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Contact Equivalence and characteristic Cones of Ordinary Differential Equations

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Contact Equivalence and Characteristic Cones of Ordinary Differential Equations

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Abstract

We study geometry of a system (F) of ordinary differential equations $x^{(k+1)} = F(t, x, x', \dots, x^{(k)})$. We show that it defines canonical fields of tangent cones on k-jets space. If all Wilczynski invariants of the system vanish then these cones project to the space of solutions of (F). Moreover, they determine locally (F) up to contact transformations.

1 Introduction

In this article we study geometry of systems of ordinary differential equations (F) in the form

$$x^{(k+1)} = F(t, x, x', \dots, x^{(k)}),$$

where $x \in \mathbb{R}^m$. The case $k = 2$ and $m = 1$ was analyzed by S-S. Chern [1] (see also [3]). He showed that if certain relation on derivatives of F (called Wuenschmann condition) holds then there is a conformal metric defined on the space of solutions of (F). In [4, 5] Y. Se-ashi considered the case of arbitrary dimension, but with linear right hand side. He has generalized results of E. J. Wilczynski [6] and reinterpreted them in terms of Cartan connections. Recently B. Doubrov [2] has extended a definition of Wilczynski invariants to non-linear systems. He has also characterized all systems (F) which are equivalent to the trivial one $x^{(k+1)} = 0$. Our aim is to present construction of invariants of (F) from a slightly different perspective. In particular in order to define Wilczynski invariants we omit the process of linearization which appears in [2]. Our results are valid for arbitrary $k > 1$ and $m \geq 1$.

The present paper begins with a study of the geometry of a pair $(\mathcal{X}, \mathcal{V})$ on manifold M, where $\mathcal{X}$ is a line field and $\mathcal{V}$ is a distribution of rank m (a subbundle of the tangent bundle). We denote $\mathcal{V}^i = (\mathrm{ad}^i_{\mathcal{X}} \mathcal{V} \mod \mathcal{X})$ and assume that $\mathcal{V}^i$ has constant rank $(i+1)m$. Provided that this assumption holds we construct invariants of $(\mathcal{X}, \mathcal{V})$. Our reasoning essentially repeats Wilczynski constructions but additionally we define fields of tangent cones on M. We call them *characteristic cones* and denote $C^i(\mathcal{X}, \mathcal{V})$, $i = 0, \dots, k$. They can be considered as curves in

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Grassmann bundle $\mathrm{Gr}_m(\mathcal{V}^i)$. If $m = 1$ then these curves are Veronese curves of order i.

In the second part of the article we apply our constructions to systems of ODE. As a base manifold M we take space $J^k(1,m)$ of k-jets of functions $\mathbb{R} \to \mathbb{R}^m$, $\mathcal{X}$ is spanned by total derivative, and $\mathcal{V}$ is Cauchy characteristic of Cartan distribution $\mathcal{C}_{k-2}$. We say that generalized Wuenschmann condition holds if all Wilczynski invariants vanish. If this is the case then characteristic cone $C^k(\mathcal{X},\mathcal{V})$ is well projected to the solution space of (F). We prove that if $k = 2$ and $m = 1$ then projection of $C^2(\mathcal{X},\mathcal{V})$ is the cone of light directions of Chern conformal metric. We summarize our main results in the following theorem (all notions will be clarified later).

Theorem 1.1 *Let (F) be a system of m ordinary differential equations of order k.*

1. *(F) defines canonical fields of tangent cones $C^i(\mathcal{X},\mathcal{V})$, $i = 0,\ldots,k$, on $J^k(1,m)$, which are invariant under contact transformations.*
2. *If the generalized Wuenschmann condition holds for (F) then $C^k(\mathcal{X},\mathcal{V})$ projects to the space of solutions of (F).*
3. *If the generalized Wuenschmann condition holds for (F) then the projection of $C^k(\mathcal{X},\mathcal{V})$ to the space of solutions determines locally system (F), uniquely up to contact transformations.*

We would like to emphasis that characteristic cones are already defined on $J^k(1,m)$. Thus, they can be used to define conformal tensors not only on solution space as in e.g. [1] but also on $J^k(1,m)$. We will clarify this in future work.

Geometry of $(\mathcal{X},\mathcal{V})$ is also applicable in different context. For instance, it can be used for construction of invariants of Pfaffian systems or, more general, invariants of arbitrary sub-manifolds of cotangent bundle.

2 Geometry of the Distribution in the Presence of the Line Field

2.1 Curvature Operators and Characteristic Cones

Let M be a differentiable manifold and let $\mathcal{V}$ be a rank m subbundle of the tangent bundle TM. In other words, for $x \in M$, $\mathcal{V}(x)$ is an m dimensional subspace of T_xM and $\mathcal{V}(x)$ depend smoothly on x. Let $\mathcal{X} \subseteq TM$ be a smooth line field. We will investigate geometry of the pair $(\mathcal{X},\mathcal{V})$. In this section we reformulate results of E. J. Wilczynski [6], but additionally we construct canonical fields of tangent cones on M. We include proofs, which are mainly reduced to calculations, for two reasons. Firstly for completeness and secondly because we need some formulas for further purposes.

