f(\cdot)).$$

This identity links the Hilbert transform to partial sums of Fourier series. By the Riesz theorem we have convergence of the partial sums $S_N(f)$ in $L^p, 1 < p < \infty$,

$$\|S_N(f) - f\|_{L^p} \to 0 \quad \text{for} \quad N \to \infty,$$

and

$$\|S_N(f)\|_{L^p} \leq 4\|Hf\|_{L^p} \leq C_p\|f\|_{L^p}, \quad (1 < p < \infty),$$

$C_p \sim p^2/(p-1)$.

1.2 The Riesz transforms

We recall the Riesz transforms, their definition as principal value integral, the L^p norm estimate and basic identities relating Riesz transforms to the Laplacian and we work out the identities relating Riesz transforms to the Helmholtz projection onto gradient vector fields

The Riesz transforms

Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. The Riesz transform of f is defined by

$$R_j(f)(x) = c_n \lim_{\epsilon \to 0} \int_{|t|>\epsilon} f(x-t)\frac{t_j}{|t|^{n+1}}dt, \quad j = 1, \dots, n, \tag{1.2}$$

with $c_n = \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}}$.

Theorem 1.2 (M. Riesz)**.** *The integral on the right-hand side in* (1.2) *exists a.e. and defines a bounded linear operator for which*

$$\|R_j\|_p \leq C_p, \quad 1 < p < \infty.$$

Let $\mathcal{F}$ denote the Fourier transform normalized so that

$$\mathcal{F}f(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(x)dx.$$

We have then the following identities identifying the Fourier multiplier of Riesz transform.

$$\mathcal{F}(\partial_j f) = i\xi_j \mathcal{F}(f) \tag{1.3}$$

$$R_j(f))(x) = i\mathcal{F}^{-1}(\frac{y_j}{|y|}\mathcal{F}(f)(.))(y) \tag{1.4}$$

Note that (1.3) is a direct consequence of the Fourier transform. Equation (1.4) however, is not at all trivial, its verification requires a delicate calculation carried out for instance in the book of Stein [Ste70].

Motivation

Standard applications of Riesz transforms, their relation to the Fourier transform and its L^p estimates pertain to regularity of solutions to the Poisson problem and to the Helmholtz projection onto gradient vector fields.

Poisson problems. We start with the regularity estimates for solutions of the Poisson problem following Stein [Ste70]. Let $u \in C_c^2(\mathbb{R}^n)$ and

$$\Delta u(x) = f(x).$$

We show that

$$f \in L^p \Longrightarrow \frac{\partial^2 u}{\partial x_j \partial x_i} \in L^p. \tag{1.5}$$

The Fourier Transformation $\mathcal{F}$ provides the link to Riesz transforms:

$$\mathcal{F}(f)(y) = \mathcal{F}(\Delta u)(y) = -|y|^2 \mathcal{F}(u)(y)$$
$$\mathcal{F}(u_{x_j,x_k})(y) = -y_j y_k \mathcal{F}(u)(y).$$

Hence,

$$\mathcal{F}(u_{x_j,x_k})(y) = \frac{y_j y_k}{|y|^2}\mathcal{F}(f)(y).$$

On the other hand

$$\mathcal{F}(R_j(f))(y) = i\frac{y_j}{|y|}\mathcal{F}(f)(y).$$

Summarizing we have

$$u_{x_j,x_k} = -R_j(R_k(f))$$

and therefore,

$$\left\|\frac{\partial^2 u}{\partial x_j \partial x_i}\right\|_p \leq \|R_j\|_p \|R_k\|_p \|f\|_p = C_p^2 \|f\|_p.$$

Gradient vector fields. Here we show how to identify the Helmholtz projections using Riesz transforms. In [Bou92] this identity was the starting point for the proof that the Banach couple of Sobolev spaces $(W^{1,\infty}, W^{1,1})$ is $K-$ closed in (L^{∞}, L^{1}).

Let $S_p = \{(f_{x_1}, \dots, f_{x_n}) \mid f \in W^{1,p}(\mathbb{R}^n)\}$. Then

$$S_p \subset L^p \oplus \ldots \oplus L^p = \bigoplus L^p$$

and especially

$$S_2 \subset L^2 \oplus \ldots \oplus L^2 = \bigoplus L^2.$$

Since S_2 is a closed subspace there exists a projection

$$P\colon \bigoplus L^2 \to S_2, P^2 = P,$$

onto S_2 The next result gives the singular integral representation of the projection P.

Theorem 1.3. *Let* $\mathbf{f} = (f_1, \dots, f_n) \in \bigoplus L^2$. *Then*

$$P(\mathbf{f}) = -\left(R_i\left(\sum_{j=1}^{n} R_j f_j\right)\right)_{i=1}^{n},$$

where R_k *is the Riesz transform.*

Proof. Use the identity

$$P(\mathbf{f}) = \left(\mathcal{F}^{-1}\left(\xi_i \sum_{j=1}^{n} \frac{\xi_j}{\mid \xi \mid^2} \mathcal{F}(f_j)\right)\right)_{i=1}^{n}$$

to show first that the image of P is in S_2:

$$\forall\, \mathbf{f} \in \bigoplus L^2 : \quad P(\mathbf{f}) \in S_2,$$

i.e.

$$\exists\, h \in W^{1,2}(\mathbb{R}^n) : \ P(\mathbf{f}) = (h_{x_1}, \dots, h_{x_n}).$$

If we take

$$h = -\sqrt{-1}\mathcal{F}^{-1}\left(\sum_{j=1}^{n} \frac{\xi_j}{\mid \xi \mid^2} \mathcal{F} f_j\right),$$

which is in $W^{1,2}(\mathbb{R}^n)$ as $f_j \in L^2(\mathbb{R}^n)$, then the following identity holds

$$(P(\mathbf{f}))_i = \sqrt{-1}\mathcal{F}^{-1}(\xi_i \mathcal{F} h) = h_{x_i}.$$

Therefore, $P(\mathbf{f}) = (h_{x_1}, \dots, h_{x_n})$, i.e. the image of $\mathbf{f}$ is the gradient of an element of $W^{1,2}(\mathbb{R}^n)$.

Next we show that

$$\forall\, \mathbf{g} \in S_2 : \quad P(\mathbf{g}) = \mathbf{g}.$$

Take any $\mathbf{g} \in S_2$. Then, by definition, there is $f \in W^{1,2}(\mathbb{R}^n)$ such that $f_{x_i} = g_i$ and

$$(\mathcal{F} f_{x_i})(\xi) = \sqrt{-1}\xi_i(\mathcal{F} f)(\xi) = (\mathcal{F} g_i)(\xi).$$

Hence, we have

$$\begin{aligned}(P(\mathbf{g}))_i &= \mathcal{F}^{-1}\left(\xi_i \sum_{j=1}^{n} \frac{\xi_j}{\mid \xi \mid^2}\mathcal{F} g_j\right) = \sqrt{-1}\,\mathcal{F}^{-1}\left(\xi_i \sum_{j=1}^{n} \frac{\xi_j}{\mid \xi \mid^2}\xi_j \mathcal{F} f\right) \\ &= \sqrt{-1}\,\mathcal{F}^{-1}\left(\xi_i \mathcal{F} f\right) = g_i.\end{aligned}$$

In summary,

$$\forall\, \mathbf{f} \in \bigoplus L^2 : \quad P(P(\mathbf{f})) = P(\mathbf{f}).$$

□

Divergence free vector fields Projections onto divergence free vector fields are given by

$$Q(\mathbf{f}) = \mathbf{f} - P(\mathbf{f}).$$

Indeed Q is a projection, since

$$Q^2 = (\text{ Id} - P)^2 = \text{ Id} - 2P + P^2 = \text{ Id} - P$$

and its range is divergence free,

$$divP(\mathbf{f}) = div\mathbf{f}.$$

1.3 Beurling transformation

Let $f : \mathbb{C} \to \mathbb{C}$, $f \in C_c^1(\mathbb{C})$. We define the *Beurling transform* as follows

$$Bf(z) = -\frac{1}{\pi}\text{pv}\int_{\mathbb{C}} \frac{f(w)}{(z-w)^2} d\lambda(w),$$

where λ denotes the Lebesgue measure on $\mathbb{C}$.

We identify first the the Fourier multiplier of the Beurling transform. View $\mathbb{R}^2$ as $\mathbb{C}$ with coordinates $z = x + iy$. Let $\zeta = \xi + i\eta$. The Fourier transform of $f : \mathbb{C} \to \mathbb{C}$ is written by

$$\hat{f}(\xi, \eta) = \int_{\mathbb{R}^2} e^{-2\pi i(\xi x + \eta y)} f(x + iy)\, dx\, dy.$$

Then the following identity holds

$$\widehat{Bf}(\xi, \eta) = \frac{\xi - i\eta}{\xi + i\eta}\hat{f}(\xi, \eta)$$

which can also be written as

$$\widehat{Bf}(\zeta) = \frac{\overline{\zeta}}{\zeta}\hat{f}(\zeta).$$

Motivation

The Beurling transform is an important tool in obtaining existence and regularity results for the Beltrami equation and for quasi conformal mappings.

Let $\mu \in L^\infty$, with $\|\mu\|_\infty < 1$ with compact support. We want to determine $f : \mathbb{C} \to \mathbb{C}$ (with high regularity) that solves the elliptic system

$$\overline{\partial} f = \mu \partial f,$$

where

$$\partial = \frac{1}{2}(\partial_x - i\partial_y), \quad \text{and} \quad \overline{\partial} = \frac{1}{2}(\partial_x + i\partial_y).$$

Solution: Try to find f of the form

$$f(z) = z + h(z), \quad \text{and} \quad h(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{g(w)}{(z-w)} d\lambda(w).$$

With this setting equation $\overline{\partial} f = \mu \partial f$ translates into conditions on h:

$$\overline{\partial} h = \mu(1 + \partial h).$$

Let $\overline{\partial} h = g$. Then $\partial h = B(g)$. Hence, we require for g

$$g = \mu(1 + \partial h) = \mu(1 + B(g)).$$

This is equivalent to

$$(\text{ Id} - \mu B) g = \mu.$$

A solution g is given by

$$g(w) = (\text{ Id} - \mu B)^{-1}(\mu)(w) = \sum_{j=1}^{\infty} (\mu B)^j (\mu)(w).$$

Beurling [Beu89, Ahl66] proves the essential estimates:

$$\|B\|_2 = 1 \quad \text{and} \quad \lim_{\epsilon \to 0} \|B\|_{2+\epsilon} = 1.$$

Hence, we get $g \in L^{1+\epsilon}$ with compact support. h is therefore continuous and tends to 0 at infinity.

1.4 Cauchy integrals on Lipschitz domains

Let $A : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function so that $\|A'\|_\infty < \infty$. Put

$$\mathcal{C}_A f(x) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(y)}{(x + iA(x)) - (y + iA(y))} dy,$$

formally. This is strongly singular for Lipschitz Domains and weakly singular when $A : \mathbb{R} \to \mathbb{R}$ is $C^{1+\eta}, \eta > 0$, see [Chr90].

Why then is it important to have estimates for Lipschitz domains? The answer is two fold: First the only the class of Lipschitz domains is invariant under and also dilations: Put

$$\delta_r f(x) = f(rx), \quad A_r(x) = rA(r^{-1}x) \quad r > 0.$$

Then

$$\mathcal{C}_A \circ \delta_r = \delta_r \circ \mathcal{C}_{A_r}.$$

And now, A_r has exactly the same Lipschitz norm as does A, i.e. $\|A_r'\|_\infty = \|A'\|_\infty$. Second if one takes two perfectly smooth domains, such as two half planes, then their intersection is at best a Lipschitz domain.

Theorem 1.4 (Coifman-McIntosh-Meyer, [CMM82b])**.** *Let $1 < p < \infty$. Then*

$$\|\mathcal{C}_A\|_p \leq C_p(C_1 + C_2\|\nabla A\|_\infty).$$

Motivation

The main reason for considering the operators $\mathcal{C}_A$ is their connection to the Cauchy integral on Lipschitz graphs. Let $\Gamma = \{x + iy : y = A(x)\} \subseteq \mathbb{C}$ and $g \in L^2(\Gamma, ds)$, where "ds" denotes the arc length. Let $z = x + iA(x)$. Then the Cauchy integral of g is given by

$$\frac{1}{2\pi i}\int_\Gamma \frac{g(w)}{(z-w)}dw = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{g(u+iA(u))}{(z-(u+iA(u)))}(1+iA'(u))du = \mathcal{C}_A G(x),$$

where $G(u) = g(u + iA(u))(1 + iA'(u))$. Thus the L^p estimates for $\mathcal{C}_A$ translate immediately into estimates for Cauchy integrals on Lipschitz graphs.

Let $E \subseteq \Gamma$, with $|E| > 0$. The L^2 estimate for the Cauchy integral on Lipschitz curves are now a corner stone in the proof of the the fact that E has then positive analytic capacity, i.e.

$$\exists\, f_E : \mathbb{C} \setminus E \to \mathbb{C} : \ \|f_E\|_\infty < \infty \text{ and } f_E'(\infty) \neq 0.$$

See [Chr90].

1.5 Double layer potential

Let $A : \mathbb{R}^{n-1} \to \mathbb{R}$ be a Lipschitz function and $x, y \in \mathbb{R}^{n-1}$. We define

$$K(x,y) = \frac{A(x) - A(y) - \sum_{j=1}^{n-1}(x_j - y_j)\frac{\partial A}{\partial y_j}(y)}{(|x-y|^2 + |A(x) - A(y)|^2)^{n/2}}.$$

K is weakly singular if $A \in C^{1+\delta}$. It is a strongly singular if $A \in C^1$ or $\nabla A \in L^\infty$. Let

$$S_A(g)(x) = \int_{\mathbb{R}^{n-1}} g(y)K(x,y)dy.$$

Coifman-McIntosh-Meyer [CMM82b, CMM82a] proved that

$$\|S_A\|_p \leq C_p(C_1 + C_2\|\nabla A\|_\infty).$$

Motivation

The solution to the Dirichlet problem by means of double layer potentials provides the main motivation for the integral operators we consider here.

Let $\Omega \subseteq \mathbb{R}^n$ be open, bordered (locally) by a Lipschitz surface. That is to say that for every $x \in \partial\Omega$ there exists a $\delta > 0$ and a Lipschitz function $A\colon \mathbb{R}^{n-1} \to \mathbb{R}$ such that (possibly after rotation of the coordinate axes)

$$B(x,\delta) \cap \Omega = \{x \in B(x,\delta) : \, x_n > A(x_1, \dots, x_{n-1})\}. \tag{1.6}$$

The method of layer potentials provides solutions for the Dirichlet problem:

$$\begin{aligned} \Delta u &= 0 \quad \text{ on } \Omega \\ u\big|_{\partial\Omega} &= f, \end{aligned} \tag{1.7}$$

where f is prescribed on $\partial\Omega$. Let $y \in \partial\Omega$ and $\nu(y)$ the exterior unit normal to $\partial\Omega$ at y. We set

$$N(x) = \frac{1}{c_n}|x|^{2-n}, \quad c_n = (n-2)\omega_n,$$

where $\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ is the volume of the surface of the unit ball in $\mathbb{R}^n$. Then $N(x)$ is the fundamental solution for the Laplacian in $\mathbb{R}^n$, $n \geq 3$.

