Lévy modelling of defaultable bonds *

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Abstract

A market with defaultable bonds modelled by equations with Lévy noise is considered. Conditions under which the market is free of arbitrage are derived.

1 Introduction

The paper is concerned the market containing defaultable bondes issued by companies. The probabilities of deafaults depend on economic conditions of the firms and are reflected by rating classes designated by rating agencies. Contrary to majority of the papers on the subject see [1], which use for modelling Brownian motion, we apply the theory of Lévy processes with discontinuous trajectories. Credit risk models with Lévy noise have been recently considered by Eberline and Özkan [6] and by Özkan and Schmidt [12].

Our main aim is to derive conditions under which the market with defaultable bonds, issued by firms with time dependent and random rating classes is free of arbitrage. Three types of recovery payments are considered: fractional recover of market value, fractional recovery of treasury value and fractional recovery of par value. The retaing classes change according to a conditional, continuous time Markov chains and the default time is equal to the moment of entering by the firm the worst rating class.

The paper starts from recalling basic facts on default-free bond market and on Lévy processes, in general, infinite dimensional. The case of two rating classes is considered in Section 3 and the general case in Section 4. Obtained theorems provide HJM conditions for the arbitrage- free property. The final section introduces the the market description in the Musiela parameterization and inbdicates how the results from earlier chapters can be applied to to the Musiela framework.

The results of the present paper extend those obtained by Özkan and Schmidt [12] for the fractional recovery case. Özkan and Schmidt approach is based on Musiela parameterization and requires more stringent conditions than ours.

2 Preliminaries

We will consider processes on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We take Levy process $Z(t)$ with values in U as source of uncertainty in model. This means that $Z(t)$ is process with independent and stationary increments having values in U which is some abstract separable Hilbert space with inner product denoted by $\langle \cdot, \cdot \rangle_U$. We can associate with $Z(t)$ measure of its jumps, denoted by μ i.e. for any

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$A \in \mathcal{B}(U)$ such that $\overline{A} \subset U \backslash \{0\}$ we have :

$$\mu([0,t],A) = \sum_{0<s\leq t} 1\!\!1_A(\Delta Z(s)).$$

The measure ν defined by:

$$\nu(A) = \mathbf{E}(\mu([0,1],A)),$$

is called Levy measure of process Z, stationarity of increments implies that we have also:

$$\mathbf{E}(\mu([0,t],A)) = t\nu(A).$$

The Lévy-Khintchine formula shows that characteristic function of Lévy process has a form:

$$\mathbf{E}e^{i\langle \lambda, Z(t)\rangle_U} = e^{t\psi(\lambda)},$$

where

$$\psi(\lambda) = i\langle a,\lambda\rangle_U - \frac{1}{2}\langle Q\lambda,\lambda\rangle_U + \int_U (e^{i\langle\lambda,x\rangle_U} - 1 - i\langle\lambda,x\rangle_U 1\!\!1_{\{|x|_U\leq 1\}}(x))\nu(dx),$$

and $a \in U$, Q is symmetric non negative nuclear operator on U, ν is a measure on U with $\nu(\{0\}) = 0$ and

$$\int_U (|x|_U^2 \wedge 1)\nu(dx) < \infty.$$

Moreover Z has a well known Lévy-Ito decomposition:

$$Z(t) = at + W(t) + \int_0^t \int_{|y|_U\leq 1} y(\mu(ds,dy) - ds\nu(dy)) + \int_0^t \int_{|y|_U>1} y\mu(ds,dy),$$

where W is a Wiener process with values in U and covariance operator Q.

Let $r(t)$, $t \geq 0$ be the short rate process. If at moment 0 one puts into the bank account 1 then at moment t one has

$$B_t = e^{\int_0^t r(\sigma)d\sigma}.$$

Let $B(t,\theta)$, $0 \leq t \leq \theta$ be the market price at moment t of a bond paying 1 at the maturity time θ. The forward rate curve is a function $f(t,\theta)$ defined for $t \leq \theta$ and such that

$$B(t,\theta) = e^{-\int_t^\theta f(t,s)ds}. \tag{1}$$

It is convenient to assume that once a bond has matured its money equivalent goes to the bank account. Thus $B(t,\theta)$, the market price at moment t of a bond paying 1 at the maturity time θ, is defined also for $t \geq \theta$ by the formula

$$B(t,\theta) = e^{\int_\theta^t r(\sigma)d\sigma}. \tag{2}$$

We postulate here the following dynamic for forward rates

$$df(t,\theta) = \alpha(t,\theta)dt + \langle\sigma(t,\theta), dZ(t)\rangle_U, \tag{3}$$

Sometimes we use another form of SDE for forward rates

$$df(t) = \tilde{\alpha}(t)dt + \tilde{\sigma}(t)dZ(t),$$

where $\tilde{\alpha}(t)$ is a function on $[t,T]$ given by $\tilde{\alpha}(t)(\theta) = \alpha(t,\theta)$ and $\tilde{\sigma}(t)$ is linear operator from U into $L^2[0,T^*]$ defined by

$$(\tilde{\sigma}(t)u)(\theta) = \langle\sigma(t,\theta),u\rangle_U.$$

For $t > \theta$ we put:

$$\alpha(t,\theta) = \sigma(t,\theta) = 0. \tag{4}$$

It follows from (3) that for $t \leq \theta$:

$$f(t,\theta) = f(0,\theta) + \int_0^t \alpha(s,\theta)ds + \int_0^t \langle \sigma(s,\theta), dZ(s)\rangle_U,$$

and for $t \geq \theta$ according to (4):

$$f(t,\theta) = f(0,\theta) + \int_0^\theta \alpha(s,\theta)ds + \int_0^\theta \langle \sigma(s,\theta), dZ(s)\rangle_U.$$

Thus the process $f(t,\theta)$ for $t \geq \theta$ is constant for each $\theta > 0$, say equal to $f(\theta,\theta)$, and it can be identified with the short rate process:

$$r(\theta) = f(\theta,\theta) = f(0,\theta) + \int_0^\theta \alpha(s,\theta)ds + \int_0^\theta \langle \sigma(s,\theta), dZ(s)\rangle_U.$$

HJM postulate states that discounted bond prices:

$$\hat{B}(t,\theta) = \frac{B(t,\theta)}{B_t},$$

are local martingales for each $\theta \in [0,T^*]$. Since for $t > u$ we have $f(u,u) = f(t,u)$, then

$$B_t = e^{\int_0^t f(u,u)du} = e^{\int_0^t f(t,u)du},$$

and thus discounted bond prices can be written as:

$$\hat{B}(t,\theta) = \frac{B(t,\theta)}{B_t} = e^{-\int_t^\theta f(t,u)du} e^{-\int_0^t f(t,u)du} = e^{-\int_0^\theta f(t,u)du},$$

and hence HJM postulate is that processes $\hat{B}(\cdot,\theta)$, $\theta \in [0,T^*]$ given by equation

$$\hat{B}(t,\theta) = e^{-\int_0^\theta f(t,u)du},$$

are local martingales. Jakubowski and Zabczyk show in [9] that under mild conditions the HJM postulate implies existence of exponential moments of Levy measure

$$\int_{|y|_U > 1} e^{\langle c,y\rangle_U} \nu(dy) < \infty, \qquad \forall c \in U. \tag{5}$$

The following theorem, under different type of conditions, goes back to the paper [2] by Björk, Di Massi, Kabanov and Runggaldier.

Theorem A. *([8]) Assume that the Lévy measure ν satisfies condition (5). Discounted bond prices are local martingales if and only if the following HJM-type condition holds for each $\theta \in [0,T^*]$ and $\forall t \leq \theta$:*

$$\begin{aligned}\int_t^\theta \alpha(t,v)dv \;=\; & -\langle \tilde{\sigma}^*(t)\mathbb{1}_{[0,\theta]}, a\rangle_U + \frac{1}{2}\langle Q\tilde{\sigma}^*(t)\mathbb{1}_{[0,\theta]}, \tilde{\sigma}^*(t)\mathbb{1}_{[0,\theta]}\rangle_U, \\ & + \int_U \left(e^{-\langle \tilde{\sigma}^*(t)\mathbb{1}_{[0,\theta]}, y\rangle_U} - 1 + \mathbb{1}_{\{|y|_U \leq 1\}}(y)\langle \tilde{\sigma}^*(t)\mathbb{1}_{[0,\theta]}, y\rangle_U \right)\nu(dy). \end{aligned} \tag{6}$$

It is convinient to express HJM condition in terms of logaritm of moment generating function of Levy process Z i.e. in terms of the functional $J : U \to \mathbb{R}$:

$$\begin{aligned} J(u) = -\langle u,a\rangle_U + \frac{1}{2}\langle Qu,u\rangle_U \;&+\; \int_{|y|_U \leq 1} e^{-\langle u,y\rangle_U} - 1 + \langle u,y\rangle_U \nu(dy) \\ &+\; \int_{|y|_U > 1} \left(e^{-\langle u,y\rangle_U} - 1\right)\nu(dy), \end{aligned}$$

Now we can use J to express "dirft vanishing" condition as:

$$\int_t^\theta \alpha(t,v)dv = J\left(\mathbb{1}_{[0,\theta]}(t)\int_t^\theta \sigma(t,v)dv\right) \quad \forall t \le \theta \quad and \quad \forall \theta \in [0,T^*]. \tag{7}$$

This transparent condition was first formulated by Eberlein and Raible in [5] for deterministic functions $\sigma(t,\theta)$ in finite dimensional setting, and under mild assumptions on $\sigma(t,\theta)$ in infinite dimensional setting by Jakubowski and Zabczyk in [8].

In what follows we assume that condition (7) is fulfilled.

