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Chern numbers for two families of noncommutative Hopf fibrations

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Abstract

We consider noncommutative line bundles associated with the Hopf fibrations of $\mathrm{SU}_q(2)$ over all Podleś spheres and with a locally trivial Hopf fibration of S^3_{pq}. These bundles are given as finitely generated projective modules associated via 1-dimensional representations of U(1) with Galois-type extensions encoding the principal fibrations of $\mathrm{SU}_q(2)$ and S^3_{pq}. We show that the Chern numbers of these modules coincide with the winding numbers of representations defining them. ***To cite this article: P.M. Hajac et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Nombres de Chern pour deux familles de fibrations de Hopf non commutatives. Nous considérons des fibrés en droites non commutatifs associés à la fibration de Hopf quantique de $\mathrm{SU}_q(2)$ sur toutes les sphères quantiques de Podleś ainsi qu'avec une fibration de Hopf localement triviale de S^3_{pq}. Ces fibrés sont construits comme des modules projectifs associés aux représentations de dimension 1 de U(1) avec des extensions galoisiennes relatives aux fibrés principaux de $\mathrm{SU}_q(2)$ et de S^3_{pq}. Nous montrons que les nombres de Chern de ces fibrés coïncident avec les degrés des représentations qui les définissent. ***Pour citer cet article : P.M. Hajac et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***
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Dans cet article, nous combinons les outils algébriques des extensions galoisiennes avec les méthodes analytiques de la formule non commutative de l'indice pour étudier deux types de fibrés quantiques. Notre résultat

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principal est la non isomorphie des fibrés en droites associés à ces fibrés principaux. Cela nous fournit une estimation du groupe K_0 des sphères quantiques de la base. Soit $B \subseteq P$ une inclusion d'algèbres telle que B est la sous-algèbre coinvariante pour une coaction $\Delta_R : P \to P \otimes C$ d'une cogèbre C. Dans le cadre de la théorie des extensions galoisiennes, on peut affirmer qu'une telle extension d'algèbres est principale. (La définition est adaptée de manière à ce qu'elle coïncide dans le cas commutatif avec les torseurs – les fibrés principaux de la géométrie algébrique.) Toute C-extension principale $B \subseteq P$ nous permet d'associer à chaque coreprésentation de dimension finie de C un module projectif de type fini à gauche sur B d'homomorphismes colinéaires $\mathrm{Hom}^C(V_\varphi, P)$ [2]. En prenant sa classe dans $K_0(B)$ et en la composant avec le caractère de Chern, on construit le caractère de Chern–Galois défini sur l'espace de toutes les coreprésentations de dimension finie et à valeurs dans l'homologie cyclique paire de B. D'autre part, le caractère de Chern en K-homologie associe des cocycles cycliques à tout module de Fredholm finiment sommable. Dans le cas 1-sommable, il prend une forme particulièrement simple, transformant un couple de représentations bornées (ρ_1, ρ_2) en une trace (un 0-cocycle cyclique) sur B donnée par $\mathrm{tr}_\rho = \mathrm{Tr} \circ (\rho_1 - \rho_2)$. L'évaluation de cette trace sur le caractère de Chern–Galois appliqué à une coreprésentation donne un invariant scalaire de la classe dans K_0 du module défini par cette corepésentation. En ce qui concerne nos exemples, l'intégralité de ces invariants (assurée par le théorème non commutatif de l'indice) les rend calculables. Notre premier exemple de fibration de Hopf non commutative se base sur les disques quantiques de Klimek–Lesniewski et sur la notion de trivialité locale. Dans ce cas, nous avons une $\mathcal{O}(\mathrm{U}(1))$-extension principale $\mathcal{O}(S^2_{pq}) \subseteq \mathcal{O}(S^3_{pq})$ [8], et toute coreprésentation de dimension 1 $\varphi_\mu(1) = 1 \otimes u^{-\mu}$ fournit un $\mathcal{O}(S^2_{pq})$-module projectif de type fini (fibré en droites quantique) $\mathcal{O}(S^3_{pq})_\mu := \mathrm{Hom}^{\mathcal{O}(\mathrm{U}(1))}(\mathbb{C}_{\varphi_\mu}, \mathcal{O}(S^3_{pq}))$. D'autre part, nous définissons les représentations involutives bornées de $\mathcal{O}(S^2_{pq})$ suivantes [4] :