One does not know the Green's function for Ω, so one hopes that

$$\langle \nu(z), \nabla_z N(x-z)\rangle \tag{1.8}$$

is a good approximation, as is the case for smooth domains. Note that up to a constant factor

$$K(x,y) = \langle \nu(z), \nabla_y N(x-z)\rangle,$$

where $z = (y, A(y))$, $y \in \mathbb{R}^{n-1}$, cf. [Tor86].

To $x \in \Omega$ and f form the harmonic function

$$u_f(x) = \int_{\partial\Omega} \langle \nu(y), \nabla_y N(x-y)\rangle \, f(y) d\sigma(y),$$

where σ denotes the surface measure on $\partial\Omega$. This is called the double layer potential of the function f, cf. [Ver84]. According to a theorem of [Ver84] boundary values of u_f exist for a.e. $p \in \partial\Omega$ and

$$\lim_{x\to p} u_f(x) = \left(\frac{1}{2}\,\mathrm{Id} + S_A\right) f(p).$$

More precisely, if $p = (y, A(y))$, $y \in \mathbb{R}^{n-1}$ then by [Ver84, Theorem 1.10],

$$\begin{aligned}\lim_{x\to p} u_f(x) &= \frac{1}{2}f(p) + \int_{\partial\Omega} \langle \nu(z), \nabla_y N(p-z)\rangle\, f(y)d\sigma(z)\\ &= \frac{1}{2}f(p) + \int_{\mathbb{R}^{n-1}} f((y, A(y))K(p,y)d(y)\\ &= (\ \mathrm{Id} + S_A)(f)(p).\end{aligned}$$

In summary, $u = u_f$ is a solution to the Dirichlet problem

$$\begin{aligned}\Delta u = 0 \quad &\text{ on } \Omega\\ u\big|_{\partial\Omega} = g,&\end{aligned} \tag{1.9}$$

where $g = (Id + S_A)(f)(p)$. In order to get a solution for (1.7), by this procedure we would like to know that the operator $(Id + S_A)$ is invertible. Should that be the case we put $h = (Id + S_A)^{-1}(f)$, and observe that $u = u_h$ solves the Dirichlet problem with boundary values

$$(Id + S_A)(h) = (Id + S_A)(Id + S_A)^{-1}(f) = f.$$

It remains to point out Verchota proved in [Ver84] that $(\ \mathrm{Id} + S_A)$ is invertible whenever A is a Lipschitz function.

1.6 Fourier multipliers

A considerable portion of present day Harmonic Analysis, Probability and PDE has its origin in the Fourier multiplier theorems of Littlewood-Paley. The classical reference to Fourier multipliers is Volume II in Zygmund [Zyg77].

Let $u \in L^p([0, 2\pi])$ with Fourier series

$$u(\theta) = \sum_{k=1}^{\infty} a_k \cos k\theta + b_k \sin k\theta.$$

We form the dyadic blocks

$$\Delta_n(u)(\theta) = \sum_{k=2^n}^{2^{n+1}-1} a_k \cos k\theta + b_k \sin k\theta$$

and define the transform

$$v(\theta) = \sum_{n=0}^{\infty} \epsilon_n \Delta_n(u)(\theta), \quad \epsilon_n \in \{-1, 1\}.$$

Theorem 1.5 (Littlewood-Paley). *For every $1 < p < \infty$ there exists a constant $C_p > 0$ so that for all $\epsilon_n \in \{-1, 1\}$,*

$$\|v\|_{L^p} \le C_p \|u\|_{L^p},$$

and $C_p \to \infty$ for $p \to \infty$ or $p \to 1$.

Use [Ste70] and [Zyg77] for reference. We next turn to describing one of the most useful tools in treating Fourier multiplier problems.

Littlewood-Paley Function

For $z \in \mathbb{D}$, let

$$P_\theta(z) = \frac{1-|z|^2}{|e^{i\theta}-z|^2}$$

denote the Poisson kernel for the unit disk. Let $u(z)$, $z \in \mathbb{D}$ denote the harmonic extension of $u \in L^p([0,2\pi])$ obtained by integration against the Poisson kernel P_θ, i.e.

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})P_\theta(z)d\theta$$

The Littlewood Paley function

$$g^2_{\mathbb{D}}(u)(\theta) = \int_{\mathbb{D}} |\nabla u(z)|^2 \log\frac{1}{|z|} P_\theta(z)dA(z)$$

plays a *central* role in proving the multiplier theorem (Theorem 1.5). Its proof (see [Ste70, p.96, p.106] or [Zyg77] for reference) consists of basically two independent components

1. Pointwise estimates between the g functions

$$g_{\mathbb{D}}(v)(\theta) \le Cg_{\mathbb{D}}(u)(\theta),$$

2. L^p-integral estimates (see again [Ste70], [Zyg77] or [Bañ86]),

$$\frac{c}{\sqrt{p}}\|v\|_{L^p} \le C_p\|g_{\mathbb{D}}(v)\|_{L^p} \le C\sqrt{p}\|v\|_{L^p}, \quad 2 \le p < \infty.$$

Uniformly bounded Littlewood Paley functions

$SL^\infty(\mathbb{T})$ denotes the space of all functions u with uniformly bounded Littlewood Paley Function, i.e.

$$SL^\infty(\mathbb{T}) = \{u : \mathbb{T} \to \mathbb{R}\,(\text{or }\mathbb{C}) : g_{\mathbb{D}} \in L^\infty(\mathbb{T})\},$$

with norm

$$\|u\|_{SL^\infty(\mathbb{T})} = \|g_{\mathbb{D}}(u)\|_\infty + \int |d|dm.$$

If we consider the definition of the Littlewood Paley function $g_{\mathbb{D}}$, then we see that the condition $\|g_{\mathbb{D}}(u)\|_\infty < \infty$ controls the growth of u and also its *oscillations.* Chang, Wilson, Wolff [CWW85] proved that there exists $c > 0$ so that

$$\int_0^{2\pi} \exp(cu^2(\theta))d\theta \le \|g_{\mathbb{D}}(u)\|_\infty < \infty.$$

Hence, by [Bañ86, Theorem 1] we have $SL^\infty \subset \mathrm{BMO}$.

On the other hand there exists $E \subseteq [0,2\pi[$ so that

$$\|g^2_{\mathbb{D}}(1_E)\|_\infty = \infty,$$

i.e. there exists functions $1_E \in L^\infty$ such that $g_{\mathbb{D}}(u) \notin L^\infty$. We get two different endpoints of the L^p scale.

$$L^2 \supset \cdots \supset L^p \supset \cdots \supset \mathrm{BMO} \supset \begin{cases} L^\infty \\ SL^\infty(\mathbb{T}) \end{cases}$$

The relation of the endpoint SL^∞ to the L^p scale is clarified by a Marcinkiewicz decomposition and by pointwise multipliers with values in $SL^\infty(\mathbb{T})$.

Multipliers into $SL^\infty(\mathbb{T})$ and Marcinkiewicz-decomposition

Definition 1.6. Let $1 \leq p \leq \infty$. Define the subspace $H^p(\mathbb{T})$ to consist of those $f \in L^p(\mathbb{T})$ with Fourier expansion of the following form

$$f(\theta) = \sum_{n=0}^{\infty} c_n e^{in\theta}.$$

The harmonic expansion of $f \in H^p(\mathbb{T})$ to the disk $\mathbb{D}$ is then given by

$$f(re^{i\theta}) = \sum_{n=0}^{\infty} c_n r^n e^{in\theta}$$

and hence, for $z \in \mathbb{D}$,

$$f(z) = \sum_{n=0}^{\infty} c_n z^n.$$

Thus demanding that $f \in H^p(\mathbb{T})$ is the same as demanding that $f \in L^p(\mathbb{T})$ and that the harmonic extension of f to the unit disc is analytic.

The next theorem gives the Marcinkiewicz decomposition for $H^p(\mathbb{T})$ spaces with $SL^\infty(\mathbb{T}) \cap H^\infty(\mathbb{T})$ as endpoint.

Theorem 1.7 ([JM04]). *To $f \in H^p(\mathbb{T})$ and $\lambda > 0$ there exists $g \in SL^\infty(\mathbb{T}) \cap H^\infty(\mathbb{T})$ so that*

$$\|g\|_{SL^\infty} + \|g\|_\infty \leq C_0 \lambda, \quad \|f - g\|_1 \leq \lambda^{1-p} \|f\|_p^p$$

Non trivial pointwise multipliers.

Theorem 1.8 ([JM04]). *To each $E \subseteq [0, 2\pi[$ there exists $0 \leq m(\theta) \leq 1$ so that*

$$\|m 1_E\|_{SL^\infty} < C_0 \quad \text{and} \quad \int_0^{2\pi} m 1_E d\theta \geq |E|/2.$$

2 Atomic decomposition and multiplication operators

In this section we review Hardy spaces and BMO. We present Fefferman's inequality and the atomic decomposition in the dyadic setting, using the Haar system as our basic tool. Given $u \in H^p$ with Haar expansion $u = \sum_{I \in \mathcal{D}} x_I h_I$ we find explicit weights $w = (\omega_I)$ such that

$$\Big\| \sum_{I \in \mathcal{D}} x_I \varphi_I h_I \Big\|_{H^p} \leq C_p \|u\|_{H^p} \Big(\sum_{I \in \mathcal{D}} |\varphi_I|^2 \omega_I \Big)^{\frac{1}{2}},$$

whenever (φ_I) is a bounded sequence of scalars. This recent application of the atomic decomposition is motivated by problems on absolutely summing operators acting on ℓ^∞.

2.1 Dyadic intervals

Definition 2.1. An interval $I \subseteq [0,1]$ is called a *dyadic interval*, if there exist non-negative integers ℓ and k with $0 \leq k \leq 2^\ell - 1$ such that

$$I = I_{\ell,k} = \Big[\frac{k}{2^\ell}, \frac{k+1}{2^\ell} \Big[.$$

The length of a dyadic interval $I_{\ell,k}$ is given by $|I_{\ell,k}| = 2^{-\ell}$. We denote by $\mathcal{D}$ the set of dyadic intervals, i.e.

$$\mathcal{D} = \{ I \subseteq [0,1] : I \text{ is dyadic interval} \}.$$

For every $N \in \mathbb{N}_0$ let $\mathcal{D}_N$ be the set of dyadic intervals with length greater than or equal to 2^{-N}, i.e.

$$\mathcal{D}_N = \{ I \in \mathcal{D} : |I| \geq 2^{-N} \} \tag{2.1}$$

or equivalently

$$\mathcal{D}_N = \{ I_{\ell,k} : 0 \leq \ell \leq N, 0 \leq k \leq 2^\ell - 1 \}. \tag{2.2}$$

Carleson constant

Let $\mathcal{C} \subseteq \mathcal{D}$. We define the Carleson constant of $\mathcal{C}$ as follows

$$[\![\mathcal{C}]\!] = \sup_{I \in \mathcal{C}} \frac{1}{|I|} \sum_{J \subseteq I, J \in \mathcal{C}} |J|. \tag{2.3}$$

If $\mathcal{C}$ is non-empty, then $[\![\mathcal{C}]\!] \geq 1$, otherwise $[\![\mathcal{C}]\!] = 0$.

Blocks of dyadic intervals

Let $\mathcal{L}$ be a collection of dyadic intervals. We say that $\mathcal{C}(I) \subseteq \mathcal{L}$ is a block of dyadic intervals in $\mathcal{L}$ if the following conditions hold:

1. The collection $\mathcal{C}(I)$ has a unique maximal interval, namely the interval I.
2. If $J \in \mathcal{C}(I)$ and $K \in \mathcal{L}$, then
$$J \subseteq K \subseteq I \text{ implies } K \in \mathcal{C}(I).$$

The Haar system

We define the L^∞-normalised *Haar system* $(h_I)_{I\in\mathcal{D}}$ indexed by the dyadic intervals $\mathcal{D}$ as follows:
$$h_I = \begin{cases} 1 & \text{on the left half of } I, \\ -1 & \text{on the right half of } I, \\ 0 & \text{otherwise.} \end{cases}$$

2.2 BMO and dyadic Hardy spaces

Let $(x_I)_{I\in\mathcal{D}}$ be a real sequence. We define $f = (x_I)_{I\in\mathcal{D}}$ to be the real vector indexed by the dyadic intervals.

The space BMO consist of vectors $f = (x_I)_{I\in\mathcal{D}}$ for which
$$\|f\|_{\mathrm{BMO}} = \sup_{I\in\mathcal{D}} \left(\frac{1}{|I|} \sum_{J\subseteq I} |x_J|^2 |J| \right)^{\frac{1}{2}} < \infty. \tag{2.4}$$

We define the square function of f as follows
$$S(f)(t) = \left(\sum_{I\in\mathcal{D}} |x_I|^2 1_I(t) \right)^{\frac{1}{2}}, \; t \in [0,1]. \tag{2.5}$$

The space H^p, $0 < p < \infty$, consists of vectors $f = (x_I)_{I\in\mathcal{D}}$ for which
$$\|f\|_{H^p} = \|S(f)\|_{L^p([0,1])} < \infty. \tag{2.6}$$

For $1 \le p < \infty$, (2.6) defines a norm and H^p is a Banach space. For $0 < p < 1$, (2.6) defines a quasi-norm and the resulting Hardy spaces H^p are quasi-Banach spaces, cf.[Woj97].

For convenience we identify $f = (x_I)_{I\in\mathcal{D}} \in \mathrm{BMO}$ resp. $f = (x_I)_{I\in\mathcal{D}} \in H^p$ with its formal Haar series
$$f = \sum_{I\in\mathcal{D}} x_I h_I.$$

Paley's theorem ([Pal32], see also [Mül05]) asserts that for all $1 < p < \infty$ there exists a constant A_p such that for all $f \in L^p([0,1])$ given by $f = \sum_{I \in \mathcal{D}} x_I h_I$ the following holds

$$\frac{1}{A_p} \|f\|_{L^p} \le \|S(f)\|_{L^p} \le A_p \|f\|_{L^p}. \tag{2.7}$$

This theorem identifies H^q as the dual space of H^p, where $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p < \infty$.