If X and Y are vector fields then the Lie bracket $[X,Y]$ is denoted $\mathrm{ad}_X Y$ and $\mathrm{ad}_X^{i+1} Y = \mathrm{ad}_X(\mathrm{ad}_X^i Y)$. If $\mathcal{V}$ is a subbundle of TM then $\mathrm{ad}_X^i\mathcal{V}$ denotes the subbundle spanned by $\mathrm{ad}_X^i Y$, for any local section Y of $\mathcal{V}$. Note that the rank of $\mathrm{ad}_X^i\mathcal{V}$ may vary even if $\mathcal{V}$ has a constant rank. We will use a matrix notation. Namely, for a tuple $V = (V_1,\ldots,V_m)$ we will denote a tuple $(\mathrm{ad}_X^i V_1,\ldots,\mathrm{ad}^i V_m)$ by $\mathrm{ad}_X^i V$.

Definition. We call $(\mathcal{X},\mathcal{V})$ *good* if there exists a number $k \geq 1$ such that the following relations hold

$$\mathrm{rk}(\mathrm{ad}_X^i\mathcal{V} \mod \mathcal{X}) \;=\; m(i+1) \tag{1}$$

for $i = 0,\dots,k$ and

$$\mathrm{rk}(\mathrm{ad}_X^{k+1}\mathcal{V} \mod \mathcal{X}) \;=\; m(k+1), \tag{2}$$

where X is an arbitrary vector field which spans $\mathcal{X}$. We denote $\mathcal{V}^i = \mathrm{ad}_X^i\mathcal{V} \mod \mathcal{X}$.

In other words $(\mathcal{X},\mathcal{V})$ is *good* if the dimensions of $\mathrm{ad}_X^i\mathcal{V}$ increase as much as possible for $i = 1,\dots,k$, and $\mathrm{ad}_X^k\mathcal{V} \mod \mathcal{X}$ is invariant with respect to the action of ad_X. The definition does not depend on the choice of X, as can be easily checked.

Proposition 2.1 *Let $(\mathcal{X},\mathcal{V})$ be a good pair and let X be an arbitrary vector field which spans $\mathcal{X}$. Locally, there exist independent vector fields $V_1,\dots,V_m$, which span $\mathcal{V}$, and are such that $\mathrm{ad}_X^{k+1}V_i = 0 \mod \mathrm{ad}_X^{k-1}\mathcal{V}$ for $i = 1,\dots,m$.*

Proof. Let $W = (W_1,\dots,W_m)$ be any tuple of vector fields which locally span $\mathcal{V}$. We will find functions $G = (g_{ij})$ such that $V_i = \sum_{j=1}^m g_{ij}W_j$, for $i = 1,\dots,m$, are desired vector fields. In the matrix notation we write $V = GW$. We have

$$\mathrm{ad}_X^{k+1}V = G\mathrm{ad}_X^{k+1}W + (k+1)X(G)\mathrm{ad}_X^k W \mod \mathrm{ad}_X^{k-1}\mathcal{V}.$$

Assume that

$$\mathrm{ad}_X^{k+1}W = H\mathrm{ad}_X^k W \mod \mathrm{ad}_X^{k-1}\mathcal{V},$$

for a certain $H = (h_{ij})$. Since $\mathrm{ad}_X^k V = G\mathrm{ad}_X^k W \mod \mathrm{ad}_X^{k-1}\mathcal{V}$, we obtain the following equation

$$GH + (k+1)X(G) = 0 \tag{3}$$

It can be solved locally and if G is a solution then $\mathrm{ad}_X^{k+1}V = 0 \mod \mathrm{ad}_X^{k-1}\mathcal{V}$. □

Definition. Vector fields $V = (V_1,\dots,V_m)$ are called *normal* (for a fixed vector field X) if

$$\mathrm{ad}_X^{k+1}V = 0 \mod \mathrm{ad}_X^{k-1}\mathcal{V}.$$

Note that (3) implies that if both V and $W = GV$ are normal then $X(G) = 0$. Hence, $\mathrm{ad}_X^i W = G\mathrm{ad}_X^i V$ for any i. We deduce that the distributions

$$\mathcal{V}_X^i = \mathrm{span}\{\mathrm{ad}_X^i V_1,\dots,\mathrm{ad}_X^i V_m\}$$

do not depend on the choice of normal vector fields $V = (V_1,\dots,V_m)$. Distribution $\mathcal{V}_X^k$ will be called *connection* and denoted $\mathcal{H}_X$. We stress that

$$\mathrm{rk}\mathcal{V}_X^i = m$$

and

$$\mathrm{ad}_X^i\mathcal{V} \;=\; \mathcal{V}_X^0\oplus\cdots\oplus\mathcal{V}_X^i,$$

for $i=0,\dots,k$. In particular

$$\mathrm{ad}_X^k\mathcal{V} \;=\; \mathcal{V}_X^0\oplus\cdots\oplus\mathcal{V}_X^{k-1}\oplus\mathcal{H}_X.$$

The last relation defines projections $\pi_X^{\mathcal{V}^i}\colon \mathrm{ad}_X^k\mathcal{V}\to\mathcal{V}_X^i$ and $\pi_X^{\mathcal{H}}\colon \mathrm{ad}_X^k\mathcal{V}\to\mathcal{H}_X$.