Remark 1. *It follows from (4) that we can write (7) as*

$$\int_0^\theta \alpha(t,v)dv = J\left(\int_0^\theta \sigma(t,v)dv\right) \quad \forall t \le \theta \quad and \quad \forall \theta \in [0,T^*]. \tag{8}$$

Using integration by parts formula and the dynamic of discounted bond i.e. $dM_t^\theta = d\hat{B}(t,\theta)$ (see [8]), we obtain

Theorem 1. *The process of price of bond solves the following stochastic differential equation:*

$$dB(t,\theta) = B(t-,\theta) \quad \Big(\quad \big(f(t,t)+\bar{a}(t,\theta)\big)dt + \int_U \left[e^{-\langle\tilde{\sigma}^*(t)\mathbb{1}_{[0,\theta]},y\rangle_U} - 1\right](\mu(dt,dy) - dt\nu(dy))$$
$$-\langle\tilde{\sigma}^*(t)\mathbb{1}_{[0,\theta]}, dW(t)\rangle_U\Big),$$

where $\bar{a}(t,\theta)$ has form

$$\bar{a}(t,\theta) \quad = \quad -\langle\mathbb{1}_{[0,\theta]},\tilde{\alpha}(t)\rangle + J(\tilde{\sigma}^*(t)\mathbb{1}_{[0,\theta]})$$

and $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2([0,T^*])$.

Corrolary 1. *Solution of above equation can be writen as*

$$B(t,\theta) = B(0,\theta)\exp\left(\int_0^t f(s,s)ds - \int_0^t \langle\mathbb{1}_{[0,\theta]},\tilde{\alpha}(s)\rangle ds - \int_0^t \langle\tilde{\sigma}^*(s)\mathbb{1}_{[0,\theta]}, dZ(s)\rangle_U\right),$$

and if HJM-type condition (7) holds, then

$$B(t,\theta) = B(0,\theta)\exp\left(\int_0^t f(s,s)ds - \int_0^t J(\tilde{\sigma}^*(s)\mathbb{1}_{[0,\theta]})ds - \int_0^t \langle\tilde{\sigma}^*(s)\mathbb{1}_{[0,\theta]}, dZ(s)\rangle_U\right).$$

3 HJM conditions for credit risk models

In default-free world by bond maturing at θ with face value 1 we mean financial instrument whose payoff is 1 at time θ. In defaultable case we have several variants describing amount and timing of so called *recovery payment* which is paid to bond holders if default has occurred before bond's maturity. If by τ we denote the moment of default, then, generally speaking, the payoff of the defaultable bond is as follows:

$$D(\theta,\theta) = \mathbb{1}_{\{\tau>\theta\}} + \mathbb{1}_{\{\tau\le\theta\}} \cdot \textit{ recovery payment }.$$

If δ is a recovery rate process, then *recovery payment* can take different forms (see eg. [1]):

- $\delta_t D(\tau-,\theta)\frac{B_\theta}{B_\tau}$ - *fractional recovery of market value* - at time of default bondholders receive a fraction of pre-default market value of defaultable bond (i.e. $D(\tau-,\theta)$):

$$D(\theta,\theta) = \mathbb{1}_{\{\tau>\theta\}} + \mathbb{1}_{\{\tau\le\theta\}} \cdot \delta_t D(\tau-,\theta)\frac{B_\theta}{B_\tau}.$$

- δ - *fractional recovery of Treasury value* - fixed fraction δ of bond's face value is paid to bondholders at maturity θ:
$$D^{\delta}(\theta,\theta) = 1\!\!1_{\{\tau>\theta\}} + 1\!\!1_{\{\tau\leq\theta\}} \cdot \delta.$$
- $\frac{\delta B_\theta}{B_\tau}$ - *fractional recovery of par value*- fixed fraction δ of bond's face value is paid to bondholders at default time τ:
$$D^{\Delta}(\theta,\theta) = 1\!\!1_{\{\tau>\theta\}} + 1\!\!1_{\{\tau\leq\theta\}} \cdot \delta \frac{B_\theta}{B_\tau}.$$

The values of defaultable bonds for $t \leq \theta$ are given by (11),(15) and (20) respectively.

We assume that the moment of default τ is a $\mathbb{G}$ stopping time, and that our filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ are filtrations generated by observing market and observing default time i.e. $\mathcal{H}_t = \sigma(\tau \leq u : u \leq t)$, respectively. Let $(H_t)_{t\geq 0}$ be a default indicator process i.e. $H_t = 1\!\!1_{\{\tau\leq t\}}$. We assume that τ admits an $\mathbb{F}$ intensity $(\lambda_t)_{t\geq 0}$ which is an $\mathbb{F}$ adapted process such that M_t given by the formula
$$M_t = H_t - \int_0^{t\wedge\tau} \lambda_u du = H_t - \int_0^t (1-H_u)\lambda_u du$$
follows a $\mathbb{G}$–martingale.

We denote by $g_1(t,u)$ the pre-default forward rate corresponding to pre-default term structure observed on the market. We postulate here that
$$dg_1(t,\theta) = \alpha_1(t,\theta)dt + \langle \sigma_1(t,\theta), dZ_1(t)\rangle_U,$$
where Z_1 is Lévy process with values in U which has the following Lévy-Ito decomposition:
$$Z_1(t) = a_1 t + W_1(t) + \int_0^t \int_{|y|_U\leq 1} y(\mu_1(ds,dy) - ds\nu_1(dy)) + \int_0^t \int_{|y|_U>1} y\mu_1(ds,dy).$$
If $D_1(t,\theta) = e^{-\int_t^\theta g_1(t,u)du}$, then by applying Itò lemma in analogous way as in the default free case we have (in theorem J_1 corresponds to Z_1 in the same way as J corresponds to Z)

Theorem 2. *Dynamic of the process $D_1(t,\theta)$ is given by*
$$dD_1(t,\theta) = D_1(t-,\theta) \quad \Bigg(\big(g_1(t,t) + \bar{a}_1(t,\theta)\big)dt + \int_U \left[e^{-\langle \tilde{\sigma}_1^*(t)1\!\!1_{[0,\theta]}, y\rangle_U} - 1\right](\mu_1(dt,dy) - dt\nu_1(dy)) - \langle \tilde{\sigma}_1^*(t)1\!\!1_{[0,\theta]}, dW_1(t)\rangle_U \Bigg),$$
where $\bar{a}_1(t,\theta)$ satisfies
$$\bar{a}_1(t,\theta) \;=\; -\langle 1\!\!1_{[0,\theta]}, \alpha_1(t)\rangle + J_1(\tilde{\sigma}_1^*(t)1\!\!1_{[0,\theta]}).$$

In what follows we use the following technical lemma:

Lemma 1. *Let τ and H_t be as above and D_t be a process of the form:*
$$D_t = (1-H_t)X_t + H_t Y_t + H_t Z_\tau,$$
where processes X_t, Y_t have local martingale parts M_t^X and M_t^Y and absolutely continuous drifts α_t^X, α_t^Y, which means that processes X_t, Y_t have decompositions:
$$\begin{aligned} dX_t &= \alpha_t^X dt + dM_t^X, \\ dY_t &= \alpha_t^Y dt + dM_t^Y. \end{aligned}$$

Then D_t is local martingale if and only if for each $t \in [0,T^*]$ the following conditions hold:

$$\alpha_t^X = \lambda_t(X_{t-} - Y_{t-} - Z_t) \qquad \text{on the set } \{\tau > t\} \tag{9}$$
$$\alpha_t^Y = 0 \qquad \text{on the set } \{\tau \le t\} \tag{10}$$

Proof. By definition of H_t

$$D_t = (1-H_t)X_t + H_tY_t + H_tZ_\tau = (1-H_t)X_t + H_tY_t + \int_0^t Z_u dH_u.$$

Since H_t is a finite variation process, the integration by parts formula implies

$$\begin{aligned} dD_t &= (1-H_t)dX_t - X_{t-}dH_t + H_tdY_t + Y_{t-}dH_t + Z_tdH_t \\ &= (1-H_t)dM_t^X + H_tdM_t^Y \\ &\quad +(1-H_t)\alpha_t^X dt + H_t\alpha_t^Y dt + (-X_{t-} + Y_{t-} + Z_t)dH_t \\ &= (1-H_t)dM_t^X + H_tdM_t^Y + (-X_{t-} + Y_{t-} + Z_t)dM_t \\ &\quad +(1-H_t)(\alpha_t^X + \lambda_t(-X_{t-} + Y_{t-} + Z_t))dt + H_t\alpha_t^Y dt, \end{aligned}$$

and hence the result follows. □

Remark 2. *In the next sections, the process X_t will correspond to predefault value, Y_t value of payments after default, and Z_t value of payments at default time τ.*

3.1 Fractional recovery of market value

Let us focus on defaultable bonds with fractional recovery of market value $D(t,\theta)$. This kind of bonds pays 1-unit cash if default didn't occurred before maturity θ i.e. if default moment $\tau > \theta$, and if bond defaults before T we have recovery payment at default time which is a fraction δ_t of it's market value just before default time, so the recovery payment is equal to $\delta_\tau D(\tau-,\theta)$. Therefore

$$D(\theta,\theta) = \mathbb{1}_{\{\tau>\theta\}} + \mathbb{1}_{\{\tau\le\theta\}}\delta_\tau D(\tau-,\theta)\frac{B_\theta}{B_\tau}$$

and for $t \le \theta$ we model a value of defaultable bond by

$$D(t,\theta) = \mathbb{1}_{\{\tau>t\}}e^{-\int_t^\theta g_1(t,u)du} + \mathbb{1}_{\{\tau\le t\}}\delta_\tau D(\tau-,\theta)\frac{B_t}{B_\tau}, \tag{11}$$

where $g_1(t,u)$ is the pre-default forward rate corresponding to pre-default term structure. Our first objective is to derive the HJM drift condition in the case of defaultable bonds with fractional recovery of market value given by (11). Using the process H_t we can represent D as

$$D(t,\theta) = (1-H_t)D_1(t,\theta) + H_t\delta_\tau D_1(\tau-,\theta)\frac{B_t}{B_\tau}.$$

