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Stable anti-Yetter-Drinfeld modules

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Abstract

We define and study a class of entwined modules (stable anti-Yetter–Drinfeld modules) that serve as coefficients for the Hopf-cyclic homology and cohomology. In particular, we explain their relationship with Yetter–Drinfeld modules and Drinfeld doubles. Among sources of examples of stable anti-Yetter–Drinfeld modules, we find Hopf–Galois extensions with a flipped version of the Miyashita–Ulbrich action. *To cite this article: P.M. Hajac et al.*, *C. R. Acad. Sci. Paris*, *Ser. I 338 (2004)*. © 2004 Académie des sciences. Published by Elsevier SAS. All rights reserved.

Résumé

Modules anti-Yetter—Drinfeld stables. Nous définissons et étudions une classe de modules enlacés (modules anti-Yetter—Drinfeld stables) qui servent de coefficients pour l'homologie et la cohomologie Hopf-cyclique. En particulier, nous expliquons leurs liens avec les modules de Yetter—Drinfeld et les doublets de Drinfeld. Parmi les sources d'exemples de modules anti-Yetter—Drinfeld stables, nous trouvons des extensions de Hopf—Galois munies d'une version transposée de l'action de Miyashita—Ulbrich. *Pour citer cet article: P.M. Hajac et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).*© 2004 Académie des sciences. Published by Elsevier SAS. All rights reserved.

1. Introduction

The aim of this paper is to define and provide sources of examples of stable anti-Yetter–Drinfeld modules. They play the role of coefficients for Hopf-cyclic theory [7]. In particular, we claim that modular pairs in involution of Connes and Moscovici are precisely 1-dimensional stable anti-Yetter–Drinfeld modules.

Throughout the paper we assume that H is a Hopf algebra with a bijective antipode. On the one hand, the bijectivity of the antipode is implied by the existence of a modular pair in involution, so that then it need not be assumed. On the other hand, some parts of arguments might work even if the antipode is not bijective. We avoid such discussions. The coproduct, counit and antipode of H are denoted by Δ , ε and S, respectively. For

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the coproduct we use the notation $\Delta(h) = h^{(1)} \otimes h^{(2)}$, for a left coaction on M we write ${}_M \Delta(m) = m^{(-1)} \otimes m^{(0)}$, and for a right coaction $\Delta_M(m) = m^{(0)} \otimes m^{(1)}$. The summation symbol is suppressed everywhere. We assume all algebras to be associative, unital and over the same ground field k. The symbol $\mathcal{O}(X)$ stands for the algebra of polynomial functions on X.

2. The transformation of Yetter-Drinfeld modules

It turns out that, in order to incorporate coefficients into cyclic theory, we need to alter the concept of a Yetter–Drinfeld module by replacing the antipode by its inverse in the Yetter–Drinfeld compatibility condition between actions and coactions. We call the modules-comodules satisfying the thus modified Yetter–Drinfeld compatibility condition *anti-Yetter–Drinfeld modules*. Just as Yetter–Drinfeld modules come in 4 different versions depending on the side of actions and coactions (see [3, p. 181] for a general formulation), so do the anti-Yetter–Drinfeld modules. All versions are completely equivalent and can be derived from one another by replacing a Hopf algebra H by H^{cop} , H^{op} , or $H^{\text{op},\text{cop}}$, respectively.

Definition 2.1. Let H be a Hopf algebra with a bijective antipode S, and M a module and comodule over H. We call M an anti-Yetter-Drinfeld module iff the action and coaction are compatible in the following sense:

$${}_{M}\Delta(hm) = h^{(1)}m^{(-1)}S^{-1}(h^{(3)}) \otimes h^{(2)}m^{(0)} \quad \text{if } M \text{ is a left module and a left comodule,}$$

$$\Delta_M(hm) = h^{(2)}m^{(0)} \otimes h^{(3)}m^{(1)}S(h^{(1)}) \qquad \text{if } M \text{ is a left module and a right comodule,}$$

$${}_{M}\Delta(mh) = S(h^{(3)})m^{(-1)}h^{(1)} \otimes m^{(0)}h^{(2)} \qquad \text{if } M \text{ is a right module and a left comodule,}$$

$$\Delta_M(mh) = m^{(0)}h^{(2)} \otimes S^{-1}(h^{(1)})m^{(1)}h^{(3)} \quad \text{if } M \text{ is a right module and a right comodule.}$$

To make cyclic theory work, we also need to assume that the action splits coaction, i.e., for all $m \in M$, $m^{(-1)}m^{(0)} = m$, $m^{(1)}m^{(0)} = m$, $m^{(0)}m^{(-1)} = m$, $m^{(0)}m^{(1)} = m$, for the left-left, left-right, right-left, and right-right versions, respectively. We call modules satisfying this condition *stable*. Let us emphasize that it is the anti-Yetter-Drinfeld condition rather than the Yetter-Drinfeld condition that makes the homomorphism $action \circ coaction H$ -linear and H-colinear. Therefore the stability condition $action \circ coaction = id$ suits the former and not the latter. The first class of examples of stable anti-Yetter-Drinfeld modules is provided by modular pairs in involution [4, p. 8]. Since such pairs occur naturally in different contexts, Lemmas 2.2 and 2.3 guarantee ample amount of examples of anti-Yetter-Drinfeld modules.