Fefferman's inequality ([FS72], see also [Mül05])

$$\left| \int fh \right| \le 2\sqrt{2}\, \|f\|_{H^1} \|h\|_{\mathrm{BMO}}, \tag{2.8}$$

and a theorem to the effect that every continuous linear functional $L : H^1 \to \mathbb{R}$ is necessarily of the form $L(f) = \int f\varphi$ with $\|\varphi\|_{\mathrm{BMO}} \le \|L\|$ identify BMO as dual space of H^1, (see [FS72],[Gar73],[Mül05]).

The Haar support

Let $f \in \mathrm{BMO}$ resp. $f \in H^p$ be given by its formal Haar series

$$f = \sum_{I \in \mathcal{D}} x_I h_I,$$

where $(x_I)_{I \in \mathcal{D}}$ is a real sequence. The set $\{I \in \mathcal{D} : x_I \neq 0\}$ is called *Haar support* of f.

2.3 Atomic decomposition

The following theorem states the decomposition of an element in H^p into absolutely summing elements with disjoint Haar support and bounded square function. The decomposition is done by a stopping time argument that may be regarded as a constructive algorithm. The decomposition originates in the work of S. Janson and P.W. Jones [JJ82].

Theorem 2.2 (Atomic decomposition)**.** *For all $0 < p \le 2$ there exists a constant A_p such that for every $u \in H^p$ with Haar expansion*

$$u = \sum_{J \in \mathcal{D}} x_J h_J,$$

there exists an index set $\mathcal{N} \subseteq \mathbb{N}$ and a sequence $(\mathcal{G}_i)_{i \in \mathcal{N}}$ of blocks of dyadic intervals such that for

$$u_i = \sum_{J \in \mathcal{G}_i} x_J h_J, \quad i \in \mathcal{N}$$

the following holds:

i) $(\mathcal{G}_i)_{i \in \mathcal{N}}$ is a disjoint partition of $\mathcal{D}$.

ii) $I_i := \bigcup_{J \in \mathcal{G}_i} J$ *is a dyadic interval*

iii)

$$\|u\|_{H^p}^p \leq \sum_{i \in \mathcal{N}} \|u_i\|_{H^p}^p \leq \sum_{i \in \mathcal{N}} |I_i| \|\mathbb{S}(u_i)\|_\infty^p \leq A_p \|u\|_{H^p}^p. \tag{2.9}$$

The family $(u_i, \mathcal{G}_i, I_i)_{i \in \mathcal{N}}$ *is called the atomic decomposition of* $u \in H^p$.

Remark 2.3. The scalar-valued decomposition procedure can be found in [Tor86]. However, note that the left-hand side inequality of (2.9) depends only on the fact that $(\mathcal{G}_i)_{i \in \mathcal{N}}$ is a sequence of disjoint blocks of dyadic intervals. Therefore, for $\varphi = (\varphi_I)_{I \in \mathcal{D}} \in \ell^\infty(\mathcal{D})$ we have

$$\left\| \sum_{I \in \mathcal{D}} \varphi_I x_I h_I \right\|_{H^p}^p \leq \sum_{i \in \mathcal{N}} \left\| \sum_{I \in \mathcal{G}_i} \varphi_I x_I h_I \right\|_{H^p}^p. \tag{2.10}$$

2.4 Multiplication operators

Given a Banach space X with an unconditional basis $\{x_n\}$ multiplication operators are certainly among the most natural and important operators to consider. Their action is given by

$$x_n \to \mu_n x_n$$

where $(\mu_k) \in \ell^\infty$.

We investigate here multiplication operators acting on the Haar system in the Hardy spaces H^p, $0 < p \leq 2$. We fix $u \in H^p$ with Haar expansion $u = \sum x_I h_I$ and define $\mathcal{M}_u : \ell^\infty(\mathcal{D}) \to H^p$ by

$$\mathcal{M}_u(\varphi) = \sum_{I \in \mathcal{D}} \varphi_I x_I h_I. \tag{2.11}$$

We frequently use the "lattice convention"

$$\varphi \cdot u = \mathcal{M}_u(\varphi) \tag{2.12}$$

to emphasize that $\varphi \in \ell^\infty(\mathcal{D})$ is acting as a multiplier on $u \in H^p$. Since the Haar basis is 1-unconditional in H^p we have

$$\|\mathcal{M}_u(\varphi)\| \leq \|u\|_{H^p} \sup_{I \in \mathcal{D}} |\varphi_I|. \tag{2.13}$$

Here we show in detail that these estimates are self improving in the following sense: If (2.13) holds true then there exists a probability measure ω on $\mathcal{D}$ such that

$$\|\mathcal{M}_u(\varphi)\|_{H^p} \leq C_p \|u\|_{H^p} \left(\int_{\mathcal{D}} |\varphi_I|^2 d\omega(I) \right)^{1/2}.$$

Expressed in the language of Grothendieck we prove now that every bounded multiplier $\mathcal{M}_u : \ell^\infty(\mathcal{D}) \to H^p$ is already 2$-$ absolutely summing. The starting point of our proof is the atomic decomposition of $u \in H^p$. It gives us the basic ingredients from which we build explicit formulas defining the probability measure ω.

Theorem 2.4 ([MP15]). *Let $0 < p \leq 2$. Let $u \in H^p$ with Haar expansion*

$$u = \sum_{I \in \mathcal{D}} x_I h_I$$

and atomic decomposition $(u_i, \mathcal{G}_i, I_i)_{i \in \mathcal{N}}$. Then the sequence $(\omega_I)_{I \in \mathcal{D}}$, defined by

$$\omega_I = \frac{1}{A_p} \frac{|I_i|^{1-\frac{p}{2}}}{\|u_i\|_2^{2-p}} \frac{|x_I|^2 |I|}{\|u\|_{H^p}^p}, \qquad I \in \mathcal{G}_i, \tag{2.14}$$

satisfies

$$\sum_{I \in \mathcal{D}} \omega_I \leq 1 \tag{2.15}$$

and there exists a constant $C_p > 0$ such that for each $\varphi \in \ell^\infty(\mathcal{D})$

$$\left\| \sum_{I \in \mathcal{D}} x_I \varphi_I h_I \right\|_{H^p} \leq C_p \|u\|_{H^p} \Big(\sum_{I \in \mathcal{D}} |\varphi_I|^2 \omega_I \Big)^{\frac{1}{2}}. \tag{2.16}$$

Proof. From

$$\|u\|_{H^p}^p \leq \sum_{i \in \mathcal{N}} \|u_i\|_{H^p}^p$$

we get the estimate

$$\|u\|_{H^p}^p \leq \sum_{i \in \mathcal{N}} \|u_i\|_2^p |I_i|^{1-\frac{p}{2}}. \tag{2.17}$$

We get from

$$\sum_{i \in \mathcal{N}} |I_i| \|S(u_i)\|_\infty^p \leq A_p \|u\|_{H^p}^p$$

that

$$\sum_{i \in \mathcal{N}} \|u_i\|_2^p |I_i|^{1-\frac{p}{2}} \leq A_p \|u\|_{H^p}^p. \tag{2.18}$$

This follows from

$$\|u_i\|_2^p |I_i|^{1-\frac{p}{2}} \leq \|S u_i\|_\infty^p |I_i|.$$

By (2.17) and equation 2.10 in Remark 2.3 we get for $\varphi \in \ell^\infty(\mathcal{D})$

$$\begin{aligned} \left\| \sum_{I \in \mathcal{D}} \varphi_I x_I h_I \right\|_{H^p}^p &= \left\| \sum_{i \in \mathcal{N}} \sum_{I \in \mathcal{G}_i} \varphi_I x_I h_I \right\|_{H^p}^p \\ &\leq \sum_{i \in \mathcal{N}} \left\| \sum_{I \in \mathcal{G}_i} \varphi_I x_I h_I \right\|_2^p |I_i|^{1-\frac{p}{2}} \\ &= \sum_{i \in \mathcal{N}} \left\| \sum_{I \in \mathcal{G}_i} \varphi_I \frac{x_I}{\|u_i\|_2} h_I \right\|_2^p \|u_i\|_2^p |I_i|^{1-\frac{p}{2}}. \end{aligned} \tag{2.19}$$

With

$$\Big\| \sum_{I \in \mathcal{G}_i} \varphi_I \frac{x_I}{\|u_i\|_2} h_I \Big\|_2^p = \Big(\sum_{I \in \mathcal{G}_i} \varphi_I^2 \frac{x_I^2}{\|u_i\|_2^2} |I| \Big)^{\frac{p}{2}}$$

we get

$$\Big\| \sum_{I \in \mathcal{D}} \varphi_I x_I h_I \Big\|_{H^p}^p \le \sum_{i \in \mathcal{N}} \Big(\sum_{I \in \mathcal{G}_i} \varphi_I^2 \frac{x_I^2}{\|u_i\|_2^2} |I| \Big)^{\frac{p}{2}} \|u_i\|_2^p |I_i|^{1-\frac{p}{2}}.$$

Applying Hölder's inequality with $\frac{p}{2} + 1 - \frac{p}{2} = 1$ to

$$\sum_{i \in \mathcal{N}} \Big(\sum_{I \in \mathcal{G}_i} \varphi_I^2 \frac{x_I^2}{\|u_i\|_2^2} |I| \|u_i\|_2^p |I_i|^{1-\frac{p}{2}} \Big)^{\frac{p}{2}} \Big(\|u_i\|_2^p |I_i|^{1-\frac{p}{2}} \Big)^{1-\frac{p}{2}}.$$

we get

$$\Big\| \sum_{I \in \mathcal{D}} \varphi_I x_I h_I \Big\|_{H^p}^p \le \Big(\sum_{i \in \mathcal{N}} \sum_{I \in \mathcal{G}_i} \varphi_I^2 \frac{x_I^2}{\|u_i\|_2^{2-p}} |I| |I_i|^{1-\frac{p}{2}} \Big)^{\frac{p}{2}} \Big(\sum_{i \in \mathcal{N}} \|u_i\|_2^p |I_i|^{1-\frac{p}{2}} \Big)^{1-\frac{p}{2}}.$$

Applying (2.18) to the second term on the right-hand side we obtain the estimate

$$\begin{aligned} \Big\| \sum_{I \in \mathcal{D}} \varphi_I x_I h_I \Big\|_{H^p}^p &\le A_p^{1-\frac{p}{2}} \|u\|_{H^p}^{p(1-\frac{p}{2})} \Big(\sum_{i \in \mathcal{N}} \sum_{I \in \mathcal{G}_i} \varphi_I^2 \frac{x_I^2}{\|u_i\|_2^{2-p}} |I| |I_i|^{1-\frac{p}{2}} \Big)^{\frac{p}{2}} \\ &= A_p \|u\|_{H^p}^p \Big(\sum_{i \in \mathcal{N}} \sum_{I \in \mathcal{G}_i} \varphi_I^2 \frac{x_I^2}{\|u_i\|_2^{2-p} \|u\|_{H^p}^p A_p} |I| |I_i|^{1-\frac{p}{2}} \Big)^{\frac{p}{2}}. \end{aligned}$$

Recall

$$\|u_i\|_2^2 = \sum_{I \in \mathcal{G}_i} x_I^2 |I|. \tag{2.20}$$

By (2.18) and (2.20) we obtain for the sequence $(\omega_I)_{I \in \mathcal{D}}$, defined by

$$\omega_I = \frac{|I_i|^{1-\frac{p}{2}}}{A_p \|u\|_{H^p}^p} \frac{|I| x_I^2}{\|u_i\|_2^{2-p}}, \qquad I \in \mathcal{G}_i,$$

the following estimate

$$\begin{aligned} \sum_{I \in \mathcal{D}} \omega_I &= \frac{1}{A_p \|u\|_{H^p}^p} \sum_{i \in \mathcal{N}} \sum_{I \in \mathcal{G}_i} \frac{|I_i|^{1-\frac{p}{2}} |I| x_I^2}{\|u_i\|_2^{2-p}} \\ &= \frac{1}{A_p \|u\|_{H^p}^p} \sum_{i \in \mathcal{N}} |I_i|^{1-\frac{p}{2}} \|u_i\|_2^p \\ &\le 1. \end{aligned}$$

□

Remark. Finding the explicit formula for the weight $(\omega_I)_{I\in\mathcal{D}}$ (and proving the required estimates) was the main problem solved in [MP15]. This paper describes also the mathematical and historical background as well as the connections to Pisier's extrapolation lattices.

Remark. The theorem admits extensions to the class of Triebel-Lizorkin spaces and also vector-valued Hardy spaces, cf. [MP15].

3 Rearrangement operators

3.1 Know your basis and its rearrangement operators

We begin with general observations describing the -potential- use of rearrangement operators acting on an unconditional bases. Let $\{x_n\}_{n=-\infty}^{\infty}$ be a unconditional basis in a Banach space E. Its biorthogonal functionals are denoted by $\{x_n^*\}_{n=-\infty}^{\infty}$. Fix a linear operator $T : E \to E$. Let

$$f = \sum_{n=-\infty}^{\infty} \langle x_n^*, f \rangle x_n$$

and

$$U_d(x_n) = x_{n+d}$$

a rearrangement. Then

$$\begin{aligned} Tf &= \sum_n \langle x_n^*, f \rangle T x_n \\ &= \sum_d \sum_n \langle x_n^*, f \rangle \langle x_{n+d}^*, T x_n \rangle x_{n+d} \\ &= \sum_d U_d \left(\sum_n \langle x_n^*, f \rangle \langle x_{n+d}^*, T x_n \rangle x_n \right). \end{aligned}$$

Hence,

$$\|Tf\| \le C \left(\sum_d \|U_d\| \sup_n |\langle x_{n+d}^*, T x_n \rangle| \right) \|f\|$$

Suppose we are now given an operator T which is adapted to the basis $\{x_n\}_{n=-\infty}^{\infty}$ in the sense that the off diagonal terms in its matrix representation $\langle x_{n+d}^*, T x_n \rangle$ decay much faster than $\|U_d\|^{-1}$. Then the above estimate estimate provides a bound for $\|T\|$. Thus rearrangement operators control the influence of the off diagonals to the operator norm.

In the next section we will see that the above guidelines provide the framework for the proof of the the David and Journé theorem on Calderón-Zygmund operators. There we use rearrangement operators acting on the Haar system $\{h_I/|I|^{1/p} : I \in \mathcal{D}\}$ normalized in L^p. Recall that the Theorem of R.E.A.C. Paley asserts that $\{h_I/|I|^{1/p} : I \in \mathcal{D}\}$ is actually an unconditional basis in L^p.