Theorem 3. *(HJM drift condition for $D(t,\theta)$) Discounted prices of defaultable bonds with fractional recovery of market value are local martingales if and only if the following conditions hold on set $\{\tau > t\}$:*
$\forall\, \theta \in [0,T^*]$ and for each $t \le \theta$

$$g_1(t,t) = f(t,t) + (1-\delta_t)\lambda_t, \tag{12}$$
$$\int_0^\theta \alpha_1(t,v)dv = J_1\bigg(\mathbb{1}_{[0,\theta]}(t)\int_t^\theta \sigma_1(t,v)dv\bigg). \tag{13}$$

Proof. Denoting by $\hat{D}(t,\theta)$ the discounted value of $D(t,\theta)$ we have

$$\hat{D}(t,\theta) = \frac{D(t,\theta)}{B_t} = (1-H_t)\frac{D_1(t,\theta)}{B_t} + H_t\delta_\tau\,\frac{D_1(\tau-,\theta)}{B_\tau}.$$

We see that this process has structure as in Lemma 1. Therefore we can apply lemma with

$$X_t = \frac{D_1(t,\theta)}{B_t}, \qquad Y_t = 0, \qquad Z_t = \delta_t\frac{D_1(t-,\theta)}{B_t},$$

where, by Itò lemma , drift of X is given by

$$\mu_t^X = \frac{D_1(t-,\theta)}{B_t}(g_1(t,t) - f(t,t) - \langle \mathbb{1}_{[0,\theta]}, \alpha_1(t)\rangle + J_1(\tilde{\sigma}_1^*(t)\mathbb{1}_{[0,\theta]})).$$

>From lemma 1 we see that $\hat{D}(t,\theta)$ is martingale if and only if (9) holds on the set $\{\tau > t\}$ for for all $\theta \in [0,T^*]$ and all $t \leq \theta$:

$$\frac{D_1(t-,\theta)}{B_t}(g_1(t,t) - f(t,t) - \langle \mathbb{1}_{[0,\theta]}, \alpha_1(t)\rangle + J_1(\tilde{\sigma}_1^*(t)\mathbb{1}_{[0,\theta]})) = \lambda_t\left(\frac{D_1(t-,\theta)}{B_t} - \delta_t\frac{D_1(t-,\theta)}{B_t}\right),$$

and this is equivalent to :

$$g_1(t,t) - f(t,t) - \langle \mathbb{1}_{[0,\theta]}, \alpha_1(t)\rangle + J_1(\tilde{\sigma}_1^*(t)\mathbb{1}_{[0,\theta]}) = \lambda_t(1-\delta_t). \tag{14}$$

(12) and (13) imply (14), so $\hat{D}(t,\theta)$ is a local martingale under $\mathbf{P}$.
If $\hat{D}(t,\theta)$ are local martingales under $\mathbf{P}$, then equality (14) holds on the set $\{\tau > t\}$ for all $t \leq \theta$ and for all $\theta \in [0,T^*]$. (12) follows from equality (14) taking for $\theta = t$ and from the fact that for $\theta = t$ we have:

$$\langle \mathbb{1}_{[0,t]}, \alpha_1(t)\rangle = 0, \qquad J_1(\tilde{\sigma}_1^*(t)\mathbb{1}_{[0,t]}) = 0.$$

(13) follows immediately from (14) and (12).

□

3.2 Fractional recovery of treasury

The holder of defaultable bond with fractional recovery of treasury receives 1 if there is no default by θ, and if default has occurred before maturity θ, then the fixed amount $\delta \in [0,1]$ is paid at maturity to bondholder. Therefore, we have the following payoff at maturity :

$$D^\delta(\theta,\theta) = \mathbb{1}_{\{\tau>\theta\}} + \mathbb{1}_{\{\tau\leq\theta\}}\cdot\delta.$$

Since paying δ at maturity θ is equivalent to paying $\delta B(\tau,T)$ at default time τ, we can write

$$D^\delta(t,\theta) = \mathbb{1}_{\{\tau>t\}}e^{-\int_t^\theta g_1(t,u)du} + \mathbb{1}_{\{\tau\leq t\}}\cdot\delta\cdot B(t,\theta),$$

so using introduced notation we have

$$D^\delta(t,\theta) = (1-H_t)D_1(t,\theta) + H_t\delta B(t,\theta). \tag{15}$$

Theorem 4. *(HJM drift condition for $D^\delta(t,\theta)$) The processes of discounted defaultable bond prices with fractional recovery of treasury are local martingales if and only if the following condition holds: on the set $\{\tau > t\}$ for each $t \in [0,\theta]$ and for all $\theta \in [0,T^*]$ we have:*

$$g_1(t,t) = f(t,t) + (1-\delta)\lambda_t \tag{16}$$

$$\int_0^\theta \alpha_1(t,v)dv = J_1\left(\mathbb{1}_{[0,\theta]}(t)\int_t^\theta \sigma_1(t,v)dv\right) + \delta\left(\frac{B(t-,\theta)}{D_1(t-,\theta)} - 1\right)\lambda_t. \tag{17}$$

Proof. From (15) we have

$$\hat{D}^\delta(t,\theta) = \frac{D^\delta(t,\theta)}{B_t} = (1-H_t)\frac{D_1(t,\theta)}{B_t} + H_t\delta\frac{B(t,\theta)}{B_t}.$$

Again, we can see that we can apply Lemma 1 with

$$X_t = \frac{D_1(t,\theta)}{B_t}, \qquad Y_t = \delta\frac{B(t,\theta)}{B_t}, \qquad Z_t = 0,$$

where drift of X is given by

$$\mu_t^X = \frac{D_1(t-,\theta)}{B_t}(g_1(t,t) - f(t,t) - \langle 1\!\!1_{[0,\theta]}, \alpha_1(t)\rangle + J_1(\tilde{\sigma}_1^*(t)1\!\!1_{[0,\theta]})),$$

and drift of Y is given by

$$\mu_t^Y = \frac{B(t-,\theta)}{B_t}(-\langle 1\!\!1_{[0,\theta]}, \alpha(t)\rangle + J(\tilde{\sigma}^*(t)1\!\!1_{[0,\theta]})).$$

Condition (10) of Lemma 1 is equivalent to

$$\int_0^\theta \alpha(t,v)dv \quad = \quad J\left(1\!\!1_{[0,\theta]}(t)\int_t^\theta \sigma(t,v)dv\right). \tag{18}$$

on the set $\{\tau \le t\}$ for each $t \in [0,\theta]$ and for all $\theta \in [0,T^*]$. One recognize this equality as HJM type condition for default-free bonds (condition 7), and we assume this condition is fulfilled. Hence the local martingale property for $\hat{D}^\delta(t,\theta)$ is equivalent to occurrence on the set $\{\tau > t\}$ for all $t \le \theta$ and for all $\theta \in [0,T^*]$ of the equality

$$\frac{D_1(t-,\theta)}{B_t}(g_1(t,t) - f(t,t) - \langle 1\!\!1_{[0,\theta]}, \alpha_1(t)\rangle + J_1(\tilde{\sigma}_1^*(t)1\!\!1_{[0,\theta]})) \quad = \quad \lambda_t\left(\frac{D_1(t-,\theta)}{B_t} - \delta\frac{B(t-,\theta)}{B_t}\right),$$

which in turn is equivalent to

$$g_1(t,t) - f(t,t) - \langle 1\!\!1_{[0,\theta]}, \alpha_1(t)\rangle + J_1(\tilde{\sigma}_1^*(t)1\!\!1_{[0,\theta]}) \quad = \quad \lambda_t\left(1 - \delta\frac{B(t-,\theta)}{D_1(t-,\theta)}\right). \tag{19}$$

Since conditions (16) and (17) imply (19), then they imply that $\hat{D}^\delta(t,\theta)$ are local martingales under $\mathbf{P}$. Conversely, if $\hat{D}^\delta(t,\theta)$ are local martingales under $\mathbf{P}$, then (19) holds. Taking $\theta = t$ we obtain from (19)

$$\big(g_1(t,t) - f(t,t) - (1-\delta)\lambda_t\big) - \langle 1\!\!1_{[0,t]}, \alpha_1(t)\rangle + J_1(\tilde{\sigma}_1^*(t)1\!\!1_{[0,t]}) + \delta\left(\frac{B(t-,t)}{D_1(t-,t)} - 1\right)\lambda_t = 0$$

which implies (16), since $\frac{B(t-,t)}{D_1(t-,t)} = 1$. From (16) and (19) we have (17). □

3.3 Fractional recovery of par

In the case of fractional recovery of par value the holder of defaultable bond receives 1 unit cash if there is no default prior to maturity and if bond has defaulted a fixed fraction δ of par value is paid at default time. Therefore the payoff at maturity has form

$$D^\Delta(\theta,\theta) = 1\!\!1_{\{\tau>\theta\}} + 1\!\!1_{\{\tau\le\theta\}} \cdot \delta\frac{B_\theta}{B_\tau},$$

and before maturity has form

$$D^\Delta(t,\theta) = 1\!\!1_{\{\tau>t\}}D_1(t,\theta) + 1\!\!1_{\{\tau\le t\}} \cdot \delta\frac{B_t}{B_\tau},$$

which is equal to

$$D^\Delta(t,\theta) = (1-H_t)D_1(t,\theta) + H_t\delta\frac{B_t}{B_\tau}. \tag{20}$$

Theorem 5. *(HJM drift condition for $D^{\Delta}(t,\theta)$)*
Discounted prices of defaultable bond with fractional recovery of par are local martingales if and only if the following conditions hold on set $\{\tau > t\}$ for each $t \in [0,\theta]$ and all $\theta \in [0,T^*]$:

$$g_1(t,t) = f(t,t) + (1-\delta)\lambda_t, \tag{21}$$

$$\int_0^\theta \alpha_1(t,v)dv = J_1\left(\mathbb{1}_{[0,\theta]}(t)\int_t^\theta \sigma_1(t,v)dv\right) + \delta\left(\frac{1}{D_1(t-,\theta)} - 1\right)\lambda_t. \tag{22}$$

Proof. By (20) we have

$$\hat{D}^{\Delta}(t,\theta) = \frac{D^{\Delta}(t,\theta)}{B_t} = (1-H_t)\frac{D_1(t,\theta)}{B_t} + H_t\delta\frac{1}{B_\tau}$$