$$\rho_1(f_0)e_k = \big(1 - p^k\big)e_k, \quad \rho_1(f_1)e_k = \sqrt{1 - p^{k+1}}\, e_{k+1}, \quad k \geqslant 0, \tag{0.1}$$

$$\rho_2(f_0)e_k = e_k, \quad \rho_2(f_1)e_k = \sqrt{1 - q^{k+1}}\, e_{k+1}, \quad k \geqslant 0, \tag{0.2}$$

où $\{e_k\}_{k \geqslant 0}$ désigne une base orthonormale d'un espace de Hilbert séparable et f_0, f_1 les générateurs de l'algèbre involutive $\mathcal{O}(S^2_{pq})$. Ces représentations nous fournissent la trace désirée et nous mènent à notre premier résultat principal :

Théorème 0.1. *Pour tout $\mu \in \mathbb{Z}$, le couplage entre le 0-cocycle cyclique tr_ρ et la classe dans K_0 de $\mathcal{O}(S^3_{pq})_\mu$ (nombre de Chern) coïncide avec le degré μ, i.e., $\langle \mathrm{tr}_\rho, [\mathcal{O}(S^3_{pq})_\mu] \rangle = \mu$.*

Notre second exemple consiste en une famille de fibrations de Hopf non commutatives de $\mathrm{SU}_q(2)$ sur toutes les sphères quantiques de Podleś $S^2_{q,s}$, $s \in [0,1]$. Etant donné que traiter les sphères de Podleś génériques nécessite d'aller au-delà de la théorie de Hopf–Galois, celles-ci furent parmi les principaux exemples motivant le développement de la théorie des extensions principales de cogèbres des anneaux non commutatifs. Dans ce cas, tout comme avant, nous associons à toute coreprésentation φ_μ un $\mathcal{O}(S^2_{q,s})$-module projectif de type fini à gauche $\mathcal{O}(\mathrm{SU}_q(2))_{\mu,s} := \mathrm{Hom}^{\mathcal{O}(\mathrm{SU}_q(2))/J_s}(\mathbb{C}_{\varphi_\mu}, \mathcal{O}(\mathrm{SU}_q(2)))$. Ici J_s est le coidéal droit idéal engendré par K, $L - s$, $L^* - s$, et K, L génèrent l'algèbre polynomiale $\mathcal{O}(S^2_{q,s})$. On peut montrer que $\mathcal{O}(\mathrm{SU}_q(2))/J_s$ coïncide avec $\mathcal{O}(\mathrm{U}(1))$ considérée comme une cogèbre, et que $\mathcal{O}(S^2_{q,s}) \subseteq \mathcal{O}(\mathrm{SU}_q(2))$ est une $\mathcal{O}(\mathrm{SU}_q(2))/J_s$-extension principale [1,12,3]. De plus, les représentations [13]

$$\pi_-(K)e_n = -s^2q^{2n}e_n, \quad \pi_-(L)e_n = \lambda_n^-(q,s)e_{n-1}, \quad \lambda_n^-(q,s) = s\sqrt{1 - (1-s^2)q^{2n} - s^2q^{4n}}, \tag{0.3}$$

$$\pi_+(K)e_n = q^{2n}e_n, \quad \pi_+(L)e_n = \lambda_n^+(q,s)e_{n-1}, \quad \lambda_n^+(q,s) = \sqrt{s^2 + (1-s^2)q^{2n} - q^{4n}}, \tag{0.4}$$

forment le module de Fredholm requis [11], et nous avons notre second résultat principal :

Théorème 0.2. *Pour tout $\mu \in \mathbb{Z}$, le couplage entre le 0-cocycle cyclique tr_π et la classe dans K_0 de $\mathcal{O}(\mathrm{SU}_q(2))_{\mu,s}$ (nombre de Chern) coïncide avec le degré μ, i.e., $\langle \mathrm{tr}_\pi, [\mathcal{O}(\mathrm{SU}_q(2))_{\mu,s}]\rangle = \mu$.*