3.2 Haar rearrangements

Here we consider general rearrangement operators of the Haar system defined by an injective map $\tau : \mathcal{D} \to \mathcal{D}$. Let $1 < p \leq \infty$, and let the rearrangement operator T_p be given by

$$T_p : \frac{h_I}{|I|^{1/p}} \to \frac{h_{\tau(I)}}{|\tau(I)|^{1/p}}.$$

Clearly T_p acts on linear combinations of the Haar systems: If $1/p + 1/q = 1$ then

$$T_p(f) = \sum_{I \in \mathcal{D}} \left\langle f, \frac{h_I}{|I|^{1/q}} \right\rangle \frac{h_{\tau(I)}}{|\tau(I)|^{1/p}}.$$

We give now an intrinsic characterisation of those injections τ for which the the associated rearrangement operators are bounded T_p in L^p. This is done in several independent steps: First we characterize the case $p = \infty$ with $T_\infty(h_I) = h_{\tau(I)}$ in BMO. Recall the that we use dyadic BMO norm given by

$$\|f\|_{\mathrm{BMO}} = \sup_{I \in \mathcal{D}} \left(\frac{1}{|I|} \sum_{J \subseteq I} \langle f, h_J \rangle^2 |J|^{-1} \right)^{1/2}.$$

Theorem 3.1 ([Mül05])**.** *Let $\tau : \mathcal{D} \to \mathcal{D}$ be injective. The rearrangement operator $T_\infty : \mathrm{BMO} \to \mathrm{BMO}$ defined by*

$$T_\infty(h_I) = h_{\tau(I)}$$

satisfies:

$$\|T_\infty\|_{BMO} \sim \left[\sup_{\mathcal{E} \subseteq \mathcal{D}} \frac{[\![\tau(\mathcal{E})]\!]}{[\![\mathcal{E}]\!]} \right]^{1/2},$$

where $[\![\mathcal{E}]\!]$ is the Carleson constant, cf. Section 2, i.e.

$$[\![\mathcal{E}]\!] = \sup_{I \in \mathcal{E}} \sum_{J \subseteq I, J \in \mathcal{E}} \frac{|J|}{|I|}.$$

We extend the above characterisation across the $L^p, (1 < p < \infty)$ and $H^p (0 < p < \infty)$ scales using interpolation and extrapolation of rearrangement operators. Note that the dependence on p is not trivial, since obviously we may rewrite the operators as

$$T_p(h_I) = h_{\tau(I)} \frac{|I|^{1/p}}{|\tau(I)|^{1/p}}.$$

Theorem 3.2 ([Mül05])**.** *Let $2 < p < \infty$. Let $\tau : \mathcal{D} \to \mathcal{D}$ be injective. The rearrangement operator $T_p : L^p \to L^p$ defined by*

$$T_p \colon \frac{h_I}{|I|^{\frac{1}{p}}} \to \frac{h_{\tau(I)}}{|\tau(I)|^{\frac{1}{p}}}$$

satisfies:

$$c_p \|T_\infty\|_{BMO}^{1/(1-2/p)} \leq \|T_p\|_p \leq C_p \|T_\infty\|_{BMO}^{1/(1-2/p)}.$$

Let T be given by the rearrangement τ then its transposed is determined by the inverse $\tau^{-1} : \tau(\mathcal{D}) \to \mathcal{D}$. Thus, by Hölder's inequality the extrapolation theorem above contains enough information to characterise the bounded rearrangement operators on L^q with $1 < q < 2$, and by Fefferman's inequality (2.8) and transposition we have $\|T_\infty\|_{\mathrm{BMO}} \approx \left\|T_1^{-1}\right\|_{H^1}$.

Combining our theorems we get therefore the complete description of the bounded rearrangements in H^p when $1 \le p < \infty$, and on BMO. However, the spaces H^p when $0 < p < 1$ cannot be reached by duality arguments. In [GMP05] we use different ideas – exploiting again the atomic decomposition for H^p spaces– to obtain the extrapolation law for rearrangement operators on the scale $H^p(0 < p < 2)$.

Theorem 3.3 ([GMP05]). *For all $0 < s < p < 2$ there exists a constant $c_{p,s} > 0$ such that*

$$\frac{1}{c_{p,s}}\|T_s\|_{H^s}^{\frac{s}{2-s}} \le \|T_p\|_{H^p}^{\frac{p}{2-p}} \le c_{p,s}\|T_s\|_{H^s}^{\frac{s}{2-s}}.$$

Remark. We saw that rearrangement operators satisfy the same extrapolation and interpolation properties as as singular integral operators. Can that be a coincidence?? The answer is "No" and the reason for that will become apparent in the section on Calderón-Zygmund operators.

3.3 Postorder rearrangements of the Haar system

Before we turn to giving application of rearrangement operators we study the postorder rearrangement in detail. In Computer Science and the Theory of Algorithms its origin dates back, at least, to J. v. Neumann's treatment of the merge-sort algorithm. We refer to [Pen14] for historic information and examples of algorithms based on the postorder rearrangement. For us it is important that the postorder induces the extremal rearrangements of the Haar system acting on finite dimensional BMO spaces. This is the topic we discuss next.

Finite dimensional dyadic spaces

Let $N \in \mathbb{N}_0$. We define the finite dimensional BMO space denoted by BMO_N and the finite dimensional Hardy spaces H^p_N, $0 < p < \infty$ as follows.

$$\mathrm{BMO}_N = \Big(\operatorname{span}\{h_I : I \in \mathcal{D}_N\},\ \|\cdot\|_{\mathrm{BMO}}\Big),$$

and

$$H^p_N = \Big(\operatorname{span}\{h_I : I \in \mathcal{D}_N\},\ \|\cdot\|_{H^p}\Big),$$

where $\|\cdot\|_{\mathrm{BMO}}$ is given by equation (2.4) and $\|\cdot\|_{H^p}$ by equation (2.5) and (2.6).

Rearrangements on the finite dimensional spaces

Recall that $\mathcal{D}_N$ is the set of dyadic intervals with length greater than or equal to 2^{-N}. Let τ be a bijective map defined on the set $\mathcal{D}_N$.

On BMO_N we study rearrangements of the L^∞-normalised Haar system $(h_I)_{I\in\mathcal{D}_N}$ given by the rearrangement operator

$$T_\tau \colon h_I \mapsto h_{\tau(I)},$$

and on H^p_N, $0 < p < \infty$, rearrangements of the L^p-normalised Haar system given by the rearrangement operator

$$T_{\tau,p} : \frac{h_I}{|I|^{\frac{1}{p}}} \mapsto \frac{h_{\tau(I)}}{|\tau(I)|^{\frac{1}{p}}}.$$

A standard argument (given below) yields the following norm estimates for rearrangement operators on BMO_N (cf. Theorem 3.1):

$$\sup_{\substack{\mathcal{C}\subseteq\mathcal{D}_N\\ \text{non-empty}}} \frac{[\![\tau(\mathcal{C})]\!]^{\frac{1}{2}}}{[\![\mathcal{C}]\!]^{\frac{1}{2}}} \le \|T_\tau\|_{\mathrm{BMO}} \le (N+1)^{\frac{1}{2}}. \tag{3.1}$$

Note that the lower bound in (3.1) is always greater than or equal to one. Let $x = \sum_{I\in\mathcal{D}_N} x_I h_I$. Then

$$\|T_\tau x\|^2_{\mathrm{BMO}} = \sup_{I\in\mathcal{D}_N} \frac{1}{|I|} \sum_{J\subseteq I} \left|x_{\tau^{-1}(J)}\right|^2 |J| \le \sup_{I\in\mathcal{D}_N} |x_I|^2 \, [\![\mathcal{D}_N]\!] \le \|x\|^2_{\mathrm{BMO}} \, [\![\mathcal{D}_N]\!].$$

Definition (2.3) yields $[\![\mathcal{D}_N]\!] = N+1$. Let $\mathcal{C} \subseteq \mathcal{D}_N$ be any non-empty collection of dyadic intervals. Let $x = \sum_{I\in\mathcal{C}} h_I$. Then

$$\|x\|_{\mathrm{BMO}} = [\![\mathcal{C}]\!]^{\frac{1}{2}} \ \text{ and } \ \|T_\tau x\|_{\mathrm{BMO}} = [\![\tau(\mathcal{C})]\!]^{\frac{1}{2}}.$$

Let $x = \sum_{I\in\mathcal{C}} x_I h_I$ for some non-empty collection of dyadic intervals $\mathcal{C} \subseteq \mathcal{D}_N$. The above argument provides the following rough upper bound

$$\|T_\tau x\|_{\mathrm{BMO}} \le \|x\|_{\mathrm{BMO}} \, [\![\tau(\mathcal{C})]\!]^{\frac{1}{2}}. \tag{3.2}$$

The adjoint operator of a rearrangement operator is again a rearrangement operator induced by the inverse rearrangement. By the duality of H^1 and BMO we have that the operator T_τ on BMO_N is the adjoint operator of $T_{\tau^{-1},1}$ on H^1_N with

$$\frac{1}{C_F}\|T_\tau\|_{\mathrm{BMO}_N} \le \|T_{\tau^{-1},1}\|_{H^1_N} \le C_F\|T_\tau\|_{\mathrm{BMO}_N}, \tag{3.3}$$

where $C_F = 2\sqrt{2}$ is the constant appearing in Fefferman's inequality (2.8). We will use the following abbreviation for equation 3.3

$$\|T_\tau\|_{\mathrm{BMO}_N} \approx_{C_F} \|T_{\tau^{-1},1}\|_{H^1_N}. \tag{3.4}$$

Remark 3.4. Observe that by Theorem 3.2 and Theorem 3.3 rearrangement operators $T_{\tau,p}$ on H_N^p, $0 < p < \infty$, induced by any bijective map τ acting on $\mathcal{D}_N$, have the norm estimate

$$\|T_{\tau,p}\|_{H_N^p} \leq c_p\,(N+1)^{\left|\frac{1}{p}-\frac{1}{2}\right|}.$$

Postorder rearrangements

Postorder on the set $\mathcal{D}_N$

See [MS97], [BP05] and [Knu05].

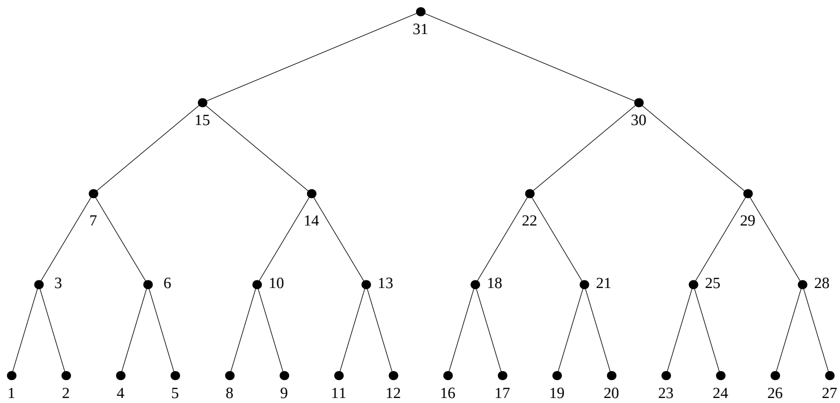

Figure 3.1: Postorder on the set $\mathcal{D}_4$.

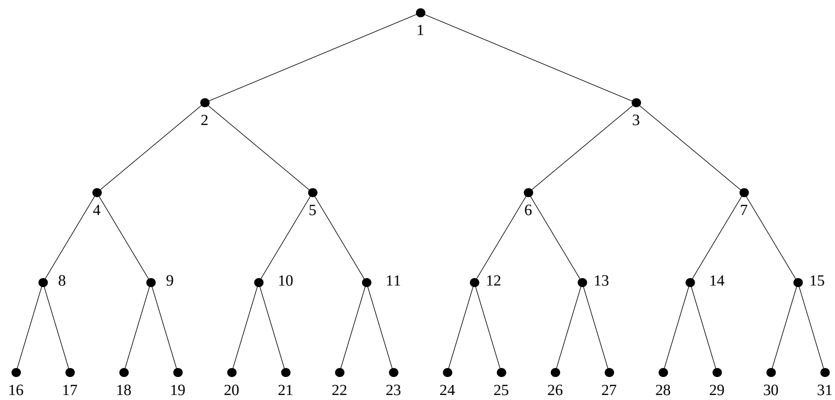

Figure 3.2: Lexicographic order on the set $\mathcal{D}_4$.

Definition 3.5 (Postorder)**.** Let $I, J \in \mathcal{D}_N$. We say $I \preceq J$ if either I and J are disjoint and I is to the left of J, or I is contained in J.

We call "$\preceq$" the *postorder* on $\mathcal{D}_N$. The natural order on the set $\mathcal{D}_N$ is the *lexicographic order*, $\leq_l$, on the set $\{(\ell, k)\}$, cf. figure 3.2. The postorder on $\mathcal{D}_N$, in contrast to the lexicographic order depends on the depth N.

The rearrangements

We denote by τ_N the bijective map on the dyadic intervals that associates to the n^{th} interval in postorder the n^{th} interval in lexicographic order, cf. figure 3.3. This rearrangement is called *postorder rearrangement*. Its inverse, which associates to the n^{th} interval in lexicographic order the n^{th} interval in postorder, is denoted by σ_N. The rearrange-

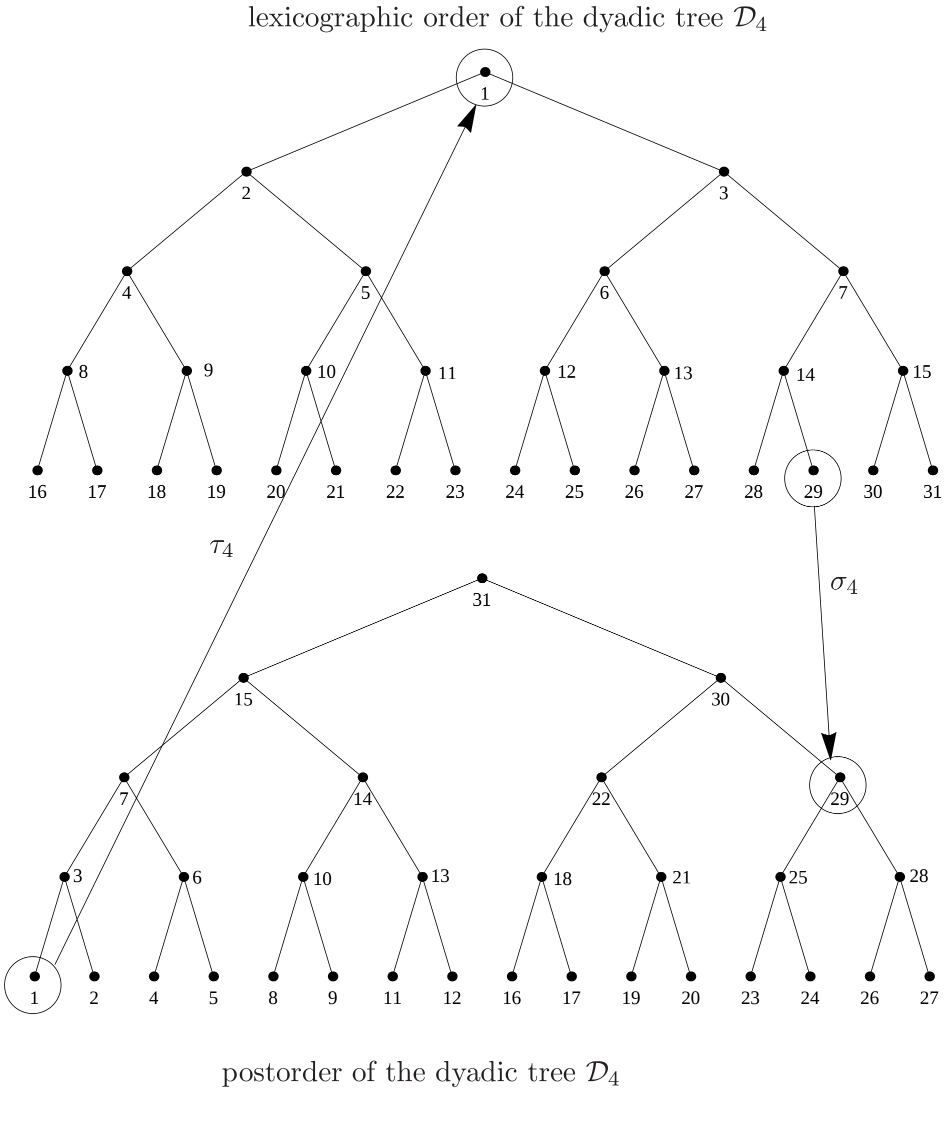

Figure 3.3: Lexicographic order and postorder of the dyadic tree $\mathcal{D}_4$. Postorder rearrangement τ_4 on $\mathcal{D}_4$ and its inverse σ_4.