Again, we see that this process has structure as in Lemma 1, therefore we can apply it with

$$X_t = \frac{D_1(t,\theta)}{B_t}, \qquad Y_t = 0, \qquad Z_t = \delta\frac{1}{B_t}.$$

By Lemma 1 we see that $\hat{D}^{\Delta}(t,\theta)$ is martingale if and only if on the set $\{\tau > t\}$ for all $t \le \theta$ and for all $\theta \in [0,T^*]$:

$$\frac{D_1(t-,\theta)}{B_t}(g_1(t,t) - f(t,t) - \langle \mathbb{1}_{[0,\theta]}, \alpha_1(t)\rangle + J_1(\tilde{\sigma}_1^*(t)\mathbb{1}_{[0,\theta]})) = \lambda_t\left(\frac{D_1(t-,\theta)}{B_t} - \delta\frac{1}{B_t}\right)$$

(it is condition (9)). In turn, this equality is equivalent to

$$g_1(t,t) - f(t,t) - \langle \mathbb{1}_{[0,\theta]}, \alpha_1(t)\rangle + J_1(\tilde{\sigma}_1^*(t)\mathbb{1}_{[0,\theta]}) = (1-\delta)\lambda_t - \delta\lambda_t\left(\frac{1}{D_1(t-,\theta)} - 1\right). \tag{23}$$

(21) and (22) imply (23), so they imply that $\hat{D}^{\Delta}(t,\theta)$ are local martingales. Conversely, if $\hat{D}^{\Delta}(t,\theta)$ are local martingales, then (23) holds on the set $\{\tau > t\}$ for $t \le \theta$ and for all $\theta \in [0,T^*]$. Hence, taking $\theta = t$, we conclude (21), and finally (22). $\square$

4 Credit rating migration case

Our objective is to generalize results of the previous section and derive HJM drift condition for models with the process $C^1(t)$ describing migration of credit ratings of bonds with different kind of recovery. Credit rating migration process C^1 which is modelled by the conditional Markov chain with values in $\mathcal{K} = \{1,\dots,K\}$ with absorbtion state K (for details see Bielecki, Rutkowski [1]). With the state i it is associated the term structure g_i. It is reasonable to avoid arbitrage to assume that

$$g_{K-1}(t,\theta) > g_{K-2}(t,\theta) > \ldots > g_1(t,\theta) > f(t,\theta)$$

for all $t \in [0,\theta]$ and all $\theta \in [0,T^*]$.

Conditional infinitesimal generator of the process C^1 at t given $\mathcal{G}_t$ has the form

$$\Lambda(t) = \begin{pmatrix} \lambda_{1,1}(t) & \lambda_{1,2}(t) & \cdots & \lambda_{1,K-1}(t) & \lambda_{1,K}(t) \\ \lambda_{2,1}(t) & \lambda_{2,2}(t) & \cdots & \lambda_{2,K-1}(t) & \lambda_{2,K}(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{K-1,1}(t) & \lambda_{K-1,2}(t) & \cdots & \lambda_{K-1,K-1}(t) & \lambda_{K-1,K}(t) \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

where off-diagonal processes $\lambda_{i,j}(t)$, $i \neq j$ are nonnegative processes adapted to $\mathbb{G}$ and diagonals elements are negative and are determined by off-diagonals by the formula

$$\lambda_{i,i}(t) = -\sum_{j\in\mathcal{K}\setminus\{i\}} \lambda_{i,j}(t)$$

We can thought of $-\lambda_{i,i}(t)$ as stochastic intensities of jumping off the i-state, and $p_{i,j}(t) = \frac{\lambda_{i,j}(t)}{\lambda_{i,i}(t)}$ as a probability of jumping from the state i to the state j given that we jump-off state i. With slight abuse of notation and we can write conditional infinitesimal generator of C^1 in an equivalent form as

$$\Lambda(t) = \begin{pmatrix} -\lambda_{1,1}(t) & \lambda_{1,1}(t)p_{1,2}(t) & \cdots & \lambda_{1,1}(t)p_{1,K-1}(t) & \lambda_{1,1}(t)p_{1,K}(t) \\ \lambda_{2,2}(t)p_{2,1}(t) & -\lambda_{2,2}(t) & \cdots & \lambda_{2,2}(t)p_{2,K-1}(t) & \lambda_{2,2}(t)p_{2,K}(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{K-1,K-1}(t)p_{K-1,1}(t) & \lambda_{K-1,K-1}(t)p_{K-1,2}(t) & \cdots & -\lambda_{K-1,K-1}(t) & \lambda_{K-1,K-1}(t)p_{K-1,K}(t) \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

For any function $h : \mathcal{K} \to \mathbb{R}$ we introduce the shorthand notation:

$$\Lambda(u)h(i) = \sum_{j=1}^{K} \lambda_{ij}(u)h(j).$$

From Bielecki, Rutkowski [1] we quote the following theorems and corollary which will be frequently used.

Theorem B. *For every function $h : \mathcal{K} \to \mathbb{R}$ the process M^h, given by the formula*

$$M^h(t) = h(C^1(t)) - \int_0^t \Lambda(u)h(C^1(u))du, \qquad \forall t \in \mathbb{R}_+,$$

is a $\mathbb{G}$ martingale.

Theorem C. *Let h be a real valued function $h : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$, then the process N^h, given by the formula*

$$N^h(t) = \sum_{0<u\leq t} h(C^1(u-), C^1(u)) - \int_0^t \sum_{k\neq C^1(u)} \lambda_{C^1(u),k}(u)h(C^1(u), k)du, \qquad \forall t \in \mathbb{R}_+,$$

is a $\mathbb{G}$ martingale.

Corrolary A. *Define the auxiliary process $H_i(t) = \mathbb{1}_{\{i\}}(C^1(t))$. By Theorem 4*

$$M_i(t) = H_i(t) - \int_0^t \lambda_{C^1(u),i}(u)du$$

is a $\mathbb{G}$ -martingale.

Corrolary B. *For $i \neq j$ we define auxiliary process $H_{i,j}(t)$ by the formula*

$$H_{i,j}(t) \triangleq \sum_{0<u\leq t} H^i(u-)H^j(u), \qquad \forall t \in \mathbb{R}_+.$$

This process $H_{i,j}$ counts the number of jumps of migration process $C^1(t)$ from the state i to the state j up to time t. For arbitrary $i, j \in \mathcal{K}, i \neq j$, the processes

$$M_{i,j}(t) = H_{i,j}(t) - \int_0^t \lambda_{i,j}(u)H_i(u)du = H_{i,j}(t) - \int_0^t \lambda_{C^1(u),j}(u)H_i(u)du,$$

and

$$M_K(t) = H_K(t) - \int_0^t \sum_{i=1}^{K-1} \lambda_{i,K}H_i(u)du = H_K(t) - \int_0^t \lambda_{C^1(u),K}(1 - H_K(u))du,$$

are $\mathbb{G}$ martingales.

To describe the credit risk we need also, beside the credit migration process C^1 defined above, the process $C^2(t)$ of previous rating. If by $\tau_1, \tau_2, \tau_3, \ldots$ we denote the consecutive moments of jumps of credit migration process C^1, then for $t \in [\tau_k, \tau_{k+1})$

$$C^1(t) = C^1(\tau_k), \qquad C^2(t) = C^1(\tau_{k-1}).$$

We denote by $C(t)$ the two dimensional credit rating process defined by

$$C(t) = (C^1(t), C^2(t)).$$

Therefore the pre-default term structure depending on $C(t)$ is given by the formula

$$g(t,u) = g_{C^1(t)}(t,u) = 1\!\!1_{\{C^1(t)=1\}} g_1(t,u) + 1\!\!1_{\{C^1(t)=2\}} g_2(t,u) + \ldots + 1\!\!1_{\{C^1(t)=K-1\}} g_{K-1}(t,u).$$

We sum up here to $K-1$, since the last K-th rating corresponds to default event

$$\tau = \inf \left\{ t > 0 \ : \ C^1(t) = K \right\}$$

It is obvious that each recovery payment depends on credit rating before default i.e.

$$\delta_t = \delta_{C^2(t)}(t) = 1\!\!1_{\{C^2(t)=1\}} \delta_1(t) + 1\!\!1_{\{C^2(t)=2\}} \delta_2(t) + \ldots + 1\!\!1_{\{C^2(t)=K-1\}} \delta_{K-1}(t),$$

where δ_i is a recovery payment connected with $i-$th rating.
Moreover, we assume that the given $K-1$ defaultable forward rates have dynamics $g_i(t,\theta)$ given by

$$dg_i(t,\theta) = \alpha_i(t,\theta)dt + \langle \sigma_i(t,\theta), dZ_i(t) \rangle_U, \quad i \in \{1,\ldots,K\},$$

where $Z_i(t)$ are Lévy processes with values in U. By Lévy-Ito decomposition, each $Z_i(t)$ has the form

$$Z_i(t) = a_i t + W_i(t) + \int_0^t \int_{|y|_U \le 1} y(\mu_i(ds,dy) - ds\nu_i(dy)) + \int_0^t \int_{|y|_U > 1} y\mu_i(ds,dy).$$

Denote by $D_i(t,\theta) = e^{-\int_t^\theta g_i(t,u),du}$, and discounted values of D_i by $\hat{D}_i(t,\theta) = \frac{D_i(t,\theta)}{B_t}$. As in previous section we consider three types of recovery payment. We investigate them separately and we use the same notion D for process of recovery payment (previously we use D, D^δ, D^Δ).