Lemma 2.2. Let the ground field k be a right module over H via a character δ and a left comodule over H via a group-like σ . Then $k = {}^{\sigma}k_{\delta}$ is a stable anti-Yetter-Drinfeld module if and only if (δ, σ) is a modular pair in involution.

The anti-Yetter–Drinfeld modules do not form a monoidal category themselves, but rather a so-called C-category over the category of Yetter–Drinfeld modules (see [11, p. 351] for details). More precisely:

Lemma 2.3. Let N be a Yetter–Drinfeld module and M an anti-Yetter–Drinfeld module. Then $N \otimes M$ is an anti-Yetter–Drinfeld module via $h(n \otimes m) = h^{(1)} n \otimes h^{(2)} m$, $_{N \otimes M} \Delta(n \otimes m) = n^{(-1)} m^{(-1)} \otimes n^{(0)} \otimes m^{(0)}$, for the left–left case, and via $h(n \otimes m) = h^{(2)} n \otimes h^{(1)} m$, $\Delta_{N \otimes M}(n \otimes m) = n^{(0)} \otimes m^{(0)} \otimes n^{(1)} m^{(1)}$, for the left–right case. Similarly, $M \otimes N$ is an anti-Yetter–Drinfeld module via $(m \otimes n)h = mh^{(2)} \otimes nh^{(1)}$, $_{M \otimes N} \Delta(n \otimes m) = m^{(-1)} n^{(-1)} \otimes m^{(0)} \otimes n^{(0)}$, for the right–left case, and via $(m \otimes n)h = mh^{(1)} \otimes nh^{(2)}$, $\Delta_{M \otimes N}(m \otimes n) = m^{(0)} \otimes n^{(0)} \otimes m^{(1)} n^{(1)}$, for the left–right case.

¹ This concept was devised independently by Ch. Voigt and, also independently, by P. Jara and D. Ştefan.

Note that, just as the right-right Yetter-Drinfeld modules are entwined modules [1] for the entwining $\psi(h' \otimes h) = h^{(2)} \otimes S(h^{(1)})h'h^{(3)}$, the right-right anti-Yetter-Drinfeld modules are entwined with respect to $\psi(h' \otimes h) = h^{(2)} \otimes S^{-1}(h^{(1)})h'h^{(3)}$. (Other cases are completely analogous.)

An intermediate step between modular pairs in involution and stable anti-Yetter–Drinfeld modules is given by matched and comatched pairs of [10]. Whenever the antipode is equal to its inverse, the difference between the Yetter–Drinfeld and anti-Yetter–Drinfeld conditions disappears. For a group ring Hopf algebra kG, a left H-comodule is simply a G-graded vector space $M = \bigoplus_{g \in G} M_g$, where the coaction is defined by $M_g \ni m \mapsto g \otimes m$. An action of G on M defines an (anti-)Yetter–Drinfeld module if and only if for all $g, h \in G$ and $m \in M_g$ we have $hm \in M_{hgh^{-1}}$. The stability condition means simply that gm = m for all $g \in G$, $m \in M_g$. A very concrete classical example of a stable (anti-)Yetter–Drinfeld module is provided by the Hopf fibration. Then $H = \mathcal{O}(SU(2))$ and $M = \mathcal{O}(S^2)$. Since $S^2 \cong SU(2)/U(1)$, we have a natural left action of SU(2) on S^2 . Its pull-back makes M a left H-comodule. On the other hand, one can view S^2 as the set of all traceless matrices of SU(2). The pull-back of this embedding $j: S^2 \hookrightarrow SU(2)$ together with the multiplication in $\mathcal{O}(S^2)$ defines a left action of H on M. It turns out that the equivariance property $j(gx) = gj(x)g^{-1}$ guarantees the anti-Yetter–Drinfeld condition, and this combined with the injectivity of j ensures the stability of M. This stability mechanism can be generalized in the following way.

Lemma 2.4. Let M be an algebra and a left H-comodule. Assume that $\pi: H \to M$ is an epimorphism of algebras and the action $hm = \pi(h)m$ makes M an anti-Yetter–Drinfeld module. Assume also that $\pi(1^{(-1)})1^{(0)} = 1$. Then M is a stable module.