ments τ_N and σ_N induce rearrangement operators on BMO_N and on the H^p_N-spaces. On

BMO_N we consider the rearrangement operators

$$T_{\tau_N} : h_I \mapsto h_{\tau_N(I)} \quad \text{and} \quad T_{\sigma_N} : h_I \mapsto h_{\sigma_N(I)}$$

and obtain the following norm estimates for these rearrangement operators applied to functions with Haar support in the sets

$$\mathcal{T}^N_{\ell,0} = \{I \in \mathcal{D}_N : I \subseteq I_{\ell,0}\}$$

and $\mathcal{D}_{N-\ell}$. We use the abbreviations

$$\mathcal{M}(\mathcal{T}^N_{\ell,0}) = \operatorname{span}\{h_I : I \in \mathcal{T}^N_{\ell,0}\}$$

and

$$\mathcal{M}(\mathcal{D}_{N-\ell}) = \operatorname{span}\{h_I : I \in \mathcal{D}_{N-\ell}\}.$$

Theorem 3.6 ([Pen14]). *Let $N \in \mathbb{N}_0$ and $0 \le \ell \le N$. Let $T = T_{\tau_N}\big|_{\mathcal{M}(\mathcal{T}^N_{\ell,0})}$ or $T = T_{\sigma_N}\big|_{\mathcal{M}(\mathcal{D}_{N-\ell})}$. Then*

$$\frac{1}{\sqrt{2}}(N-\ell+1)^{\frac12} \le \|T\|_{BMO} \le (N-\ell+1)^{\frac12}. \tag{3.5}$$

This theorem in combination with the general upper bound in (3.1) reveals the extremal nature of the rearrangements τ_N and σ_N in the sense that for $T = T_{\tau_N}$ resp. $T = T_{\sigma_N}$ we have

$$\frac{1}{\sqrt{2}} R(\mathrm{BMO}_N) \le \|T\|_{\mathrm{BMO}} \le R(\mathrm{BMO}_N),$$

where

$$R(\mathrm{BMO}_N) = \sup\Big\{\|T_\tau \colon \mathrm{BMO}_N \to \mathrm{BMO}_N\| : \tau \colon \mathcal{D}_N \to \mathcal{D}_N \text{ bijective}\Big\}.$$

Obviously, the lower bound in (3.5) is the important one for this result and the statement of Theorem 3.6. The upper bound in (3.5) is the trivial one that originates from the depth (in the sense of dyadic trees) of the sets $\mathcal{D}_{N-\ell}$ resp. $\mathcal{T}^N_{\ell,0}$.

The duality relation of H^1_N and BMO_N, in particular the norm equivalence in equation (3.3), and the interpolation resp. extrapolation statements in Theorem 3.2 and Theorem 3.3 give equivalent norm estimates as in Theorem 3.6 for the rearrangement operators on H^p_N, $0 < p < \infty$, given by

$$T_{\tau_N,p} \colon \frac{h_I}{|I|^{\frac1p}} \mapsto \frac{h_{\tau_N(I)}}{|\tau_N(I)|^{\frac1p}} \quad \text{resp.} \quad T_{\sigma_N,p} \colon \frac{h_I}{|I|^{\frac1p}} \mapsto \frac{h_{\sigma_N(I)}}{|\sigma_N(I)|^{\frac1p}}.$$

Corollary 3.7. *For all $0 < p < \infty$ there exists a constant C_p such that for all $N \in \mathbb{N}_0$, $0 \le \ell \le N$ and $T = T_{\tau_N,p}\big|_{\mathcal{M}(\mathcal{T}^N_{\ell,0})}$ or $T = T_{\sigma_N,p}\big|_{\mathcal{M}(\mathcal{D}_{N-\ell})}$ the following holds*

$$\frac{2^{-\left|\frac1p-\frac12\right|}}{C_p}(N-\ell+1)^{\left|\frac1p-\frac12\right|} \le \|T\|_{H^p_N} \le C_p\,(N-\ell+1)^{\left|\frac1p-\frac12\right|}. \tag{3.6}$$

Remark 3.8. By the convexification method for the concrete specialisation to Hardy spaces (see [MP15]) one obtains the same result as in Corollary 3.7 for the more general Triebel-Lizorkin spaces.

Corollary 3.7 gives, considering the general upper bound in Remark 3.4, the same extremality statement for the rearrangement operators $T = T_{\tau_N,p}$ resp. $T = T_{\sigma_N,p}$ on the spaces H^p_N, $0 < p < \infty$. For all $0 < p < \infty$ there exists a constant B_p such that

$$\frac{2^{-\left|\frac{1}{p}-\frac{1}{2}\right|}}{B_p} R^N(H^p_N) \leq \|T\|_{H^p_N} \leq R^N(H^p_N),$$

where $R^N(H^p_N) = \sup\Big\{\|T_\tau\colon H^p_N \to H^p_N\| : \tau\colon \mathcal{D}_N \to \mathcal{D}_N \text{ bijective}\Big\}$.

4 Singular integral operators: Central estimates

We give two independent proofs for the L^p estimates for scalar valued singular integral operators. The first one extends to $L^p(X)$ when X is a UMD spaces, whereas the second one to operators on $L^p(L^1)$ with values in $L^p(L^\alpha)$ where $\alpha < 1$.

4.1 The theorem of David and Journé

In this section we discuss the $T(1)$ theorem of David and Journé, asserting that singular integral operators are bounded in L^p for $1 < p < \infty$ provided that $T(1), T^*(1) \in$ BMO. We present Figiel's proof of the $T(1)$ theorem. His idea was to expand the integral kernels along the isotropic Haar basis and thereby obtain representations of the associated integral operator as a series rearrangement operators, Haar multipliers and paraproducts. For sake of notational simplicity we consider integral operator on $\mathbb{R}$ and kernels in $\mathbb{R}^2$. We assume that the kernel $K : \mathbb{R}^2 \to \mathbb{R}$ satisfies the following properties

1. $|K(x,y)| \le A|x-y|^{-1}$ and $|\nabla K(x,y)| \le A|x-y|^{-2}$,
2. $|\langle K, h_I \otimes h_I\rangle| + |\langle K, h_I \otimes 1_I\rangle| + |\langle K, 1_I \otimes h_I\rangle| \le A|I|$,
3. $K \in C_c(\mathbb{R}^2)$,

and consider the integral operator

$$T_K(u) = \int_{\mathbb{R}} K(x,y)u(y)dy.$$

The singular integral operators introduced in Section 1 can be approximated by the kernel operators defined above. Hence the L^p $(1 < p < \infty)$ estimates in Section 1 are a consequence of the following theorem.

Theorem 4.1 (G. David, J.L. Journé [DJ84]). *Let $1 < p < \infty$. Let*

$$T(1)(x) = \int_{\mathbb{R}} K(x,y)dy, \quad T^*(1)(y) = \int_{\mathbb{R}} K(x,y)dx.$$

Then

$$\|T_K\|_p \le C_p(A + \|T(1)\|_{BMO} + \|T^*(1)\|_{BMO}).$$

Proof. (T. Figiel. [Fig90])

Step 1: (The kernel-expansion.) We define a basis in $L^2(\mathbb{R}^2)$. Let $\mathcal{A} = \{0,1\}^2 \backslash \{(0,0)\}$ and $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathcal{A}$. Let $I, J \in \mathcal{D}$ with $|I| = |J|$ and define

$$h^{(\varepsilon)}_{I\times J}(x,y) = h^{\varepsilon_1}_I(x) h^{\varepsilon_2}_J(y).$$

The family of functions

$$\{h^{(\varepsilon)}_{I\times J} : \varepsilon \in \mathcal{A};\ I, J \in \mathcal{D},\ |I| = |J|\}$$

forms a complete orthogonal system in $L^2(\mathbb{R}^2)$. Hence,

$$K = \sum \left\langle K, h^{(\varepsilon)}_{I\times J} \right\rangle h^{(\varepsilon)}_{I\times J} |I \times J|^{-1},$$

and

$$T^*_K(u) = \sum \left\langle K, h^{(\varepsilon)}_{I\times J} \right\rangle \langle u, h^{\varepsilon_1}_I \rangle h^{\varepsilon_2}_J |I \times J|^{-1},$$

where the index set is $\{\varepsilon \in \mathcal{A};\ I, J \in \mathcal{D},\ |I| = |J|\}$.

Step 2: (Decomposition.) We decompose T^*_K as follows: $T^*_K = T^{(1,1)}_K + T^{(1,0)}_K$
Let $I, J \in \mathcal{D}$ with $|I| = |J|$. Then there is $m \in \mathbb{Z}$, so that $J = I + m|I|$ or $I = J - m|J|$. We define

$$T^{(1,1)}_K(u) = \sum_{m\in\mathbb{Z}} \sum_{I\in\mathcal{D}} \frac{\langle K, h^{(1,1)}_{I\times(I+m|I|)}\rangle}{|I|} \frac{\langle u, h_I\rangle}{|I|} h_{I+m|I|},$$

$$T^{(1,0)}_K(u) = \sum_{m\in\mathbb{Z}} \sum_{I\in\mathcal{D}} \frac{\langle K, h^{(1,0)}_{I\times(I+m|I|)}\rangle}{|I|} \frac{\langle u, h_I\rangle}{|I|} 1_{I+m|I|},$$

$$T^{(0,1)}_K(u) = \sum_{m\in\mathbb{Z}} \sum_{J\in\mathcal{D}} \frac{\langle K, h^{(0,1)}_{(J-m|J|)\times J}\rangle}{|J|} \frac{\langle u, 1_{J-m|J|}\rangle}{|J|} h_J.$$

Step 3. Here we identify the Haar coefficients of $T(1)$ and $T*(1)$ and we isolate the rearrangement operators hidden in the above decomposition of operators Remarks: $I, J \in \mathcal{D}$ fix.

$$\begin{aligned}\sum_{m\in\mathbb{Z}} \langle K, h^{(1,0)}_{I\times(I+m|I|)}\rangle &= \int\int K(x,y) h_I(x) dy dx \\ &= \langle h_I, T(1)\rangle.\end{aligned}$$

$$\begin{aligned}\sum_{m\in\mathbb{Z}} \langle K, h^{(0,1)}_{(J-m|J|)\times J}\rangle &= \int\int K(x,y) h_J(y) dx dy \\ &= \langle T^*(1), h_J\rangle.\end{aligned}$$

The relevant operators are then this:

$$T_m(h_I) = h_{I+m|I|}, \qquad \text{(Permutation operator)}$$
$$U_m(h_I) = 1_{I+m|I|} - 1_I, \qquad \text{(Shift operator)}$$
$$P_a(u) = \sum_{I\in\mathcal{D}} \frac{\langle a, h_I\rangle}{|I|}\frac{\langle u, h_I\rangle}{|I|} 1_I, \qquad \text{(Paraproduct)}$$
$$P_a^*(u) = \sum_{I\in\mathcal{D}} \frac{\langle a, h_I\rangle}{|I|}\frac{\langle u, 1_I\rangle}{|I|} h_I. \qquad \text{(Transposed)}$$

Step 4. We rewrite the expansion of the operator using T_m U_m and paraproducts.

$$T_K^{(1,1)}(u) = \sum_{m\in\mathbb{Z}}\sum_{I\in\mathcal{D}} \frac{\langle K, h^{(1,1)}_{I\times(I+m|I|)}\rangle}{|I|}\frac{\langle u, h_I\rangle}{|I|} T_m(h_I)$$
$$T_K^{(1,0)}(u) = \sum_{m\in\mathbb{Z}}\sum_{I\in\mathcal{D}} \frac{\langle K, h^{(1,0)}_{I\times(I+m|I|)}\rangle}{|I|}\frac{\langle u, h_I\rangle}{|I|} U_m(h_I) + P_{T(1)}(u)$$
$$T_K^{(0,1)}(u) = \sum_{m\in\mathbb{Z}}\sum_{J\in\mathcal{D}} \frac{\langle K, h^{(0,1)}_{J\times(J-m|J|)}\rangle}{|J|}\frac{\langle u, U_{-m}(h_J)\rangle}{|J|} h_J + P^*_{T^*(1)}(u).$$

Step 5: (Paley, Triangle inequality.) The off- diagonal decay of the kernel K translates into decay estimates for its matrix representation along the Haar system. The matrix $\langle K, h^{(\varepsilon)}_{I\times(I+m|I|)}\rangle$ is almost diagonal:

$$|\langle K, h^{(\varepsilon)}_{I\times(I+m|I|)}\rangle| \le CA|I|(1+|m|)^{-2}.$$

As the Haar system is an unconditional basis in L^p (Paley's inequality (2.7)) we obtain with the triangle inequality that

$$\|T_K^{(1,1)}(u)\|_p \le \sum_{m\in\mathbb{Z}} (1+|m|)^{-2}\|T_m(u)\|_p$$
$$\|T_K^{(1,1)}(u)\|_p \le \sum_{m\in\mathbb{Z}} (1+|m|)^{-2}\|T_m(u)\|_p$$
$$\|T_K^{(1,0)}(u)\|_p \le C\sum_{m\in\mathbb{Z}} (1+|m|)^{-2}\|U_m(u)\|_p + \|P_{T(1)}(u)\|_p.$$

Step 6: It remains to bound the paraproducts and to find estimates for the growth of the rearrangement operators T_m, U_m such that $(1+|m|)^{-2}(\|T_m\| + \|U_m\|)$ remains summable, For scalar valued paraproducts we have the estimates of R. Coifman, and Y. Meyer, [CM78])

$$\|P_a(u)\|_p \le C_p\|a\|_{\mathrm{BMO}}\|u\|_p.$$

and for permutation operators T. Figiel [Fig88] proved that

$$\|T_m\|_p + \|U_m\|_p \le C_p \log(1+|m|).$$

□

Remark. Figiel's proof outlined above applies to vector valued L^p spaces. provided the target space X is in the UMD class. Recall that X is a UMD space if there exists a constant $C > 0$ such that for every finite choice of vectors $x_I \in X$ and signs $\epsilon_I \in \{+1, -1\}$

$$\int_0^1 \| \sum \epsilon_I h_I(t) x_I \|_X^2 dt \le C \int_0^1 \| \sum h_I(t) x_I \|_X^2 dt.$$

We refer to Burkholder [Bur83] or Figiel [Fig90, Fig88] for for more information and references to Banach spaces with the UMD property. Now, if X is a UMD space and $L^p(X)$ the Bochner space of X valued $p-$ integrable random variables then Figiel [Fig90] proves that

$$T_K \otimes \text{ Id}_X : L^p(X) \to L^p(X)$$

is bounded if X is a UMD space, $T(1) \in \text{BMO}$, $T^*(1) \in \text{BMO}$ and $1 < p < \infty$. (A detailed exposition of Figiel's treatment of vector valued paraproducts is contained in [Mül05].)