1. Introduction

Herein we study two families of noncommutative deformations of the Hopf fibration $S^3 \to S^2$. The first one is based on the idea of local triviality. We can view S^2 as the gluing of two discs along their boundaries, and S^3 as the gluing of two solid tori along their boundaries (a Heegaard splitting of S^3). Then the classical discs can be replaced by Klimek–Lesniewski quantum discs [9], and subsequently a two-parameter family of noncommutative Hopf fibrations can be constructed [5,8]. The other family originates from the theory of quantum groups. First, one considers S^3 as $\mathrm{SU}(2)$ and deforms it into the quantum group $\mathrm{SU}_q(2)$ [14]. Then the classification of $\mathrm{SU}_q(2)$-quantum homogeneous spaces yields a two-parameter family of noncommutative two-spheres [13]. The latter form the base of the Hopf fibrations of $\mathrm{SU}_q(2)$ [3].

Function algebras of total spaces of principal U(1)-bundles always decompose into direct sums of sections of all associated line bundles. The same phenomenon occurs for both of the aforementioned deformations, i.e., the coordinate algebras $\mathcal{O}(S^3_{pq})$ and $\mathcal{O}(\mathrm{SU}_q(2))$ are direct sums of associated finitely generated projective modules. The aim of this paper is to prove that these modules are mutually non-isomorphic.

To achieve this, we take an advantage of the K-homology Chern character [6] and the Chern–Galois character [2]. The former produces cyclic cocycles out of bounded representations of the base algebra, and the latter cyclic cycles out of finite dimensional corepresentations of the structure coalgebra that define associated modules. The evaluation of these cocycles on these cycles gives K_0-invariants of the modules. The noncommutative index formula [6] shows that these invariants are indices of Fredholm operators, so that they have to be integers. This fact is essentially used in carrying out the computations.

Throughout this paper we work with unital algebras over a field (complex numbers in the studied examples) and adopt the standard Hopf-algebraic notation Δ, ε, S for the comultiplication, counit and antipode, respectively. The symbolic notation $\mathcal{O}$(quantum space) means a polynomial algebra defined by generators and relations, m stands for the multiplication map, and Tr denotes the operator trace.

2. Principal extensions and summable Fredholm modules

Let C be a coalgebra and P an algebra and a right C-comodule via $\Delta_R : P \to P \otimes C$. Put $B = P^{coC} := \{b \in P \mid \Delta_R(bp) = b\Delta_R(p),\ \forall p \in P\}$. We say that the inclusion $B \subseteq P$ is a C-extension. A C-extension $B \subseteq P$ is called *principal* [2] iff

(1) the Galois map $can : P \otimes_B P \to P \otimes C,\ p \otimes p' \mapsto p\Delta_R(p')$ is bijective;
(2) there exists a left B-linear right C-colinear splitting of the multiplication map $B \otimes P \to P$;
(3) the canonical entwining map $\psi : C \otimes P \to P \otimes C$, $c \otimes p \mapsto can(can^{-1}(1 \otimes c)p)$ is bijective;
(4) there is a group-like $e \in C$ such that $\Delta_R(p) = \psi(e \otimes p)$, $\forall p \in P$.

In order to define a strong connection on a principal extension, first we need to define a bicomodule structure on $P \otimes P$. The tensor product $P \otimes P$ is a right C-comodule via $\Delta_R^{\otimes} := \mathrm{id} \otimes \Delta_R$. Since ψ is bijective, it is also a left C-comodule via $\Delta_L^{\otimes} := (\psi^{-1} \circ (\mathrm{id} \otimes e)) \otimes \mathrm{id}$. The two coactions evidently commute. Now, let $\pi_B : P \otimes P \to P \otimes_B P$ be the canonical surjection. A linear map $\ell : C \to P \otimes P$ is called a *strong connection* [2] iff it satisfies $can \circ \pi_B \circ \ell = 1 \otimes \mathrm{id}$, $\Delta_R^{\otimes} \circ \ell = (\ell \otimes \mathrm{id}) \circ \Delta$, $\Delta_L^{\otimes} \circ \ell = (\mathrm{id} \otimes \ell) \circ \Delta$, and $\ell(e) = 1 \otimes 1$. If $B \subseteq P$ is a principal C-extension and $\varphi : V_\varphi \to V_\varphi \otimes C$ is a finite-dimensional corepresentation, then the left B-module $\mathrm{Hom}^C(V_\varphi, P)$ of all colinear maps from V_φ to P is finitely generated projective [2]. Denote by