3. Hopf-Galois extensions and the opposite Miyashita-Ulbrich action

Another source of examples is provided by Hopf–Galois theory. These examples are purely quantum in the sense that the employed actions are automatically trivial for commutative algebras. To fix the notation and terminology, recall that an algebra and an H-comodule is called a comodule algebra if the coaction is an algebra homomorphism. An H-extension $B := \{p \in P \mid \Delta_P(p) = p \otimes 1\} \subseteq P$ is called Hopf–Galois iff the canonical map $can : P \otimes_B P \to P \otimes H$, $can(p \otimes p') = p\Delta(p')$, is bijective. The bijectivity assumption allows us to define the translation map $T : H \to P \otimes_B P$, $T(h) := can^{-1}(1 \otimes h) =: h^{[1]} \otimes_B h^{[2]}$ (summation suppressed). It can be shown that when everything is over a field (our standing assumption), the centralizer $Z_B(P) := \{p \in P \mid bp = pb, \forall b \in B\}$ of B in B is a subcomodule of B. On the other hand, the formula B0 is a right action on B1 called the Miyashita–Ulbrich action. This action and coaction satisfy the Yetter–Drinfeld compatibility condition [6, (3.11)]. The following proposition modifies the Miyashita–Ulbrich action so as to obtain stable anti-Yetter–Drinfeld modules.

Proposition 3.1. Let $B \subseteq P$ be a Hopf–Galois H-extension such that B is central in P. Then P is a right–right stable anti-Yetter–Drinfeld module via the action $ph = (S^{-1}(h))^{[2]} p(S^{-1}(h))^{[1]}$ and the right coaction on P.

The simplest examples are obtained for P = H. A broader class is given by the so-called Galois objects [2]. Then quantum-group coverings at roots of unity provide examples with central coinvariants bigger than the ground field (see [5] and examples therein). Finally, one can generalize Proposition 3.1 to arbitrary Hopf–Galois extensions by replacing P by P/[B, P] [8, Remark 4.2].

4. The Drinfeld double comodule algebra

For finite-dimensional Hopf algebras, the Yetter–Drinfeld modules can be understood as modules over the Drinfeld double [9, p. 220]. Much in the same way, the anti-Yetter–Drinfeld modules can also be understood as modules over a certain algebra. This makes the usual notions and operations for modules, like projectivity

or induction, directly available for anti-Yetter–Drinfeld modules. To this end, the comodule structure of an anti-Yetter–Drinfeld module has to be converted into a module structure over the dual Hopf algebra H^* , so that from now on we assume that the Hopf algebra H is finite-dimensional.

Proposition 4.1. Let H be a finite-dimensional Hopf algebra. The formula

$$(\varphi \otimes h)(\varphi' \otimes h') = \varphi'^{(1)} \left(S^{-1}(h^{(3)}) \right) \varphi'^{(3)} \left(S^{2}(h^{(1)}) \right) \varphi \varphi'^{(2)} \otimes h^{(2)} h'$$
(5)

turns the vector space $A(H) := H^* \otimes H$ into an associative algebra with the unit $\varepsilon \otimes 1$.

Note that the above product differs from the product in the Drinfeld double of H [9, p. 214] only by the additional squared antipode in the second factor. To relate the modules over A(H) with anti-Yetter-Drinfeld modules, recall first that every right H-comodule M becomes a left H^* -module via $\varphi m := \varphi(m^{(1)})m^{(0)}$. Conversely, any left H^* -module yields a right H-comodule via $\Delta_M(m) = \sum_{i=1}^n h_i^* m \otimes h_i$. Here $\{h_1, \ldots, h_n\}$ is a basis of H and $\{h_1^*, \ldots, h_n^*\}$ is the dual basis. (Of course, this comodule structure does not depend on the choice of a basis.) Using this, we get the following connection between the modules over A(H) and anti-Yetter-Drinfeld modules:

Proposition 4.2. Let H be a finite-dimensional Hopf algebra. If M is a left-right anti-Yetter-Drinfeld module, it becomes a left A(H)-module by $(\varphi \otimes h)m := \varphi((hm)^{(1)})(hm)^{(0)}$. Conversely, if M is a left A(H)-module, it becomes a left-right anti-Yetter-Drinfeld module by $hm := (\varepsilon \otimes h)m$, $\Delta_M(m) := \sum_{i=1}^n (h_i^* \otimes 1)m \otimes h_i$. Here $\{h_1, \ldots, h_n\}$ is a basis of H and $\{h_1^*, \ldots, h_n^*\}$ its dual basis.

The claim of Lemma 2.3 is reflected in the fact that although A(H) is not a Hopf algebra itself, it can be shown that the formula $(\varphi \otimes h) \mapsto (\varphi^{(2)} \otimes h^{(1)}) \otimes (\varphi^{(1)} \otimes h^{(2)})$ makes A(H) a right comodule algebra over the Drinfeld double D(H).

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