Remark. The Lebesgue spaces L^q and the sequence spaces ℓ^q when $1 < q < \infty$ are UMD spaces. The spaces L^1, ℓ^1 and c_0 are not.

4.2 Operators on $L^p(L^1)$

In this section we prove Bourgain's estimates for singular integral operators on $L^p(L^1)$. We present Kislyakov's approach based on good-lambda-inequalities [Kis91] and encourage the reader to consult [Kis91] for numerous deep applications of Bourgain's estimates.

For sake of simplicity we further specialize the discussion to the case of the Hilbert transform. Recall from Section 1.1 that it is given as a Fourier multiplier, by

$$H(y^n) = (-i)\text{sign}(n) y^n, \qquad y \in \mathbb{T}.$$

Kolmogorov's theorem ([Koo98]) asserts that the Hilbert transform maps L^1 to weak L^1, that is

$$m\{|Hf| > \lambda\} \le \frac{C\|f\|_1}{\lambda}.$$

For $g \in L^p(\mathbb{T})$ we let Mg denote its Hardy Littlewood maximal function, thus $Mg(x) = \sup\{\int_I |g| dm/|I|\}$ where the supremum is extended over all intervals in $\mathbb{T}$ containing x. By the maximal function theorem of Hardy and Littlewood, M preserves the spaces $L^p(1 < p \le \infty)$ with $\|Mg\|_p \le Cp/(p-1)\|g\|_p$. See also [Koo98].

In this Section we are concerned with the vector-valued interpretation of Kolmogorov's theorem. Let (Ω, μ) be a probability space. For $0 < \alpha < 1$ let

$$J_\Omega : L^1(\Omega, \mu) \to L^\alpha(\Omega, \mu)$$

denote the inclusion $J_\Omega(f) = f$. On the algebraic tensor product $L^p(\mathbb{T}) \otimes L^1(\Omega, \mu)$ define $\mathcal{H} = H \otimes J_\Omega$ by putting

$$\mathcal{H}(f \otimes \varphi) = (Hf) \otimes \varphi, \qquad f \in L^p(\mathbb{T}), \quad \varphi \in L^1(\Omega, \mu)$$

and linear extension. Put

$$L^p(L^1) = L^p(\mathbb{T}, L^1(\Omega)) \quad \text{and} \quad L^p(L^\alpha) = L^p(\mathbb{T}, L^\alpha(\Omega)).$$

The proof presented in the previous section shows that the operator $\mathcal{H} = H \otimes J_\Omega$ extends as follows:

$$\|\mathcal{H} \colon L^p(L^q) \to L^p(L^q)\| < C_{p,q},$$

where $C_{p,q} \to \infty$ as $p \to 1$ or $q \to 1$. The same for $p \to \infty$ or $q \to \infty$. (See [Bur83]. See also [Fig90].)

In the case $q = 1$ the following theorem of J. Bourgain [BD86, DG85] provides a powerful vector valued extension of Kolmogorov's inequality.

Theorem 4.2. *Let $1 < p < \infty$ and $0 < \alpha < 1$. There exists $A = A(p, \alpha)$ so that for any probability space (Ω, μ) the operator*

$$\mathcal{H} = H \otimes J_\Omega$$

extends to a bounded operator with

$$\|\mathcal{H} : L^p(L^1) \to L^p(L^\alpha)\| \leq A.$$

We present here S. Kislyakov's [Kis91] proof of Theorem 4.2. Establishing the following "Good Lambda Inequality" is the central idea of Kislyakov's approach, which led to striking simplification over the arguments in [BD86] or [DG85].

Theorem 4.3. *Let $F \in L^1(\mathbb{T}) \otimes L^1(\Omega)$ and $G = \mathcal{H}F$. For $z \in \mathbb{T}$ and $0 < \alpha < 1$ we write*

$$f(z) = \|F(z)\|_{L^1} \quad \text{and} \quad g(z) = \|G(z)\|_{L^\alpha},$$

where $L^\alpha = L^\alpha(\Omega, \mu)$ and $L^1 = L^1(\Omega, \mu)$. Then there exists $A = A(\alpha)$ such that

$$\sup_{\lambda > 0} \lambda m\{g > \lambda\} \leq A \int_{\mathbb{T}} f(y) dm(y), \tag{4.1}$$

and $A_1 = A_1(\alpha, A)$ such that for any $\eta > 0$ and $\lambda > 0$,

$$m\{g > A_1 \lambda, Mf < \eta\lambda\} \leq \eta m\{Mg > \lambda\}, \tag{4.2}$$

where Mf denotes the Hardy Littlewood maximal function of f.

Proof. The proof consists in extrapolating the weak type estimates for the Hilbert transform.

Step 1. (Weak Type estimates.) Fix $\lambda > 0$. We prove the following weak type $(1, 1)$ estimate:

$$\lambda m(\{g > \lambda\}) \leq A \int_{\mathbb{T}} f(y) dm(y). \tag{4.3}$$

Recall the convention $L^1 = L^1(\Omega, \mu)$. Let $B = \{g > \lambda\}$. By arithmetic and Fubini's theorem

$$\begin{aligned} \lambda m(B) &\leq \lambda^{(1-\alpha)} \int_B g(y)^\alpha dm(y) \\ &\leq \lambda^{(1-\alpha)} \| \left(\int_B |G(y)|^\alpha dm(y) \right) \|_{L^1}. \end{aligned} \tag{4.4}$$

For $\omega \in \Omega$ fixed, the weak type estimate for the Hilbert transform yields

$$\int_B |G(y,\omega)|^\alpha dm(y) \leq (\int_{\mathbb{T}} |F(y,\omega)| dm(y))^\alpha (m(B))^{1-\alpha} \tag{4.5}$$

Inserting (4.5) to (4.4) and applying Hölder's inequality gives

$$\begin{aligned} &\lambda^{(1-\alpha)} \| \int_B |G(y)|^\alpha dm(y) \|_{L^1} \\ &\qquad \leq (\lambda m(B))^{(1-\alpha)} \left(\int_{\mathbb{T}} \|F(y)\|_{L^1} dm(y) \right)^\alpha \\ &\qquad = (\lambda m(B))^{(1-\alpha)} \left(\int_{\mathbb{T}} f(y) dm(y) \right)^\alpha . \end{aligned} \tag{4.6}$$

It remains to combine (4.4) and (4.6) to obtain (4.3) and hence, (4.1) .

Step 2. (Reduction.) Since Mg is lower semicontinuous, $\{Mg > \lambda\}$ is open. There exists a sequence of disjoint intervals I_j so that $\{Mg > \lambda\} = \bigcup I_j$. Fix j and put $I = I_j$. The expanded interval $(2I)$ intersects $\{Mg \leq \lambda\}$, hence

$$\int_{2I} g dm \leq 2m(I) \inf_{x \in 2I} Mg(x) \leq 2m(I)\lambda. \tag{4.7}$$

We prove that for any $\eta > 0$ and $\lambda > 0$,

$$m(I \cap \{g > A_1 \lambda,\ Mf < \eta\lambda\}) \leq \eta m(I). \tag{4.8}$$

In order to show (4.8) we will verify the following implication:

$$\text{If} \quad I \cap \{Mf < \eta\lambda\} \neq \emptyset \quad \text{then} \quad m(I \cap \{g > A_1\lambda\}) \leq \eta m(I). \tag{4.9}$$

Step 3. Put $F_2 = F 1_{\mathbb{T} \setminus 2I}$, $G_2 = \mathcal{H} F_2$, and $g_2(x) = \|G_2(x)\|_{L^\alpha}$. We show the following implication: If there exists $x_0 \in I$ with $Mf(x_0) > \eta\lambda$, then there exists $x_1 \in I$ so that

$$g_2(x_1) \leq C_1 \lambda.$$

To this end let $F_1 = F - F_2, G_1 = \mathcal{H} F_1$, and

$$f_1(x) = \|F_1(x)\|_{L^1}, \quad g_1(x) = \|G_1(x)\|_{L^1}.$$

Since $F_1 = F1_{2I}$, we have

$$\int_{\mathbb{T}} f_1 dm = \int_{\mathbb{T}\setminus 2I} f dm \le 2m(I) \inf_{x\in 2I} Mf(x) \le 2m(I)Mf(x_0) \le 2\eta\lambda.$$

The weak type inequality (4.1) applied to f_1, g_1 yields

$$m(I\cap\{g_1 > 6A\eta\lambda\}) \le \frac{A}{6A\eta\lambda}\int_{\mathbb{T}} f_1 dm \le \frac{1}{3}m(I). \tag{4.10}$$

Consequently by (4.10) and (4.7),

$$m(I\cap\{g_1 \le 6A\eta\lambda\}) \ge \frac{2}{3}m(I), \quad \text{and} \quad m(I\cap\{g\le\lambda\}) \ge \frac{2}{3}m(I).$$

Comparing the lower measure estimates shows that their intersection is non-empty. Pick $x_1 \in I\cap\{g\le\lambda\}\cap\{g_1 > 6A\eta\lambda\} \ne \emptyset$. By the quasi triangle inequality in $L^\alpha = L^\alpha(\Omega,\mu)$, we get

$$g_2^\alpha(x_1) \le g^\alpha(x_1) + g_1^\alpha(x_1) \le (6\lambda)^\alpha(1+(A\eta)^\alpha),$$

and hence

$$g_2(x_1) \le C_1\lambda, \quad \text{where} \quad C_1 = 6(1+(A\eta)^\alpha)^{1/\alpha}. \tag{4.11}$$

Step 4. Here we show that there exists $C_2 > 0$ so that for any $x\in I$,

$$g_2^\alpha(x) \le \lambda^\alpha(C_1^\alpha + C_2^\alpha).$$

We first use the kernel estimates for the Hilbert transform to show that there exists $C_2 > 0$ so that for any $x \in I$,

$$\|G_2(x) - G_2(x_1)\|_{L^\alpha} \le C_2\eta\lambda. \tag{4.12}$$

As $\alpha < 1$ the $L^\alpha(\Omega,\mu)$ quasi-norm on the left-hand side of (4.12) is bounded by the $L^1(\Omega,\mu)$. In the latter space we use the triangle inequality. Thus, the kernel estimates for the Hilbert transform yield the following

$$\begin{aligned} \|G_2(x) - G_2(x_1)\|_{L^1} &= \left\| \int_{\mathbb{T}\setminus 2I} (K(x,y) - K(x_1,y))F(y)dm(y) \right\|_{L^1} \\ &\le \int_{\mathbb{T}\setminus 2I} \frac{|x-x_1|}{|x-y|^2} f(y)dm(y) \\ &\le C_2 \inf_{x\in I} Mf(x). \end{aligned} \tag{4.13}$$

Since $\inf_{x\in I} Mf(x) \le Mf(x_0) \le \eta\lambda$, (4.12) follows directly from (4.13). In summary for $x\in I$ we have

$$\|G_2(x)\|_{L^\alpha}^\alpha \le \|G_2(x_1)\|_{L^\alpha}^\alpha + \|G_2(x) - G_2(x_1)\|_{L^\alpha}^\alpha,$$

and obtain

$$g_2^\alpha(x) \le \lambda^\alpha(C_1^\alpha + C_2^\alpha), \quad x \in I \tag{4.14}$$

Step 5. (Putting it together). Put $C_3 = (C_1^\alpha + C_2^\alpha + 2^\alpha A^\alpha)^{1/\alpha}$. By (4.14)

$$g^\alpha(x) \le g_1^\alpha(x) + g_2^\alpha(x) \le g_1^\alpha(x) + \lambda^\alpha C_3^\alpha \tag{4.15}$$

With $A_1 = 2^\alpha C_3$ we get $A_1^\alpha - C_3^\alpha = C_3^\alpha$ and with (4.15),

$$I \cap \{g > A_1\lambda\} \subseteq I \cap \{g^\alpha > A_1^\alpha\lambda^\alpha\} \subseteq I \cap \{g_1^\alpha > (A_1^\alpha - C_3^\alpha)\lambda^\alpha\}.$$

Applying again the weak type estimate (4.7) and (4.10) to f_1, g_1 and using that $2A \le C_3$ gives

$$m(I \cap \{g_1 > C_3\lambda\}) \le \eta m(I).$$

This completes the verification of the implication (4.9). Finally we invoke that $\{Mg \le \lambda\} \cap \{g > A_1\lambda\} = \emptyset$ and obtain for any $\eta > 0$ and $\lambda > 0$,

$$\begin{aligned} m\{g > A_1\lambda, Mf < \eta\lambda\} &= \sum_{j=1}^{\infty} m(I_j \cap \{g > A_1\lambda, Mf < \eta\lambda\}) \\ &\le \eta \sum_{j=1}^{\infty} m(I_j) \\ &\le \eta m\{Mg > \lambda\} \end{aligned}$$

□

We finally describe the well known path from good lambda inequalities to L^p estimates.