4.1 Fractional recovery of market value with ratings migrations

The price process of defaultable bond with credit migrations and fractional recovery of market value should satisfy

$$D(\theta,\theta) = 1\!\!1_{\{\tau>\theta\}} + 1\!\!1_{\{\tau\le\theta\}} \delta_{C^2(\tau)}(\tau) D(\tau-,\theta) \frac{B_\theta}{B_\tau},$$

where $\tau = \inf\{t > 0 : C^1(t) = K\}$. Hence we have

$$\begin{aligned} D(t,\theta) &= 1\!\!1_{\{C^1(t)\neq K\}} D_{C^1(t)}(t,\theta) + 1\!\!1_{\{C^1(t)=K\}} \delta_{C^2(\tau)}(\tau) D_{C^2(\tau)}(\tau,\theta) \frac{B_t}{B_\tau} \\ &= \sum_{i=1}^{K-1} 1\!\!1_{\{C^1(t)\neq K\}} 1\!\!1_{\{C^1(t)=i\}} D_i(t,\theta) + \sum_{i=1}^{K-1} 1\!\!1_{\{C^1(t)=K\}} 1\!\!1_{\{C^2(t)=i\}} \delta_i(\tau) D_i(\tau-,\theta) \frac{B_t}{B_\tau} \end{aligned}$$

or equivalently

$$D(t,\theta) = \sum_{i=1}^{K-1} \left(H_i(t) D_i(t,\theta) + H_{i,K}(t) \delta_i(\tau) D_i(\tau-,\theta) \frac{B_t}{B_\tau} \right).$$

Theorem 6. *The processes of discounted prices of defaultable bond with credit migrations and fractional recovery of market value are local martingale if and only if following conditions hold on the set* $\{C^1(t) \neq K\}$

$$g_{C^1(t)}(t,t) = f(t,t) + (1 - \delta_{C^1(t)}(t)) \lambda_{C^1(t),K}(t)), \tag{24}$$

$$\begin{aligned} \int_0^\theta \alpha_{C^1(t)}(t,v)dv &= J_{C^1(t)} \left(1\!\!1_{[0,\theta]}(t) \int_t^\theta \sigma_{C^1(t)}(t,v)dv \right) \\ &+ \sum_{i=1, i\neq C^1(t)}^{K-1} \left[\frac{D_i(t-,\theta)}{D_{C^1(t)}(t-,\theta)} - 1 \right] \lambda_{C^1(t),i}(t). \end{aligned} \tag{25}$$

Proof. We have

$$d\Big(\frac{D(t,\theta)}{B_t}\Big) = \sum_{i=1}^{K-1} \Big(d\Big(H_i(t)\frac{D_i(t,\theta)}{B_t}\Big) + d\Big(H_{i,K}(t)\delta_i(\tau)\frac{D_i(\tau-,\theta)}{B_\tau}\Big)\Big). \tag{26}$$

Notice that the differential of the second term in this sum has the form

$$d\Big(H_{i,K}(t)\delta_i(\tau)\frac{D_i(\tau-,\theta)}{B_\tau}\Big) = \delta_i(t)\frac{D_i(t-,\theta)}{B_t}d\Big(H_{i,K}(t)\Big).$$

Since the process

$$M_{i,K}(t) = H_{i,K}(t) - \int_0^t \lambda_{i,K}(u)H_i(u)du$$

follows $\mathbb{G}$ - martingale, then we have

$$\delta_i(t)\frac{D_i(t-,\theta)}{B_t}d\Big(H_{i,K}(t)\Big) = \frac{D_i(t-,\theta)}{B_t}\delta_i(t)dM_{i,K}(t) + \frac{D_i(t-,\theta)}{B_t}\delta_i(t)\lambda_{i,K}(t)H_i(t)dt.$$

The first term in the sum in (26) is equal to

$$d\Big(H_i(t)\frac{D_i(t,\theta)}{B_t}\Big) = d\Big(H_i(t)\Big)\frac{D_i(t-,\theta)}{B_t} + H_i(t)d\Big(\frac{D_i(t,\theta)}{B_t}\Big) + \underbrace{d\left[H_i(\cdot), \frac{D_i(\cdot,\theta)}{B.}\right]_t^c}_{=0},$$

and since the process

$$M_i(t) = H_i(t) - \int_0^t \lambda_{C^1(u),i}(u)du$$

follows $\mathbb{G}$-martingale, then we have

$$\begin{aligned} d\Big(H_i(t)\frac{D_i(t,\theta)}{B_t}\Big) = \frac{D_i(t-,\theta)}{B_t}\Big(&\quad dM_i(t) + \lambda_{C^1(t),i}(t)dt \\ &+ \quad H_i(t)(g_i(t,t) - f(t,t) + a_i(t,\theta))dt \\ &+ \quad H_i(t)\int_U \left[e^{\langle \tilde{\sigma}_i^*(t)\mathbb{1}_{[0,\theta]},y\rangle_U} - 1\right](\mu_i(dt,dy) - dt\nu_i(dy)) \\ &- \quad H_i(t)\langle \tilde{\sigma}_i^*(t)\mathbb{1}_{[0,\theta]}, dW_i(t)\rangle_U\Big). \end{aligned}$$

If we gather these results we obtain that differential of the single term in the sum in (26) is given by

$$\begin{aligned} d\Big(H_i(t)\frac{D_i(t,\theta)}{B_t} &+ \quad H_{i,K}(t)\delta_i(\tau)\frac{D_i(\tau,\theta)}{B_\tau}\Big) \\ = \frac{D_i(t-,\theta)}{B_t}\Big(&\quad dM_i(t) + \delta_i(t)dM_{i,K}(t) + \lambda_{C^1(t),i}(t)dt \\ &+ \quad H_i(t)(g_i(t,t) - f(t,t) + a_i(t,\theta) + \delta_i(t)\lambda_{i,K}(t))dt \\ &+ \quad H_i(t)\int_U \left[e^{\langle \tilde{\sigma}_i^*(t)\mathbb{1}_{[0,\theta]},y\rangle_U} - 1\right](\mu_i(dt,dy) - dt\nu_i(dy)) \\ &- \quad H_i(t)\langle \tilde{\sigma}_i^*(t)\mathbb{1}_{[0,\theta]}, dW_i(t)\rangle_U\Big). \end{aligned}$$

Therefore the drift term of the sum in (26) is given by

$$I = \sum_{i=1}^{K-1} H_i(t)\frac{D_i(t-,\theta)}{B_t}(g_i(t,t) - f(t,t) + a_i(t,\theta) + \delta_i(t)\lambda_{i,K}(t))dt + \sum_{i=1}^{K-1}\frac{D_i(t-,\theta)}{B_t}\lambda_{C^1(t),i}(t)dt.$$

We can represent I in the following way

$$\begin{aligned} I = \quad & (1-H_K(t))\frac{D_{C^1(t)}(t-,\theta)}{B_t}(g_{C^1(t)}(t,t)-f(t,t)+a_{C^1(t)}(t,\theta)+\delta_{C^1(t)}(t)\lambda_{i,K}(t))dt \\ & + \sum_{i=1}^{K-1}\frac{D_i(t-,\theta)}{B_t}\lambda_{C^1(t),i}(t)dt, \end{aligned}$$

Since $D_{C^1(t)} > 0$ (it has exponential form) and

$$\begin{aligned} \sum_{i=1}^{K-1}\frac{D_i(t-,\theta)}{D_{C^1(t)}(t-,\theta)}\lambda_{C^1(t),i}(t) &= \sum_{i=1,i\neq C^1(t)}^{K-1}\frac{D_i(t-,\theta)}{D_{C^1(t)}(t-,\theta)}\lambda_{C^1(t),i}(t)+\lambda_{C^1(t),C^1(t)}(t) \\ &= \sum_{i=1,i\neq C^1(t)}^{K-1}\left[\frac{D_i(t-,\theta)}{D_{C^1(t)}(t-,\theta)}-1\right]\lambda_{C^1(t),i}(t)-\lambda_{C^1(t),K}(t) \end{aligned}$$

we can write

$$\begin{aligned} I = (1-H_K(t))\frac{D_{C^1(t)}(t-,\theta)}{B_t}\Bigg(\quad & g_{C^1(t)}(t,t)-f(t,t)+a_{C^1(t)}(t,\theta)+(\delta_{C^1(t)}(t)-1)\lambda_{C^1(t),K}(t) \\ & + \sum_{i=1,i\neq C^1(t)}^{K-1}\left[\frac{D_i(t-,\theta)}{D_{C^1(t)}(t-,\theta)}-1\right]\lambda_{C^1(t),i}(t)\Bigg)dt. \end{aligned}$$

Now we split I into two parts, the first one $I_1(t)$, which is not depending on θ and the second one $I_2(t,\theta)$ depending on both t and θ i.e. we have

$$I = (1-H_K(t))\frac{D_{C^1(t)}(t-,\theta)}{B_t}(I_1(t)+I_2(t,\theta))dt,$$

where

$$I_1(t) = \Big(g_{C^1(t)}(t,t)-f(t,t)-(1-\delta_{C^1(t)}(t))\lambda_{C^1(t),K}(t)\Big),$$

and

$$I_2(t,\theta) = \left(a_{C^1(t)}(t,\theta)+\sum_{i=1,i\neq C^1(t)}^{K-1}\left[\frac{D_i(t-,\theta)}{D_{C^1(t)}(t-,\theta)}-1\right]\lambda_{C^1(t),i}(t)\right).$$

If (24) and (25) hold, then $I_1(t)=0$ and $I_2(t,\theta)=0$, which implies that the drift term I vanish. Conversly, if the drift term I vanish, then on the set $\{C^1(t)\neq K\}$:

$$I_1(t)+I_2(t,\theta)=0 \qquad \forall t\leq\theta \text{ and } \forall\theta\in[0,T^*].$$

Since for $\theta=t$ we have $I_2(t,t)=0$, then we obtain that $I_1(t)=0$ which is equivalent to (24), and if $I_1(t)=0$ then we must have $I_2(t,\theta)=0$ which is equivalent to (25). □