$\mathrm{Corep}_f(C)$ the space of all finite-dimensional corepresentations of C. Then a principal C-extension $B \subseteq P$ yields a map $\varphi \mapsto [\mathrm{Hom}^C(V_\varphi, P)]$ from $\mathrm{Corep}_f(C)$ to $K_0(B)$. The composition of this map with the Chern character $K_0(B) \to HC_{\mathrm{even}}(B)$ (e.g., see [10]) is called the *Chern–Galois character* [2] and is denoted by *chg*. Explicitly, in degree 0, we have $chg_0(\varphi) = [c_\varphi^{\langle 2\rangle} c_\varphi^{\langle 1\rangle}]$. Here c_φ is the character of φ, i.e., $c_\varphi = \sum_{i=1}^{\dim V} e_{ii}$, $\varphi(e_j) = \sum_{i=1}^{\dim V} e_i \otimes e_{ij}$, $\{e_i \mid i \in \{1, \ldots, \dim V\}\}$ a basis of V_φ, and $\ell(c) =: c^{\langle 1\rangle} \otimes c^{\langle 2\rangle}$ (summation understood).

Let us now recall the analytic tool of Fredholm modules that are used in the sequel. A *p-summable Fredholm module* over a $*$-algebra B can be viewed as a pair (ρ_1, ρ_2) of bounded $*$-representations of B such that $\mathrm{Tr}\,|\rho_1(b) - \rho_2(b)|^p < \infty$ for all $b \in B$ (see [6, p. 88] for related details). The K-homology Chern character assigns to finitely summable Fredholm modules cyclic cocycles [6]. For $p = 1$ it takes a particularly simple form, notably it turns (ρ_1, ρ_2) into a trace (cyclic 0-cocycle) on B via the formula $\mathrm{tr}_\rho(b) = \mathrm{Tr}(\rho_1(b) - \rho_2(b))$.

3. A locally trivial quantum Hopf bundle

Let us consider the two-parameter family [5] of $*$-algebras $\mathcal{O}(S^3_{pq})$, $0 \leqslant p, q \leqslant 1$, generated by a and b satisfying $a^*a - qaa^* = 1 - q$, $b^*b - pbb^* = 1 - p$, $ab = ba$, $a^*b = ba^*$, $(1 - aa^*)(1 - bb^*) = 0$. The $*$-subalgebra of $\mathcal{O}(S^3_{pq})$ generated by ab and bb^* can be identified with the $*$-algebra $\mathcal{O}(S^2_{pq})$ generated by f_0 and f_1 satisfying [4] $f_0 = f_0^*$, $f_1^* f_1 - q f_1 f_1^* = (p-q) f_0 + 1 - p$, $f_0 f_1 - p f_1 f_0 = (1-p) f_1$, $(1 - f_0)(f_1 f_1^* - f_0) = 0$. The isomorphism is given by $f_0 \mapsto bb^*$ and $f_1 \mapsto ab$. It follows from [8, Lemma 4.2] that $\mathcal{O}(S^2_{pq}) \subseteq \mathcal{O}(S^3_{pq})$ is a principal $\mathcal{O}(\mathrm{U}(1))$-extension. Moreover, we have

Lemma 3.1 [8]. *Let u be the unitary generator of $\mathcal{O}(\mathrm{U}(1))$. The linear map $\ell : \mathcal{O}(\mathrm{U}(1)) \to \mathcal{O}(S^3_{pq}) \otimes \mathcal{O}(S^3_{pq})$ given on the basis elements u^μ, $\mu \in \mathbb{Z}$, by the formulas*

$$\ell(1) = 1 \otimes 1, \quad \ell(u) = a^* \otimes a + qb(1 - aa^*) \otimes b^*, \quad \ell(u^*) = b^* \otimes b + pa(1 - bb^*) \otimes a^*, \tag{3.1}$$

$$\ell\big(u^\mu\big) = u^{[1]} \ell\big(u^{\mu-1}\big) u^{[2]}, \quad \ell\big(u^{*\mu}\big) = u^{*[1]} \ell\big(u^{*(\mu-1)}\big) u^{*[2]}, \quad \mu > 0, \tag{3.2}$$

defines a strong connection on $\mathcal{O}(S^2_{pq}) \subseteq \mathcal{O}(S^3_{pq})$.