Proof of Theorem 4.2. Starting from Kislyakov's distributional estimate (4.2),

$$m\{g > A_1\lambda, Mf < \eta\lambda\} \le \eta m\{Mg > \lambda\},$$

we obtain

$$m\{g > A_1\lambda\} \le m\{Mf > \eta\lambda\} + \eta m\{Mg > \lambda\},$$

which holds uniformly in λ and η. Multiplying both sides by $p\lambda^{p-1}$ and integrating over $0 < \lambda < \infty$ gives

$$\int_{\mathbb{T}} g^p dm \le \frac{A_1^p}{\eta^p} \int_{\mathbb{T}} (Mf)^p dm + \eta A_1^p \int_{\mathbb{T}} (Mg)^p dm.$$

Invoking the Hardy-Littlewood maximal function estimates we obtain $C > 0$ so that

$$\int_{\mathbb{T}} g^p dm \le \left(\frac{CpA_1^p}{(p-1)\eta^p}\right) \int_{\mathbb{T}} f^p dm + \eta \left(\frac{CpA_1^p}{p-1}\right) \int_{\mathbb{T}} g^p dm.$$

The above estimate holds uniformly over $0 < \eta < 1$. Given $1 < p < \infty$, we select $\eta > 0$ so small that the factor in front of the second integral on the right hand side is bounded by $1/2$. Thus we obtain, $A = A(p, A_1)$ so that

$$\int_{\mathbb{T}} g^p dm \le A \int_{\mathbb{T}} f^p dm.$$

□

4.3 Reflexive subspaces of L^1 and vector valued SIOs

We obtain here Kislyakov's embedding using Bourgain's estimate for singular integrals on $L^p(L^1)$, and finally we deduce Bourgain 's theorem on the extension operators from reflexive subspaces of L^1. Throughout this section we follow Kislyakov [Kis91].

4.3.1 Uniform integrability

For $f \in L^1(\mathbb{T})$ we denote $[f] = \{f + h : h \in H^1(\mathbb{T})\}$ and we call

$$q : L^1 \to L^1/H^1, \qquad f \to [f]$$

the canonical projection onto the quotient space L^1/H^1 where

$$L^1/H^1 = L^1(\mathbb{T})/H^1(\mathbb{T}) = \{[f] : f \in L^1(\mathbb{T})\}.$$

Recall the fact that the unit ball of a reflexive subspaces in L^1 is equi-integrable. The following result is a simple and useful consequence thereof.

Theorem 4.4. *If a closed linear subspace $Y \subseteq L^1(\Omega)$ is reflexive, then for any $0 < \alpha < 1$, there exists $C = C(Y, \alpha)$ so that*

$$\|y\|_{L^1} \le C\|y\|_{L^\alpha}, \qquad y \in Y. \tag{4.16}$$

If a closed linear subspace $Y \subseteq \overline{H^1(\mathbb{T})}$ is reflexive, then the restriction of q to Y is invertible on its range, and there exists $C = C(Y)$ so that

$$\|y\|_{L^1} \le C\|q(y)\|_{L^1/H^1}, \qquad y \in Y. \tag{4.17}$$

Proof. If the assertion (4.16) fails to hold then there exists a sequence $y_n \in Y$ so that $\|y_n\|_{L^1} = 1$ and $\|y_n\|_{L^\alpha} \longrightarrow 0$. For any $\epsilon > 0$ we have $\mu\{|y_n| > \epsilon\} \le \epsilon^{-\alpha}\|y_n\|^\alpha_{L^\alpha}$. Thus we obtained a sequence $y_n \in Y$ so that

$$\|y_n\|_{L^1} = 1 \quad \text{and} \quad \mu\{|y_n| > \epsilon\} \longrightarrow 0. \tag{4.18}$$

On the other hand a bounded set in a reflexive subspace of $L^1(\Omega)$ is equi-integrable. Hence (4.18) contradicts the reflexivity of Y.

Next we turn to proving (4.17). If the assertion would not hold true then we could find a sequence $y_n \in Y$ and $h_n \in H^1(\mathbb{T})$ so that

$$\|y_n\|_{L^1} = 1 \quad \text{and} \quad \|y_n - h_n\|_{L^1} \longrightarrow 0. \tag{4.19}$$

Let $0 < \alpha < 1$. By Kolmogorov's theorem the analytic Riesz-projection $P : L^2(\mathbb{T}) \to H^2(\mathbb{T})$ extends boundedly as follows,

$$\|P : L^1(\mathbb{T}) \to L^\alpha(\mathbb{T})\| \le A_\alpha. \tag{4.20}$$

By assumption $\overline{y_n} \in H^1(\mathbb{T})$ and $h_n \in H^1(\mathbb{T})$, hence $P(\overline{y_n} - \overline{h_n}) = \overline{y_n}$ and by (4.20)

$$\|y_n\|_{L^\alpha} = \|P(\overline{y_n} - \overline{h_n})\|_{L^\alpha} \le A_\alpha\|y_n - h_n\|_{L^1}. \tag{4.21}$$

By (4.19) and (4.21) we obtained a sequence $y_n \in Y$ satisfying $\|y_n\|_{L^1} = 1$ and $\|y_n\|_{L^\alpha} \longrightarrow 0$. In view of (4.16) this contradicts the reflexivity of Y. □

We fix a bounded linear operator

$$R : L^p(\mathbb{T}) \to L^p(\mathbb{T}), \qquad 1 < p < \infty,$$

and assume also that for any probability space (Ω, μ), $0 < \alpha < 1$ and $1 < p < \infty$ there exists $A = A(\alpha, p, \Omega)$ so that $\mathcal{R} = R \otimes J_\Omega$ extends to a bounded operator

$$\|\mathcal{R} : L^p(L^1) \to L^p(L^\alpha)\| \leq A. \tag{4.22}$$

where $J_\Omega : L^1(\Omega) \to L^\alpha(\Omega)$ denotes the formal inclusion map, and

$$L^p(L^1) = L^p(\mathbb{T}, L^1(\Omega, \mu)), \qquad L^p(L^\alpha) = L^p(\mathbb{T}, L^\alpha(\Omega, \mu)).$$

Given $F \in L^1(L^1)$ we let $F(z) : \Omega \to \mathbb{C}$ be defined by

$$F(z)(\omega) = F(z, \omega).$$

If Y be a closed subspace of $L^1(\Omega, \mu)$, we define $L^1(Y) \subseteq L^1(L^1)$ by the condition $F(z) \in Y$, where $z \in \mathbb{T}$. The norm on $L^p(Y)$ is the one induced by $L^p(L^1)$.

Theorem 4.5. *Let Y be a reflexive subspace of $L^1(\Omega, \mu)$ and $1 < p < \infty$. There exists $A = A(Y, p, \alpha)$ so that the restriction of $\mathcal{R}$ to $L^p(Y)$ is bounded as follows*

$$\|\mathcal{R}_{|L^p(Y)} : L^p(Y) \to L^p(L^1)\| \leq A. \tag{4.23}$$

Proof. The algebraic tensor product $L^p \otimes Y$ is dense in $L^p(Y)$. Fix $g \in L^p \otimes Y$. Then also $\mathcal{R}(g) \in L^p \otimes Y$ or equivalently $\mathcal{R}(g)(z) \in Y$ for any $z \in \mathbb{T}$. Applying Theorem 4.4 to $\mathcal{R}(g)(z) \in Y$ gives $C = C(Y)$ such that

$$\int_\Omega |\mathcal{R}(g)(z)| d\mu \leq C \left(\int_\Omega |\mathcal{R}(g)(z)|^\alpha d\mu \right)^{1/\alpha}.$$

Taking the L^p norm with respect to $z \in \mathbb{T}$ we obtain finally

$$\|\mathcal{R}g\|_{L^p(L^1)} \leq C \|\mathcal{R}(g)\|_{L^p(L^\alpha)}. \tag{4.24}$$

By (4.22), we have

$$\|\mathcal{R}(g)\|_{L^p(L^\alpha)} \leq A \|g\|_{L^p(L^1)}. \tag{4.25}$$

Combining (4.24) and (4.25) gives (4.23) where $A(Y, p, \alpha) = C(Y) A(\alpha, p, \Omega)$ □

Remarks.

1. By Theorem 4.2 the Hilbert transform and the Riesz projection satisfy the assumption (4.22) with

$$A(\alpha, p, \Omega) \leq C_p/(1 - \alpha), \quad C_p \leq Cp^2/(p - 1).$$

2. The constant $A = A(Y, p)$ depends expressly on the reflexive space Y. It is not derived from quantitative properties of Y.

4.3.2 Kislyakov's embedding theorem

Let (Ω, μ) be a complete probability space. We write

$$L^1(L^1) = L^1(\mathbb{T}, L^1(\Omega, \mu)) = L^1(\mathbb{T} \times \Omega, dm \otimes d\mu).$$

Given $F \in L^1(L^1)$ we put $F \in H^1(\mathbb{T}, L^1(\Omega, \mu))$ if

$$F(z) : \Omega \to \mathbb{C}, \quad \omega \to F(z, \omega), \quad \text{satisfies} \quad \int_{\mathbb{T}} F(z) z^n dm(z) = 0, \ n \in \mathbb{N}.$$

Fix next a closed linear subspace Y of $L^1(\Omega, \mu)$. We say that $F \in L^1(L^1)$ belongs to $L^1(\mathbb{T}, Y)$ if

$$F(z) \in Y, \quad z \in \mathbb{T}.$$

Finally we put

$$H^1(\mathbb{T}, Y) = H^1(\mathbb{T}, L^1(\Omega, \mu)) \cap L^1(\mathbb{T}, Y).$$

We use throughout the notational convention

$$L^1(Y) = L^1(\mathbb{T}, Y), \quad H^1(L^1) = H^1(\mathbb{T}, L^1(\Omega)), \quad H^1(Y) = H^1(\mathbb{T}, Y).$$

By definition $L^1(Y)$ respectively $H^1(Y)$ are closed subspaces in $L^1(L^1)$ respectively $H^1(L^1)$ hence when equipped with the quotient norm the spaces

$$L^1(Y)/H^1(Y) \quad \text{and} \quad L^1(L^1)/H^1(L^1)$$

are indeed Banach spaces. Given $F_1 \in L^1(Y)$ and $F_2 \in L^1(L^1)$, the corresponding equivalence classes are defined by

$$[F_1]_1 = \{F_1 + u : u \in H^1(Y)\}, \quad [F_2]_2 = \{F_2 + v : v \in H^1(L^1)\}$$

For any $F \in L^1(Y)$ we have clearly $[F]_1 \subseteq [F]_2$. Hence for any closed subspace $Y \subseteq L^1(\Omega)$ the inclusion map

$$I : L^1(Y)/H^1(Y) \to L^1(L^1)/H^1(L^1), \quad [F]_1 \to [F]_2$$

is a well defined linear contraction.

Kislyakov's embedding theorem asserts that under the additional hypothesis that Y is a reflexive subspace of $L^1(\Omega)$, the inclusion map I is invertible on its range (it is an embedding). It is our first application of Theorem 4.5. Define R such that $\mathrm{Id} - R$ is the Riesz projection. By Theorem 4.2 for $0 < \alpha < 1$.

$$\mathcal{R} = R \otimes J_\Omega, \quad \text{with} \quad J_\Omega : L^1(\Omega) \to L^\alpha(\Omega), \quad J_\Omega(f) = f,$$

extends then to a bounded operator $\mathcal{R} : L^2(L^1) \to L^2(L^\alpha)$.

Theorem 4.5 is hence applicable $\mathcal{R}$.

Theorem 4.6. *If Y is a reflexive subspace of $L^1(\Omega)$ then the inclusion operator*

$$I : L^1(Y)/H^1(Y) \to L^1(L^1)/H^1(L^1), \quad [F]_1 \to [F]_2$$

is invertible on its range. There exists $C > 0$ so that for any $F \in L^1(Y)$

$$\|[F]_1\|_{Z_1} \leq C\|[F]_2\|_{Z_2}. \tag{4.26}$$

where

$$Z_1 = L^1(Y)/H^1(Y), \qquad Z_2 = L^1(L^1)/H^1(L^1).$$

Consequently for any $u_1 \in Z_1^*$ there exists $u_2 \in Z_2^*$ so that for any $F \in L^1(Y)$,

$$u_2([F]_2) = u_1([F]_1), \quad F \in L^1(Y), \quad \textit{and} \quad \|u_2\|_{Z_2^*} \leq C\|u_1\|_{Z_1^*}. \tag{4.27}$$

Proof. **Step 1.** Let $F \in L^1(Y)$. Since the algebraic tensor product $L^1(\mathbb{T}) \otimes Y$ is dense in $L^1(Y)$ we may assume that

$$F = \sum_{j=1}^{n} f_j x_j, \qquad f_j \in L^1(\mathbb{T}), \quad x_j \in Y (\subseteq L^1(\Omega)). \tag{4.28}$$

To $[F]_2$ select a lifting $A : \mathbb{T} \times \Omega \to \mathbb{C}$ such that

$$\int_{\mathbb{T}} \|A(z)\|_{L^1(\Omega)} dm(z) \leq 2\|[F]_2\|_{Z_2}. \tag{4.29}$$

Fix $z \in \mathbb{T}$ and put $\varphi(z) = \|A(z)\|_{L^1}$. Let $\epsilon > 0$ and let $g \in H^1(\mathbb{T})$ be the outer function satisfying $|g| = \varphi + \epsilon$. Explicitly

$$g = \exp(\ln(\varphi + \epsilon) + iH \ln(\varphi + \epsilon)),$$

where H denotes the Hilbert transform on $\mathbb{T}$. We will show below that

$$G = g^{1/2}\mathcal{R}(Ag^{-1/2}) \tag{4.30}$$

is a lifting of $[F]_1$, and that there exists a constant $C = C(Y)$ so that

$$\int_{\mathbb{T}} \|G(z)\|_{L^1} dm \leq C \int_{\mathbb{T}} \|A(z)\|_{L^1} dm. \tag{4.31}$$

If G is a lifting of $[F]_1$, and A is a lifting of $[F]_2$ the estimates (4.31) and (4.29) imply (4.26).