4.2 Fractional recovery of treasury value with ratings migrations

In the case of fractional recovery of treasury value with ratings migrations we have

$$D(\theta,\theta) = 1\!\!1_{\{\tau>\theta\}}+1\!\!1_{\{\tau\leq\theta\}}\delta_{C^2(t)},$$

hence

$$\begin{aligned} D(t,\theta) &= 1\!\!1_{\{C^1(t)\neq K\}}D_{C^1(t)}(t,\theta)+1\!\!1_{\{C^1(t)=K\}}\delta_{C^2(t)}B(t,\theta) \\ &= \sum_{i=1}^{K-1}1\!\!1_{\{C^1(t)\neq K\}}1\!\!1_{\{C^1(t)=i\}}D_i(t,\theta)+\sum_{i=1}^{K-1}1\!\!1_{\{C^1(t)=K\}}1\!\!1_{\{C^2(t)=i\}}\delta_iB(t,\theta) \end{aligned}$$

or equivalently

$$D(t,\theta)=\sum_{i=1}^{K-1}\Big(H_i(t)D_i(t,\theta)+H_{i,K}(t)\delta_i B(t,\theta)\Big).$$

Theorem 7. *The process of discounted prices of defaultable bond with fractional recovery of treasury value are local martingales if and only if the following two conditions hold:*
on the set $\{C^1(t)\neq K\}$ *for all* $t\leq\theta$ *and for all* $\theta\in[0,T^*]$ *we have :*

$$g_{C^1(t)}(t,t)=f(t,t)+(1-\delta_{C^1(t)})\lambda_{C^1(t),K} \tag{27}$$

$$\int_0^\theta \alpha_{C^1(t)}(t,u)du=J_{C^1(t)}\Big(\mathbb{1}_{[0,\theta]}(t)\int_t^\theta\sigma_{C^1(t)}(t,v)dv\Big) \tag{28}$$

$$+\,\delta_{C^1(t)}\Big[\frac{B(t-,\theta)}{D_{C^1(t)}(t-,\theta)}-1\Big]\lambda_{C^1(t),K}(t)+\sum_{j=1,j\neq C^1(t)}^{K-1}\Big[\frac{D_j(t-,\theta)}{D_{C^1(t)}(t-,\theta)}-1\Big]\lambda_{C^1(t),j}(t).$$

Proof. Discounted value of defaultable bonds with fractional recovery of market value equals

$$\frac{D(t,\theta)}{B_t}=\sum_{i=1}^{K-1}\Big(H_i(t)\frac{D_i(t,\theta)}{B_t}+H_{i,K}(t)\delta_i\frac{B(t,\theta)}{B_t}\Big).$$

By integration by parts formula we have:

$$\begin{aligned}
&d\Big(H_{i,K}(t)\delta_i\frac{B(t,\theta)}{B_t}\Big)\\
&=\delta_i\frac{B(t-,\theta)}{B_t}\Big(dM_{i,K}(t)+\big(\lambda_{i,K}(t)H_i(t)+\bar a(t,\theta)H_{i,K}(t)\big)dt\\
&+H_{i,K}(t)\Big(\int_U\Big[e^{-\langle\tilde\sigma^*(t)\mathbb{1}_{[0,\theta]},y\rangle_U}-1\Big](\mu(dt,dy)-dt\nu(dy))-\langle\tilde\sigma^*(t)\mathbb{1}_{[0,\theta]},dW(t)\rangle_U\Big)\Big)
\end{aligned}$$

and

$$\begin{aligned}
d\Big(H_i(t)\frac{D_i(t,\theta)}{B_t}\Big)=\frac{D_i(t-,\theta)}{B_{t-}}\Big(&\quad dM_i(t)+\lambda_{C^1(t),i}(t)dt\\
&+\quad H_i(t)(g_i(t,t)-f(t,t)+a_i(t,\theta))dt\\
&+\quad H_i(t)\int_U\Big[e^{\langle\tilde\sigma_i^*(t)\mathbb{1}_{[0,\theta]},y\rangle_U}-1\Big](\mu_i(dt,dy)-dt\nu_i(dy))\\
&-\quad H_i(t)\langle\tilde\sigma_i^*(t)\mathbb{1}_{[0,\theta]},dW_i(t)\rangle_U\Big).
\end{aligned}$$

Therefore the drift term I is given by

$$\begin{aligned}
I=\quad&\sum_{i=1}^{K-1}\frac{D_i(t-,\theta)}{B_t}\Big(\lambda_{C^1(t),i}(t)+H_i(t)(g_i(t,t)-f(t,t)+a_i(t,\theta))\Big)dt\\
&+\sum_{i=1}^{K-1}\frac{B(t-,\theta)}{B_t}\delta_i\big(\lambda_{i,K}(t)H_i(t)+\bar a(t,\theta)H_{i,K}(t)\big)dt=I_1+I_2+I_3,
\end{aligned}$$

where

$$I_1\quad=\quad\sum_{i=1}^{K-1}H_i(t)\frac{D_i(t-,\theta)}{B_t}\Big((g_i(t,t)-f(t,t)+a_i(t,\theta))+\delta_i\frac{B(t-,\theta)}{D_i(t-,\theta)}\lambda_{i,K}(t)\Big)dt,$$

$$
\begin{aligned}
I_2 &= \sum_{j=1}^{K-1} \frac{D_j(t-,\theta)}{B_t} \lambda_{C^1(t),j}(t)dt = \sum_{j=1}^{K-1} \frac{D_j(t-,\theta)}{B_t} \sum_{i=1}^{K-1} H_i(t)\lambda_{i,j}(t)dt \\
&= \sum_{i=1}^{K-1} H_i(t) \frac{D_i(t-,\theta)}{B_t} \Big(\sum_{j\neq i}^{K-1} \frac{D_j(t-,\theta)}{D_i(t-,\theta)} \lambda_{i,j}(t) + \lambda_{i,i}(t) \Big) dt \\
&= \sum_{i=1}^{K-1} H_i(t) \frac{D_i(t-,\theta)}{B_t} \Big(\sum_{j\neq i}^{K-1} \Big[\frac{D_j(t-,\theta)}{D_i(t-,\theta)} - 1 \Big] \lambda_{i,j}(t) - \lambda_{i,K}(t) \Big) dt,
\end{aligned}
$$

$$
I_3 = \frac{B(t-,\theta)}{B_t} \bar{a}(t,\theta) \sum_{i=1}^{K-1} \delta_i H_{i,K}(t)dt = H_K(t) \Big(\frac{B(t-,\theta)}{B_t} \bar{a}(t,\theta) \sum_{i=1}^{K-1} \delta_i 1\!\!1_{\{C^2(t)=i\}} dt \Big).
$$

We assume HJM type condition for default-free bonds (condition 7), so $I_3 = 0$. It is easy to see that

$$
\begin{aligned}
I_1 + I_2 = \sum_{i=1}^{K-1} H_i(t) \frac{D_i(t-,\theta)}{B_t} \Big(& (g_i(t,t) - f(t,t) + a_i(t,\theta)) + \delta_i \Big[\frac{B(t-,\theta)}{D_i(t-,\theta)} - 1 \Big] \lambda_{i,K}(t) \\
& + \sum_{j\neq i}^{K-1} \Big[\frac{D_j(t-,\theta)}{D_i(t-,\theta)} - 1 \Big] \lambda_{i,j}(t) - (1-\delta_i)\lambda_{i,K}(t) \Big) dt.
\end{aligned}
$$

Since $H_i(t) = 1$ on the set $\{C^1(t) = i\}$ and zero on its complement we can write this as:

$$
\begin{aligned}
I_1 + I_2 = (1 - H_K(t)) \frac{D_{C^1(t)}(t-,\theta)}{B_t} \Big(& g_{C^1(t)}(t,t) - f(t,t) - (1 - \delta_{C^1(t)})\lambda_{C^1(t),K}(t) + a_{C^1(t)}(t,\theta) \\
& + \delta_i \Big[\frac{B(t-,\theta)}{D_{C^1(t)}(t-,\theta)} - 1 \Big] \lambda_{C^1(t),K}(t) \\
& + \sum_{j\neq C^1(t)}^{K-1} \Big[\frac{D_j(t-,\theta)}{D_{C^1(t)}(t-,\theta)} - 1 \Big] \lambda_{C^1(t),j}(t) \Big) dt,
\end{aligned}
$$

where we have also used a fact that we have summation only up to $K - 1$. Arguing as before we obtain thesis of theorem. □

4.3 Fractional recovery of par value with ratings migrations

In the case of fractional recovery of par value with ratings migrations we have

$$
D(\theta,\theta) = 1\!\!1_{\{\tau>\theta\}} + 1\!\!1_{\{\tau\leq\theta\}} \delta_{C^2(t)} \frac{B_\theta}{B_\tau},
$$

hence

$$
\begin{aligned}
D(t,\theta) &= 1\!\!1_{\{C^1(t)\neq K\}} D_{C^1(t)}(t,\theta) + 1\!\!1_{\{C^1(t)=K\}} \delta_{C^2(t)} \frac{B_t}{B_\tau} \\
&= \sum_{i=1}^{K-1} 1\!\!1_{\{C^1(t)\neq K\}} 1\!\!1_{\{C^1(t)=i\}} D_i(t,\theta) + \sum_{i=1}^{K-1} 1\!\!1_{\{C^1(t)=K\}} 1\!\!1_{\{C^2(t)=i\}} \delta_i \frac{B_t}{B_\tau}
\end{aligned}
$$

or equivalently

$$
D(t,\theta) = \sum_{i=1}^{K-1} \Big(H_i(t) D_i(t,\theta) + H_{i,K}(t) \delta_i \frac{B_t}{B_\tau} \Big).
$$

Theorem 8. *The processes of discounted prices of defaultable bond with fractional recovery of par value are local martingales if and only if the following two conditions hold:*
on the set $\{C^1(t) \neq K\}$ for all $t \leq \theta$ and for all $\theta \in [0, T^*]$ we have :

$$g_{C^1(t)}(t,t) = f(t,t) + (1 - \delta_{C^1(t)})\lambda_{C^1(t),K}(t), \tag{29}$$

$$\int_0^\theta \alpha_{C^1(t)}(t,u)du = J_{C^1(t)}\bigg(\mathbb{1}_{[0,\theta]}(t)\int_t^\theta \sigma_{C^1(t)}(t,v)dv\bigg) \tag{30}$$

$$+ \delta_{C^1(t)}\left[\frac{1}{D_{C^1(t)}(t-,\theta)} - 1\right]\lambda_{C^1(t),K}(t) + \sum_{j=1, j\neq C^1(t)}^{K-1} \left[\frac{D_j(t-,\theta)}{D_{C^1(t)}(t-,\theta)} - 1\right]\lambda_{C^1(t),j}(t).$$