The one-dimensional corepresentations of $\mathcal{O}(\mathrm{U}(1))$ are labelled by integers. Explicitly, we have $\varphi_\mu(1) = 1 \otimes u^{-\mu}$. Each φ_μ yields a finitely generated projective $\mathcal{O}(S^2_{pq})$-module (quantum line bundle) $\mathcal{O}(S^3_{pq})_\mu := \mathrm{Hom}^{\mathcal{O}(\mathrm{U}(1))}(\mathbb{C}_{\varphi_\mu}, \mathcal{O}(S^3_{pq}))$. On the other hand, we have the following irreducible bounded $*$-representations of $\mathcal{O}(S^2_{pq})$ [4, Proposition 19]:

$$\rho_1(f_0) e_k = (1 - p^k) e_k, \quad \rho_1(f_1) e_k = \sqrt{1 - p^{k+1}}\, e_{k+1}, \quad k \geqslant 0; \tag{3.3}$$

$$\rho_2(f_0) e_k = e_k, \quad \rho_2(f_1) e_k = \sqrt{1 - q^{k+1}}\, e_{k+1}, \quad k \geqslant 0. \tag{3.4}$$

Here $\{e_k\}_{k \geqslant 0}$ is an orthonormal basis of a separable Hilbert space. Moreover, we have

Lemma 3.2. *The pair of representations (ρ_2, ρ_1) given by* (3.3)–(3.4) *yields a* 1*-summable Fredholm module over $\mathcal{O}(S^2_{pq})$, so that $\mathrm{tr}_\rho(f) := \mathrm{Tr}(\rho_2(f) - \rho_1(f))$ defines a trace on $\mathcal{O}(S^2_{pq})$.*

Theorem 3.3. *For all $\mu \in \mathbb{Z}$, the pairing between the cyclic* 0*-cocycle tr_ρ and the K_0-class of $\mathcal{O}(S^3_{pq})_\mu$ (Chern number) coincides with the winding number μ, i.e., $\langle \mathrm{tr}_\rho, [\mathcal{O}(S^3_{pq})_\mu] \rangle = \mu$.*

Proof outline. The pairing of cyclic cohomology and K-theory is given by the evaluation of a cyclic cocycle on the image of the Chern character. In our case (see Section 1) it gives $\langle \mathrm{tr}_\rho, [\mathcal{O}(S^3_{pq})_\mu] \rangle = \mathrm{tr}_\rho(chg_0(\varphi_\mu)) =$

$\mathrm{tr}_\rho((u^{-\mu})^{\langle 2\rangle}(u^{-\mu})^{\langle 1\rangle})$. The last expression can be computed explicitly as a function of p or q, depending on whether μ is positive or negative. Surprisingly, these functions can be identified with a certain expression appearing in the index computation carried out in [7]. Since the latter is proven therein to be the constant μ, the assertion of the theorem follows. □