Step 2. Here we show that $[G]_1 = [F]_1$. Since A is a lifting of $[F]_2$ and $\mathcal{R}$ annihilates $H^1(L^1)$ we have

$$g^{1/2}\mathcal{R}(Ag^{-1/2}) = \sum_{j=1}^{n} g^{1/2}\mathcal{R}(f_j g^{-1/2})x_j. \tag{4.32}$$

Moreover,

$$g^{1/2}\mathcal{R}(f_j g^{-1/2}) - f_j \in H^1(\mathbb{T}), \tag{4.33}$$

hence combining (4.33) with (4.32) and (4.28) gives $G - F \in H^1(Y)$, or equivalently $q_1(G) = q_1(F)$.
Step 3. Next we prove the $L^1(L^1)$ norm estimates for G. First we fix $z \in \mathbb{T}$, and let

$$\mathcal{R}(Ag^{-1/2})(z) : \Omega \to \mathbb{C}, \quad \omega \to \mathcal{R}(Ag^{-1/2})(z, \omega).$$

Observe that by (4.30), $\|G(z)\|_{L^1} = |g^{1/2}(z)| \cdot \|\mathcal{R}(Ag^{-1/2})(z)\|_{L^1}$. Applying the Cauchy Schwarz inequality we obtain the reduction of the $L^1(L^1)$ to $L^2(L^1)$ estimates,

$$\int_{\mathbb{T}} \|G(z)\|_{L^1} dm \leq \left(\int_{\mathbb{T}} |g(z)| dm\right)^{1/2} \left(\int_{\mathbb{T}} \|\mathcal{R}(Ag^{-1/2})(z)\|_{L^1}^2 dm\right)^{1/2}.$$

By (4.32), $\mathcal{R}(Ag^{-1/2})(z) \in Y$ and since Y is reflexive, Theorem 4.5 implies that there exists a constant $C = C(Y)$ so that

$$\int_{\mathbb{T}} \|\mathcal{R}(Ag^{-1/2})(t)\|_{L^1}^2 dm \leq C \int_{\mathbb{T}} |g(z)|^{-1} \|A(z)\|_{L^1}^2 dm = C \int_{\mathbb{T}} \|A(z)\|_{L^1} dm.$$

Recall that $|g(z)| = \|A(z)\|_{L^1} + \epsilon$. With $\epsilon \to 0$ the previous two inequalities yield

$$\int_{\mathbb{T}} \|G(z)\|_{L^1} dm \leq C \int_{\mathbb{T}} \|A(z)\|_{L^1} dm.$$

In Step 2 we proved that G is a lifting of $[F]_1$ hence the above estimate and (4.29) give (4.26). Finally, by the Hahn Banach theorem (4.26) yields (4.27) . $\square$

We use the embedding to prove that bounded operators into H^∞ that are defined on reflexive subspaces of L^1 can be lifted to bounded operator on L^1. Thus we deduce Bourgain's lifting theorem from Kislyakov's embedding theorem.

For $f \in L^1(\mathbb{T})$ define the equivalence class $[f] = \{f + h : h \in H^1(\mathbb{T})\}$ and the quotient space

$$L^1/H^1 = L^1(\mathbb{T})/H^1(\mathbb{T}) = \{[f] : f \in L^1(\mathbb{T})\}.$$

Its dual space is $H_0^\infty(\mathbb{T}) = \{g \in L^\infty(\mathbb{T}) : \widehat{g(n)} = 0,\ n \in \mathbb{Z} \setminus \mathbb{N}\}$ where the duality is provided by the bilinear form

$$\langle [f], g \rangle = \int_{\mathbb{T}} fg dm, \quad f \in L^1(\mathbb{T}), g \in H_0^\infty(\mathbb{T}).$$

Theorem 4.7. *Let Y be a reflexive subspace of $L^1(\Omega)$. There exists a constant $C = C(Y)$ so that : Any bounded linear operator $T : Y \to H^\infty(\mathbb{T})$ admits an extension to a bounded linear operator $T_1 : L^1(\Omega) \to H^\infty(\mathbb{T})$ such that*

$$T_1(y) = T(y), \quad y \in Y, \quad \text{and} \quad \|T_1\| \leq C\|T\|,$$

where $C = C(Y)$ is given by Kislyakov's embedding theorem.

Proof. Given $f \in L^1(\mathbb{T})$, $y \in Y$ and $z \in \mathbb{C}$, we define

$$u(f \otimes y)(z) = g(z)(Ty)(z).$$

By linearity u has a well defined extension to the algebraic tensor product $L^1(\mathbb{T}) \otimes Y$ and a uniquely defined, and bounded extension to $L^1(Y)$ so that

$$\|u : L^1(\mathbb{T}, Y) \to L^1(\mathbb{T})\| \le \|T\|.$$

Next define the linear functional

$$u_0 : L^1(\mathbb{T}, Y) \to \mathbb{C}, \quad F \to \int_{\mathbb{T}} u(F)(z) dm(z).$$

Clearly $\|u_0\| \le \|u\|$, and $H^1(Y) \subseteq \ker(\mathrm{u}_0)$. Hence u_0 induces a well defined functional on the quotient space $Z_1 = L^1(Y)/H^1(Y)$,

$$u_1 : Z_1 \to \mathbb{C}, \quad [F]_1 \to u_0(F)$$

so that $\|u_1\| \le \|u_0\|$ and

$$u_1([f \otimes y]_1) = \langle [f], Ty \rangle, \quad f \in L^1(\mathbb{T}),\ y \in Y.$$

By Theorem 4.6 there exists an extension of u_1 to the quotient space $Z_2 = L^1(L^1)/H^1(L^1)$

$$u_2 : Z_2 \to \mathbb{C},$$

satisfying (4.27). In particular u_2 inherits from u_1 the identities

$$u_2([f \otimes y]_2) = u_1([f \otimes y]_1) = \langle [f], Ty \rangle, \quad f \in L^1(\mathbb{T}),\ y \in Y.$$

Moreover for $x \in L^1(\Omega)$ and $f \in L^1(\mathbb{T})$ we get

$$|u_2([f \otimes x]_2)| \le \|u_2\|_{Z_2^*} \|[f \otimes x]_2\|_{Z_2} \le C \|u_1\|_{Z_1^*} \|[f]\|_{L^1/H^1} \|x\|_{L^1(\Omega)}$$

Define finally the extension $T_1 : L^1(\Omega) \to H^\infty(\mathbb{T})$ by the duality relation,

$$\langle [f], T_1 x \rangle = u_2([f \otimes y]_2), \quad [f] \in L^1(\mathbb{T})/H^1(\mathbb{T}),\ x \in L^1(\Omega).$$

Thus defined T_1 is an extension of T since

$$\langle [f], T_1 y \rangle = u_2([f \otimes y]_2) = u_1([f \otimes y]_1) = \langle [f], Ty \rangle, \quad y \in Y.$$

and bounded as follows,

$$\|T_1 x\| \le C \|u_1\|_{Z_1^*} \|x\|_{L^1(\Omega)} \le C \|T\| \cdot \|x\|_{L^1(\Omega)}, \quad x \in L^1(\Omega).$$

□

Remark By a theorem of Pisier the above extension theorem implies that L^1/H^1 is Cotype 2 space which satisfies Grothendieck's theorem. For the significance of this assertion we refer again to Kislyakov's paper [Kis91].

5 Haar projections

In this Section we discuss interpolatory estimates between directional Haar projections and Riesz transforms. Their significance for problems in the calculus of variation –in particular for the famous open problem if rank-one convexity implies quasiconvexity for all 2×2 matrices– was recognized by Stefan Müller in [Mül99]. The interpolatory estimates proven by S. Müller [Mül99] are the analytic backbone of his solution to a long standing conjecture of L. Tartar that rank-one convexity implies quasiconvexity for *diagonal* matrices.

5.1 Interpolatory estimates

We use an isotropic system that is supported on dyadic cubes. **Haar basis in $L^2(\mathbb{R}^n)$.** Let $x \in \mathbb{R}^n$, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{0,1\}^n \setminus \{0\}$,

$$h^{(\varepsilon)}_{I_1\times\cdots\times I_n}(x) = h^{\varepsilon_1}_{I_1}(x_1)\cdots h^{\varepsilon_n}_{I_n}(x_n),$$

where $|I_1| = \cdots = |I_n|$. **Interpolatory Estimates**. Fix $i_0 \le n$, a Riesz transform,

$$\widehat{R_{i_0}(w)}(y) = \frac{y_{i_0}}{|y|}\hat{w}(y)$$

and select directions

$$\varepsilon \in \{0,1\}^n \setminus \{0\} \text{ so that } \varepsilon_{i_0} = 1.$$

Let

$$\mathcal{H}^{(\varepsilon)} = \{h^{(\varepsilon)}_{I_1\times\cdots\times I_n} : |I_1| = \cdots = |I_n|\}.$$

Then

$$P^{(\varepsilon)}w(x) = \sum \langle w, h^{(\varepsilon)}_{I_1\times\cdots\times I_n}\rangle \frac{h^{(\varepsilon)}_{I_1\times\cdots\times I_n}(x)}{|I_1\times\cdots\times I_n|}$$

is the orthogonal projection onto $\operatorname{span}\mathcal{H}^{(\varepsilon)}$. Directional Haar projections $P^{(\varepsilon)}$ are pointwise dominated by the Riesz transform R_{i_0}. We get

Theorem 5.1 ([LMM11])**.** *Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. For $1 \le i_0 \le n$ define*

$$\mathcal{A}_{i_0} = \{\varepsilon \in \{0,1\}^n \setminus \{0\} : \varepsilon_{i_0} = 1.\}$$

Let $w \in L^p(\mathbb{R}^n)$. If $\varepsilon \in \mathcal{A}_{i_0}$, then $P^{(\varepsilon)}$ and R_{i_0} are related by interpolatory estimates in L^p:

$$\|P^{(\varepsilon)}w\|_p \le C_p \|R_{i_0}w\|_p^{1/2}\|w\|_p^{1/2}, \quad p \ge 2$$

and

$$\|P^{(\varepsilon)}w\|_p \le C_p \|R_{i_0}w\|_p^{1/q}\|w\|_p^{1/p}, \quad p \le 2.$$

Remark. The estimates of the above theorem are meaningful only when $\|R_{i_0}w\|_p << \|w\|_p$. We remark as well that the exponents are sharp in the following sense: For any $\delta < 0$,

$$\sup_{w \in L^p} \frac{\|P^{(\varepsilon)}w\|_p}{\|R_{i_0}w\|_p^{1/2+\delta}\|w\|_p^{1/2-\delta}} = \infty$$

Remark. Extensions to the case where the Haar projections are replaced by directional wavelet projections are obtained in [MM15]. The case of vector valued operators ranging in a UMD space was solved by R. Lechner [Lec14] in his dissertation.

5.2 Interpolatory estimates are useful

We describe here in broad strokes the argument that links interpolatory estimates to problems in compensated compactness. A detailed discussion may be found in [LMM11]. To a vector field $v : \mathbb{R}^n \to \mathbb{R}^n$ define

$$\mathcal{A}_0(v) = \nabla v - \mathrm{diag}\nabla v,$$

Explicitly,

$$\mathcal{A}_0(v)_{i,j} = \partial_i v_j \quad \text{if} \quad i \neq j$$

and $\mathcal{A}_0(v)_{i,i} = 0$. If $\mathcal{A}_0(v) = 0$ then v splits as

$$v(x) = (v_1(x_1), \ldots, v_n(x_n)).$$

Theorem 5.2 ([LMM11]). *Let $1 < p < \infty$. Let*

$$f : \mathbb{R}^n \to \mathbb{R}^+ \text{ separately convex,}$$

and of moderate growth, i.e. $0 \leq f(x) \leq (1 + |x|^p)$.
If

$$\begin{cases} v_r \rightharpoonup v & \text{in} \quad L^p, \\ \mathcal{A}_0(v_r) & \text{precompact in} \quad W^{-1,p}, \end{cases}$$

then

$$\int_{\mathbb{R}^n} f(v(x))\varphi(x)dx \leq \liminf_{r\to\infty} \int_{\mathbb{R}^n} f(v_r(x))\varphi(x)dx, \quad \varphi \in C_c^+(\mathbb{R}^n).$$

Theorem 5.2 is a consequence of the following decomposition.
Decomposition. For sequences $(v_r : \mathbb{R}^n \to \mathbb{R}^n)$ supported in the unit cube satisfying

$$\begin{cases} v_r \rightharpoonup 0 & \text{weakly in} \quad L^p, \\ \mathcal{A}_0(v_r) \to 0 & \text{in} \quad W^{-1,p} \end{cases}$$

there exists a decomposition

$$v_r = u_r + w_r,$$

so that :

1. the separately convex f satisfies Jensen's inequality on u_r :

$$\int_{[0,1]^n} f(a+u_r(x))dx \geq f(a).$$

2. w_r converges in L^p norm, $\|w_r\|_{L^p} \to 0$.

We next discuss the role of interpolatory estimates in the proof of the decomposition stated above. Recall $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{0,1\}^n \setminus \{0\}$,

$$h^{(\varepsilon)}_{I_1\times\cdots\times I_n}(x) = h^{\varepsilon_1}_{I_1}(x_1)\cdots h^{\varepsilon_n}_{I_n}(x_n),$$

Let

$$e_j = (0,\ldots,1,\ldots 0) \qquad \mathcal{H}^j = \{h^{(e_j)}_{I_1\times\cdots\times I_n} :\}.$$

$P^{(j)}$ is the Projection onto span$\mathcal{H}^j$.

$$P(v) = \left(P^{(1)}(v_1),\ldots,P^{(n)}(v_n)\right) \quad v = (v_1,\ldots,v_n).$$

These are the central properties of P :

1. Jensen's inequality for **separately convex** functions in the range of P.
2. $\|v - P(v)\|_p \leq C_p \|\mathcal{A}_0 v\|^{1-\alpha}_{W^{-1,p}} \|v\|^{\alpha}_p + \|T(v)\|_p$, with T compact.

For sequences of vector fields (v_r) supported in the unit cube satisfying

$$\begin{cases} v_r \rightharpoonup 0 \quad \text{weakly in} \quad L^p, \\ \mathcal{A}_0(v_r) \to 0 \quad \text{in} \quad W^{-1,p} \end{cases}$$

decompose $u_r = P(v_r)$ and $w_r = v_r - P(v_r)$. Interpolatory estimates imply $\|w_r\|_p \to 0$: Indeed for $p \geq 2$ we have

$$\begin{aligned} \|v - P(v)\|_p &\leq C_p \sum_{i=1}^n \sum_{j=1, j\neq i}^n \|R_j v_i\|^{1/2}_p \|v_i\|^{1/2}_p \\ &\leq C_p \|\mathcal{A}_0 v\|^{1/2}_{W^{-1,p}} \|v\|^{1/2}_p + \|T(v)\|_p, \end{aligned}$$

where T is a compact operator in L^p. (In addition to interpolatory estimates we used the link between Riesz transforms and $W^{-1,p}$ asserting that $\|R_j w\|_p \leq C\|\partial_j w\|_{W^{-1,p}} + \|Tw\|_p$, where T a compact in L^p .

Finally we verify that the range of P and separately convex functions are linked by Jensen's inequality as follows: If f is separately convex then

$$f\left(\int_{[0,1]^n} P(v)(x)dx\right) \leq \int_{[0,1]^n} f(P(v(x))dx.$$

Indeed, since

$$h^{(e_j)}_Q(x) = h_{I_j}(x_j), \quad x \in Q = I_1 \times \cdots \times I_n$$

the restriction of $h_Q^{(e_j)}$ to Q is **just** a function of x_j, and with Jensen's inequality applied to each variable separately, we have

$$\begin{aligned}
&\int_Q f(a_1 + c_1 h_Q^{(e_1)}(x), \dots, a_n + c_n h_Q^{(e_n)}(x))dx \\
&= \int_Q f(a_1 + c_1 h_{I_1}(x_1), \dots, a_n + c_n h_{I_n}(x_n))dx. \\
&\geq |Q| f(a).
\end{aligned}$$

It remains to iterate over levels.

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