Proof. We have that discounted value of this bond is given by:

$$\frac{D(t,\theta)}{B_t} = \sum_{i=1}^{K-1}\ \Big(H_i(t)\frac{D_i(t,\theta)}{B_t} + H_{i,K}\frac{\delta_i}{B_\tau}\Big) = \sum_{i=1}^{K-1}\ \Big(H_i(t)\frac{D_i(t,\theta)}{B_t} + \int_0^t \frac{\delta_i}{B_u}dH_{i,K}(u)\ \Big).$$

The differential of the first part was calculated before, and the differential of the second part can be written using martingale $M_{i,K}$:

$$\frac{\delta_i}{B_t}dH_{i,K}(t) = \frac{\delta_i}{B_t}dM_{i,K}(t) + \frac{\delta_i}{B_t}H_i(t)\lambda_{i,K}(t)dt.$$

Hence drift term I is given by

$$I = \sum_{i=1}^{K-1}\ \bigg(\frac{D_i(t-,\theta)}{B_t}\Big(\lambda_{C^1(t),i}(t) + H_i(t)(g_i(t,t) - f(t,t) + a_i(t,\theta))\Big)dt + \frac{\delta_i}{B_t}H_i(t)\lambda_{i,K}(t)dt\ \bigg).$$

The sum $\sum_{j=1}^{K-1}\frac{D_j(t-,\theta)}{B_t}\lambda_{C^1(t),j}(t)dt$ we calculate in the proof of previous theorem, so we write the drift term I in the form

$$I = \sum_{i=1}^{K-1} H_i(t)\frac{D_i(t-,\theta)}{B_t}\quad\bigg(\quad g_i(t,t) - f(t,t) + a_i(t,\theta) + \delta_i\left[\frac{1}{D_i(t-,\theta)} - 1\right]\lambda_{i,K}(t)$$

$$+ \sum_{j\neq i}^{K-1}\left[\frac{D_j(t-,\theta)}{D_i(t-,\theta)} - 1\right]\lambda_{i,j}(t) - (1-\delta_i)\lambda_{i,K}(t)\bigg)dt.$$

By similar arguments as before

$$I = (1 - H_K(t))\tfrac{D_{C^1(t)}(t-,\theta)}{B_t}\bigg(g_{C^1(t)}(t,t) - f(t,t) + a_{C^1(t)}(t,\theta) +$$

$$+\delta_{C^1(t)}\left[\tfrac{1}{D_{C^1(t)}(t-,\theta)} - 1\right]\lambda_{C^1(t),K}(t) + \textstyle\sum_{j\neq i}^{K-1}\left[\tfrac{D_j(t-,\theta)}{D_{C^1(t)}(t-,\theta)} - 1\right]\lambda_{C^1(t),j}(t) - (1-\delta_{C^1(t)})\lambda_{C^1(t),K}(t)\bigg)dt.$$

Arguing as before we obtain thesis of theorem. □

4.4 HJM condition in terms of derivative of functional J

In the series of lemmas we present the form of derivative of functional J. First we recall the well known lemma

Lemma 2. *Let J be a linear-quadratic functional i.e.*

$$J(x) = -\langle a, x\rangle_U + \frac{1}{2}\langle Qx, x\rangle_U,$$

where $a \in U$, Q is a linear symmetric bounded linear operator, then J is differentiable for each $x \in U$ and

$$DJ(x) = -a + Qx.$$

Lemma 3. *Let J be a functional of the form :*

$$J(x) = \int_U \left(e^{-\langle x,y\rangle_U} - 1 + 1\!\!1_{|y|_U\le 1}(y)\langle x,y\rangle_U\right)\nu(dy),$$

where ν is a Levy measure which has exponential moments, then J is differentiable at each $x \in U$ and

$$DJ(x) = -\int_U \left(e^{-\langle x,y\rangle_U} - 1\!\!1_{|y|_U\le 1}(y)\right) y\,\nu(dy).$$

Proof. The proof is straightforward. We use the existence of exponential moments of Levy measure ν . □

Lemma 4. *Let $J : U \to \mathbb{R}$ be a differentiable functional and u be a smooth curve i.e. smooth mapping $u : \mathbb{R} \to U$, then the mapping $J(u(\cdot)) : \mathbb{R} \to \mathbb{R}$ is differentiable and*

$$\left.\frac{\partial J(u(\theta))}{\partial\theta}\right|_{\theta=\theta_0} = \langle DJ(u(\theta_0)), du(\theta_0)\rangle_U,$$

where by du we denote differential of curves $u : \mathbb{R} \to U$.

Corrolary 2. *In a view of above lemmas HJM type condition (7) can be written as*

$$\alpha(t,\theta) = \left\langle DJ\left(\int_0^\theta \sigma(t,v)dv\right), \sigma(t,\theta)\right\rangle_U,$$

where $DJ(x)$ is given by:

$$DJ(x) = -a + Qx - \int_U \left(e^{-\langle x,y\rangle_U} - 1\!\!1_{|y|_U\le 1}(y)\right) y\,\nu(dy),$$

so HJM type condition has the following form:

$$\begin{aligned}\alpha(t,\theta) &= -\langle a,\sigma(t,\theta)\rangle_U + \left\langle Q\int_0^\theta \sigma(t,v)dv, \sigma(t,\theta)\right\rangle_U \\ &\quad - \int_U \left(e^{-\langle\int_0^\theta \sigma(t,v)dv, y\rangle_U} - 1\!\!1_{|y|_U\le 1}(y)\right)\langle y,\sigma(t,\theta)\rangle_U \nu(dy).\end{aligned}$$

Remark 3. *For calculating HJM type conditions for models with credit risk we will also need the following derivatives:*
i) for fractional recovery of treasury

$$\frac{\partial}{\partial\theta}\left(\frac{B(t-,\theta)}{D_1(t-,\theta)} - 1\right) = \Big(g_1(t-,\theta) - f(t-,\theta)\Big)e^{\int_t^\theta (g_1(t-,u)-f(t-,u))du},$$

ii) for fractional recovery of par value

$$\frac{\partial}{\partial\theta}\left(\frac{1}{D_1(t-,\theta)} - 1\right) = g_1(t-,\theta)e^{\int_t^\theta g_1(t-,u)du},$$

iii) for fractional recovery of market value with rating migrations

$$\frac{\partial}{\partial\theta}\left(\frac{D_i(t-,\theta)}{D_{C^1(t)}(t-,\theta)} - 1\right) = \Big(g_{C^1(t)}(t-,\theta) - g_i(t-,\theta)\Big)e^{\int_t^\theta (g_{C^1(t)}(t-,u)-g_i(t-,u))du},$$

iv) for fractional recovery of treasury value with rating migrations

$$\frac{\partial}{\partial\theta}\left(\frac{B(t-,\theta)}{D_{C^1(t)}(t-,\theta)} - 1\right) = \Big(g_{C^1(t)}(t-,\theta) - f(t-,\theta)\Big)e^{\int_t^\theta (g_{C^1(t)}(t-,u)-f(t-,u))du},$$

v) for fractional recovery of par value with rating migrations

$$\frac{\partial}{\partial\theta}\left(\frac{1}{D_{C^1(t)}(t-,\theta)} - 1\right) = g_{C^1(t)}(t-,\theta)e^{\int_t^\theta g_{C^1(t)}(t-,u)du}.$$

From these lemmas and previous results we obtain a series of propositions. For models without ratings we have

Theorem 9. *i) Condition (13) for fractional recovery of market value has the form*

$$\alpha_1(t,\theta) \;=\; \Big\langle DJ_1\Big(\int_0^\theta \sigma_1(t,v)dv\Big), \sigma_1(t,\theta)\Big\rangle_U.$$

ii) Condition (17) for fractional recovery of treasury has the form

$$\alpha_1(t,\theta) \;=\; \Big\langle DJ_1\Big(\int_0^\theta \sigma_1(t,v)dv\Big), \sigma_1(t,\theta)\Big\rangle_U + \delta\lambda_t\Big(g_1(t-,\theta)-f(t-,\theta)\Big)e^{\int_t^\theta (g_1(t-,u)-f(t-,u))du}.$$

iii) Condition (22) for fractional recovery of par value has the form

$$\alpha_1(t,\theta) \;=\; \Big\langle DJ_1\Big(\int_0^\theta \sigma_1(t,v)dv\Big), \sigma_1(t,\theta)\Big\rangle_U + \delta\lambda_t g_1(t-,\theta)e^{\int_t^\theta g_1(t-,u)du}.$$

And for models with ratings we have:

Theorem 10. *i) Condition (25) for fractional recovery of market value has the form*

$$\begin{aligned}\alpha_{C^1(t)}(t,\theta) \;=\;& \Big\langle DJ_{C^1(t)}\Big(\int_0^\theta \sigma_{C^1(t)}(t,v)dv\Big), \sigma_{C^1(t)}(t,\theta)\Big\rangle_U \\ &+ \sum_{i=1,i\neq C^1(t)}^{K-1} \lambda_{C^1(t),i}(t)\Big(g_{C^1(t)}(t-,\theta)-g_i(t-,\theta)\Big)e^{\int_t^\theta (g_{C^1(t)}(t-,u)-g_i(t-,u))du}.\end{aligned}$$

ii) Condition (28) for fractional recovery of treasury has the form

$$\begin{aligned}\alpha_{C^1(t)}(t,\theta) \;=\;& \Big\langle DJ_{C^1(t)}\Big(\int_0^\theta \sigma_{C^1(t)}(t,v)dv\Big), \sigma_{C^1(t)}(t,\theta)\Big\rangle_U \\ &+ \sum_{i=1,i\neq C^1(t)}^{K-1} \lambda_{C^1(t),i}(t)\Big(g_{C^1(t)}(t-,\theta)-g_i(t-,\theta)\Big)e^{\int_t^\theta (g_{C^1(t)}(t-,u)-g_i(t-,u))du} \\ &+ \delta_{C^1(t)}\lambda_{C^1(t),K}\Big(g_{C^1(t)}(t-,\theta)-f(t-,\theta)\Big)e^{\int_t^\theta (g_{C^1(t)}(t-,u)-f(t-,u))du}.\end{aligned}$$

iii) Condition (30) for fractional recovery of par value has the form

$$\begin{aligned}\alpha_{C^1(t)}(t,\theta) \;=\;& \Big\langle DJ_{C^1(t)}\Big(\int_0^\theta \sigma_{C^1(t)}(t,v)dv\Big), \sigma_{C^1(t)}(t,\theta)\Big\rangle_U \\ &+ \sum_{i=1,i\neq C^1(t)}^{K-1} \lambda_{C^1(t),i}(t)\Big(g_{C^1(t)}(t-,\theta)-g_i(t-,\theta)\Big)e^{\int_t^\theta (g_{C^1(t)}(t-,u)-g_i(t-,u))du} \\ &+ \delta_{C^1(t)}\lambda_{C^1(t),K}\, g_{C^1(t)}(t-,\theta)e^{\int_t^\theta g_{C^1(t)}(t-,u)du}.\end{aligned}$$

5 Musiela parameterization and HJM equations

Results similar to those of Section 3.1 were obtained by Özkan and Schmidt in [12]. In [12] HJM conditions are formulated in terms of Musiela parameterization and to obtain them the authors used an Ito formula in Hilbert spaces. To do so some technical conditions were needed which are not required in the direct approach presented here. In this final section we clarify a connection between direct and Musiela approaches and give some additional information on the latter.

Assume that $T^* = +\infty$ and start with the following form of HJM equation:

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \langle \sigma(s,T), dZ(s)\rangle_U. \tag{31}$$

If we want to work under Musiela parametrization we have to put $T = t + x$ in (31), and then we have

$$f(t,t+x) = f(0,t+x) + \int_0^t \alpha(s,t+x)ds + \int_0^t \langle \sigma(s,t+x), dZ(s)\rangle_U.$$

Therefore for each t we have defined the real function $f(t, t+\cdot)$ which is the current forward curve. If we introduce right-shift operator $S(t)$:

$$S(t)\phi(x) = \phi(t+x), \qquad t \geq 0, \qquad x \geq 0.$$

then we can write equation (31) in the form

$$f(t,t+x) \;=\; S(t)f(0,x) + \int_0^t S(t-s)\alpha(s,s+x)ds + \int_0^t \langle S(t-s)\sigma(s,s+x), dZ(s)\rangle_U.$$

Introducing new objects:

$$\begin{aligned} \hat{f}_t(x) &= f(t,t+x) \\ \widehat{\alpha}_s(x) &= \alpha(s,s+x) \\ \big[\widehat{\sigma}_s(u)\big](x) &= \langle \sigma(s,s+x), u\rangle_U \qquad \forall u \in U, \end{aligned}$$

note that $\widehat{\alpha}_s$ is a process with values in the space of curves and $\widehat{\sigma}_s$ is a process with values in the space of operators acting from the space where $Z(s)$ lives into the space of forward curves. We obtain equation in the space of forward curves:

$$\widehat{f}_t \;=\; S(t)\widehat{f}_0 + \int_0^t S(t-s)\widehat{\alpha}_s ds + \int_0^t S(t-s)\widehat{\sigma}_s dZ(s). \tag{32}$$

The formulea defining $\hat{f}$, $\widehat{\alpha}$ and $\big[\widehat{\sigma}(u)\big]$ establish a one-to-one correspondence between the classical and Musiela parameterization and the HJM conditions formulated in one language can be rewritten in the other one.

We give now some background material to treat equation (32) in a precise way. We start from a relationship between Hilbert-Schmidt operators and integral operators with square integrable kernel (see [4] Sect II.2):

Theorem D. *A linear operator $B : \mathcal{L}^2(\Theta_1, \mu_1) \to \mathcal{L}^2(\Theta_2, \mu_2)$, is Hilbert-Schmidt operator if and only if it is an integral operator with square integrable kernel i.e.*

$$Bh(y) = \int_{\Theta_1} b(x,y)h(x)\mu_1(dx), \qquad \|B\|_{HS} = \left(\int_{\Theta_1}\int_{\Theta_2} b^2(x,y)\mu_1(dx)\mu_2(dy)\right)^{\frac{1}{2}} < \infty.$$

It is easy to see that to a given HS operator B corresponds kernel $b(z,x)$ with the following series representation

$$b(z,x) = \sum_n [B^*(e_n)](z)e_n(x),$$

where B^* is adjoint operator of B. Indeed. For any orthonomal basis $\{e_n\}$ of $\mathcal{L}^2(\Theta, \mu)$ we have for $h \in \mathcal{L}^2(\Theta,\mu)$ and $x \in R$:

$$[B(h)](x) \;=\; \sum_n e_n(x)\int_\Theta [B^*(e_n)](z)h(z)\mu(dz) = \int_\Theta \left(\sum_n [B^*(e_n)](z)e_n(x)\right)h(z)\mu(dz).$$

Therefore, if $\widehat{\sigma}_s$ is a Hilbert-Schmidt operator from $U = \mathcal{L}^2(\Theta_1, \mu_1)$ into $\mathcal{L}^2(\Theta_2, \mu_2)$ with the kernel b_s we have

$$\int_{\Theta_1} b_s(x,v)u(v)\mu_1(dv) = \big[\widehat{\sigma}_s(u)\big](x) = \langle \sigma(s, s+x), h\rangle_U.$$

To move from Musiela parametrization (32) to classical HJM equation (31) we identify:

$$\sigma(s, s+x, \xi) = b_s(x, \xi),$$

hence

$$\sigma(t, T, \xi) = b_t(T-t, \xi).$$

For any sequence $(\tilde{\rho}_k)$ of positive numbers denote by $U = l^2(\tilde{\rho})$ the Hilbert space of all sequences $u = (u_k)$ such that

$$|u|_U = \sum_{k=1}^{\infty} u_k^2 \tilde{\rho}_k < \infty$$

If $\tilde{\rho}_k = 1,\ k = 1, \ldots$ one writes simply l^2.
Assume that $Z_1(t), Z_2(t), \ldots$ are zero mean, uncorrelated, real Lévy processes such that:

$$\mathbf{E}|Z_i(t)|^2 = t, \qquad i = 1, 2 \ldots$$

Then $Z(t) = (Z_1(t), Z_2(t), \ldots)$ is a Lévy process in any space $U = l^2(\tilde{\rho})$ where $\sum_{k=1}^{\infty} \tilde{\rho}_k < \infty$. (In fact, by considering expansions with respect to eigenvectors of the covariance operator, arbitrary Hibert space valued Lévy process, with finite second moments, can be identified with a sequence $Z_1(t), Z_2(t), \ldots$). If H is an arbitrary Hilbert space then

$$\mathbf{E}\left|\int_0^T \phi(s)dZ(s)\right|_H^2 = \mathbf{E}\int_0^T \|\phi(s)\|_{HS}^2\, ds$$

where $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm from l^2 to H. Note that any Hilbert-Schmidt operator $\phi : l^2 \to H$ is of the form

$$\phi(u) = \sum_{i=1}^{\infty} u_i h_i, \qquad u = (u_i) \in l^2$$

where

$$\|\phi\|_{HS} = \sum_{i=1}^{\infty} |h_i|^2$$

in particular if $H = L^2([0,\infty), \rho)$, where ρ is positive weight, then $h_i = h_i(\theta),\ \ i = 1, 2, \ldots,\ \ \theta \geq 0$, and

$$\sum_{i=1}^{\infty} |h_i|_H^2 = \int_0^{\infty} \sum_{i=1}^{\infty} |h_i(\theta)|^2 \rho(\theta) d\theta$$

and

$$\phi(u)(\theta) = \sum_{i=1}^{\infty} h_i(\theta) u_i,\ \ u \in l^2,\ \ \theta \geq 0.$$

Assume in particulary, that the positive function ρ is such that for each $t \geq 0$

$$\sup_{x \geq 0} \frac{\rho(x)}{\rho(x+t)} = M(t) < +\infty, \qquad t \geq 0,$$

where $M(t) \leq M_0 e^{wt},\ \ t \geq 0$ for some $M_0 > 0$ and $w > 0$.

Then $(S(t), t \geq 0)$ is a C_0 - semigroup on $L^2([0,\infty),\rho)$. In fact :

$$\begin{aligned} |S(t)h|_H^2 &= \int_0^\infty |h(t+x)|^2\rho(x)dx = \int_0^\infty |h(t+x)|^2 \frac{\rho(x)}{\rho(x+t)}\rho(t+x)dx \\ &\leq M(t)\int_0^\infty |h(t+x)|^2\rho(t+x)dx \leq M(t)|h|_H^2 \end{aligned}$$

and therefore

$$|S(t)| \leq M^{\frac{1}{2}}(t), \qquad t \geq 0.$$

The generator A of S is of the form

$$Ah(\theta) = \frac{\partial h}{\partial \theta},$$

and $h \in D(A)$ if and only if h absolutely continuous on $[0,+\infty)$ and $\int_0^\infty |\frac{\partial h}{\partial \theta}|^2\rho(\theta)d\theta < \infty$. Thus (32) can be written in the mild form:

$$d\hat{f}_t = \left(\frac{\partial}{\partial \theta}\hat{f}_t + \hat{\alpha}_t\right)dt + \hat{\sigma}_t dZ(t).$$

If $\hat{\alpha}_t = F(\hat{f}_t)$, $\hat{\sigma}_t = G(\hat{f}_t)$, then $(\hat{f}_t)$ is a solution of the stochastic evolution equation:

$$d\hat{f} = (A\hat{f} + F(\hat{f}))dt + G(\hat{f})dZ(t)$$

For some information about the proper state space for the bond (LIBOR) curves, see e.g. [15].

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