The trace tr_ρ computes the Chern numbers of $\mathcal{O}(S^3_{pq})_\mu$. In order to determine the rank of these modules, we can employ any character of $\mathcal{O}(S^3_{pq})$. Indeed, let δ be an algebra homomorphism from $\mathcal{O}(S^3_{pq})$ to $\mathbb{C}$. (See [8] for the classification of irreducible representations of $\mathcal{O}(S^3_{pq})$, including one-dimensional ones.) Then $\langle\delta, [\mathcal{O}(S^3_{pq})_\mu]\rangle = \delta((u^{-\mu})^{\langle 2\rangle}(u^{-\mu})^{\langle 1\rangle}) = \delta((u^{-\mu})^{\langle 1\rangle}(u^{-\mu})^{\langle 2\rangle}) = \varepsilon(u^{-\mu}) = 1$. The last two equalities follow from the general properties $m\circ\ell = \varepsilon$ (see [2]) and $\varepsilon(\text{group-like}) = 1$, respectively. The characters always pair integrally with K_0 (e.g., see [10]). On the other hand, as tr_ρ comes from a 1-summable Fredholm module, its pairing with K_0 is an index of a Fredholm operator [6, p. 60], whence also an integer. Therefore, it follows from the linearity of the pairing that we have a group homomorphism $(\delta, \mathrm{tr}_\rho) : K_0(\mathcal{O}(S^2_{pq})) \to \mathbb{Z}\oplus\mathbb{Z}$, $[p]\mapsto (\langle\delta,[p]\rangle, \langle\mathrm{tr}_\rho,[p]\rangle)$. The point here is that for any $\mu\in\mathbb{Z}$ there exists a rank one projective module with its Chern number equal to μ. More formally, we have

Corollary 3.4. *The image of the positive cone of* $K_0(\mathcal{O}(S^2_{pq}))$ *under* $(\delta, \mathrm{tr}_\rho) : K_0(\mathcal{O}(S^2_{pq})) \to \mathbb{Z}\times\mathbb{Z}$ *contains* $\mathbb{Z}_+\times\mathbb{Z}$.

4. Hopf fibrations of $\mathrm{SU}_q(2)$ over generic Podleś spheres

In this section, we work with the Hopf $*$-algebra $\mathcal{O}(\mathrm{SU}_q(2))$, $q\in\,]0,1[$, generated by elements α and γ satisfying [14] $\alpha\gamma = q\gamma\alpha$, $\alpha\gamma^* = q\gamma^*\alpha$, $\gamma\gamma^* = \gamma^*\gamma$, $\alpha^*\alpha + \gamma^*\gamma = 1$, $\alpha\alpha^* + q^2\gamma\gamma^* = 1$, and with the Podleś spheres [13]. A uniform description of all Podleś spheres can be obtained by rescaling generators used in [13]. Then the coordinate $*$-algebras of the quantum spheres $S^2_{q,s}$, $s\in[0,1]$, are defined by generators K and L satisfying the relations $K = K^*$, $LK = q^2KL$, $L^*L + K^2 = (1-s^2)K + s^2$, $LL^* + q^4K^2 = (1-s^2)q^2K + s^2$. We can view $\mathcal{O}(S^2_{q,s})$ as a $*$-subalgebra of $\mathcal{O}(\mathrm{SU}_q(2))$ via the formulas $K = s(\gamma\alpha + \alpha^*\gamma^*) + (1-s^2)\gamma^*\gamma$, $L = s(\alpha^2 - q\gamma^{*2}) + (1-s^2)\alpha\gamma^*$. Next, let us define the quotient coalgebra $\mathcal{O}(\mathrm{SU}_q(2))/J_s$, where J_s is the coideal right ideal generated by K, $L-s$, L^*-s. One can show that it coincides with $\mathcal{O}(\mathrm{U}(1))$ viewed as a coalgebra, and that $\mathcal{O}(S^2_{q,s}) = \mathcal{O}(\mathrm{SU}_q(2))^{co\mathcal{O}(\mathrm{SU}_q(2))/J_s}$ [1,12]. Moreover, one can prove that $\mathcal{O}(S^2_{q,s})\subseteq\mathcal{O}(\mathrm{SU}_q(2))$ is a principal $\mathcal{O}(\mathrm{SU}_q(2))/J_s$-extension [3]. With the help of [2], the latter follows from an explicit construction of a strong connection:

Lemma 4.1 [3]. *Let* $i : \mathcal{O}(\mathrm{U}(1)) \to \mathcal{O}(\mathrm{SU}_q(2))$ *be the linear map defined on the basis elements* u^μ, $\mu\in\mathbb{Z}$, *by the formulas*

$$i\big(u^{-\mu}\big) := \begin{cases} \prod_{j=0}^{-\mu-1} h_j & \text{for } \mu<0\\ 1 & \text{for } \mu = 0\\ \prod_{j=0}^{\mu-1} k_j & \text{for } \mu>0,\end{cases} \quad (\text{products increase from left to right}), \tag{4.1}$$

$$h_j := \frac{\alpha + q^j s(\gamma - q\gamma^*) + q^{2j}s^2\alpha^*}{1+q^{2j}s^2}, \qquad k_j := \frac{\alpha^* - q^{-j}s(\gamma - q\gamma^*) + q^{-2j}s^2\alpha}{1+q^{-2j}s^2}. \tag{4.2}$$

Then $\ell = (S\otimes\mathrm{id})\circ\Delta\circ i$ *is a strong connection on* $\mathcal{O}(S^2_{q,s})\subseteq\mathcal{O}(\mathrm{SU}_q(2))$.

Much as before, we associate to every corepresentation φ_μ a finitely generated projective left $\mathcal{O}(S^2_{q,s})$-module $\mathcal{O}(\mathrm{SU}_q(2))_{\mu,s} := \mathrm{Hom}^{\mathcal{O}(\mathrm{SU}_q(2))/J_s}(\mathbb{C}_{\varphi_\mu}, \mathcal{O}(\mathrm{SU}_q(2)))$. On the other hand, using the representations [13]

$$\pi_-(K)e_n = -s^2q^{2n}e_n, \quad \pi_-(L)e_n = \lambda_n^-(q,s)e_{n-1}, \quad \lambda_n^-(q,s) = s\sqrt{1-(1-s^2)q^{2n} - s^2q^{4n}}, \tag{4.3}$$

$$\pi_+(K)e_n = q^{2n}e_n, \quad \pi_+(L)e_n = \lambda_n^+(q,s)e_{n-1}, \quad \lambda_n^+(q,s) = \sqrt{s^2+(1-s^2)q^{2n} - q^{4n}}, \tag{4.4}$$

one can prove

Lemma 4.2 [11]. *For any $q \in\,]0,1[$, $s \in [0,1]$, the pair of representations (π_-, π_+) yields a 1-summable Fredholm module over $\mathcal{O}(S^2_{q,s})$, so that $\mathrm{tr}_\pi := \mathrm{Tr}\circ(\pi_- - \pi_+)$ is a trace on $\mathcal{O}(S^2_{q,s})$.*

Theorem 4.3. *For all $\mu \in \mathbb{Z}$, the pairing between the cyclic 0-cocycle tr_π and the K_0-class of $\mathcal{O}(\mathrm{SU}_q(2))_{\mu,s}$ (Chern number) coincides with the winding number μ, i.e., $\langle \mathrm{tr}_\pi, [\mathcal{O}(\mathrm{SU}_q(2))_{\mu,s}]\rangle = \mu$.*

Proof outline. The proof rests on the following three facts: $\langle \mathrm{tr}_\pi, [\mathcal{O}(\mathrm{SU}_q(2))_{\mu,s}]\rangle$ is a rational function of q and s, it is an integer, and $\langle \mathrm{tr}_\pi, [\mathcal{O}(\mathrm{SU}_q(2))_{\mu,s}]\rangle(q,0) = \mu$. The first claim can be proven with the help of the Chern–Galois character [2], the second follows from the noncommutative index formula [6, p. 60], and the third has been obtained in [7, Theorem 2.1]. Since an integer-valued rational function on a connected set has to be constant, we can conclude that $\langle \mathrm{tr}_\pi, [\mathcal{O}(\mathrm{SU}_q(2))_{\mu,s}]\rangle(q,s) = \langle \mathrm{tr}_\pi, [\mathcal{O}(\mathrm{SU}_q(2))_{\mu,s}]\rangle(q,0) = \mu$. □

Corollary 4.4. *The image of the positive cone of $K_0(\mathcal{O}(S^2_{q,s}))$ under $(\varepsilon, \mathrm{tr}_\pi): K_0(\mathcal{O}(S^2_{q,s})) \to \mathbb{Z}\times\mathbb{Z}$ contains $\mathbb{Z}_+ \times \mathbb{Z}$. (Here ε is the counit of $\mathcal{O}(\mathrm{SU}_q(2))$.)*

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