Let us notice that operators $I_X^{ij}\colon \mathcal{V}_X^i\to\mathcal{V}_j^X$, for $i=0,\dots,k$ and $j=i,\dots,k$ defined by formula

$$I_X^{ij}=\pi_{\mathcal{V}_j}^X\circ\mathrm{ad}_X^{j-i}$$

are vector bundle isomorphisms. By I_X^{ji} we denote an inverse of I_X^{ij}. Additionally, for $i=0,\dots,k-1$, we have the following homomorphism of vector bundles $J_X^i\colon\mathcal{H}_X\to\mathcal{V}_X^i$

$$J_X^i=\pi_{\mathcal{V}_i}^X\circ\mathrm{ad}_X.$$

Definition. An i-th *curvature operator* $K_X^i\in\mathrm{End}(\mathcal{V})$ is

$$K_X^i=-I_{i0}^X\circ J_i^X\circ I_{0k}^X\colon\mathcal{V}\to\mathcal{V}.$$

In a fixed base of the space $\mathcal{V}_X^i(x)$ operator $K_X^i(x)$ is represented by $s\times s$ matrix, also denoted $K_X^i(x)$. Note that, if $V=(V_1,\dots,V_m)$ are normal, then curvature matrices in the base V are defined by the following equation

$$\mathrm{ad}_X^{k+1}V+K_X^{k-1}\mathrm{ad}_X^{k-1}V+\cdots+K_X^1\mathrm{ad}_XV+K_X^0V=0. \tag{4}$$

Proposition 2.2

$$\mathrm{tr}K_{k-1}^{fX}=f^2\mathrm{tr}K_X^{k-1}-mc_kS^X(f), \tag{5}$$

where $c_k=-\frac{1}{24}k(k+1)(k+2)$ *is a constant, and*

$$S^X(f)=2fX^2(f)-X(f)^2. \tag{6}$$

Proof. Let V be normal for X, and GV be normal for fX. We compute directly

$$\begin{aligned}\mathrm{ad}_{fX}^{k+1}GV \;&=\; Gf^{k+1}\mathrm{ad}_X^{k+1}V+\sum_{i=0}^{k}f^{k+1-i}X(f^iG)\mathrm{ad}_X^kV\\ &+\;\sum_{i=0}^{k-1}\sum_{j=0}^{k-1-i}f^{k-i-j}X(f^{j+1}X(f^iG))\mathrm{ad}_X^{k-1}V \mod \mathrm{ad}_X^{k-2}\mathcal{V},\ \mathcal{X}.\end{aligned} \tag{7}$$

From normality of V and GV it follows

$$\sum_{i=0}^{k}f^{k+1-i}X(f^iG)=0.$$

Application of Leibnitz rule gives

$$fX(G) = -\tfrac{k}{2}X(f)G, \tag{8}$$

and by differentiation we obtain

$$f^2X^2(G) = \Big(\big(\tfrac{k}{2}+\tfrac{k^2}{4}\big)X(f)^2 - \tfrac{k}{2}fX^2(f)\Big)G. \tag{9}$$

From (8) and (9) we get

$$\begin{aligned}\sum_{i=0}^{k-1}\sum_{j=0}^{k-1-i} f^{k-i-j}X(f^{j+1}X(f^iG)) \;&=\; \sum_{i=0}^{k-1}\sum_{j=0}^{k-1-i} f^{k+1}X^2(G) \;+\; \sum_{i=0}^{k-1}\sum_{j=0}^{k-1-i} if^kX^2(f)G \\ &\quad + \sum_{i=0}^{k-1}\sum_{j=0}^{k-1-i}(2i+j+1)f^kX(f)X(G) \\ &\quad + \sum_{i=0}^{k-1}\sum_{j=0}^{k-1-i} i(i+j)f^{k-1}X(f)^2G \\ &= Gf^{k-1}\sum_{i=0}^{k}(k-i)\Big(\big(-\tfrac{i}{2}+\tfrac{i^2}{2}+\tfrac{k}{4}-\tfrac{ki}{4}\big)X(f)^2 \\ &\quad + (i-\tfrac{k}{2})fX^2(f)\Big).\end{aligned}$$

One can see that

$$\sum_{i=0}^{k}(k-i)(i-\tfrac{k}{2}) = -2\sum_{i=0}^{k}(k-i)(-\tfrac{i}{2}+\tfrac{i^2}{2}+\tfrac{k}{4}-\tfrac{ki}{4}).$$

We denote $c_k = \frac{1}{2}\sum_{i=0}^{k}(k-i)(i-\frac{k}{2})$. Then (7) imply

$$\mathrm{ad}_{fX}^{k+1}GV = Gf^{k+1}\mathrm{ad}_X^{k+1}V + Gf^{k-1}c_kS^X(f)\mathrm{ad}_X^{k-1}V \mod \mathrm{ad}_X^{k-2}\mathcal{V},\ \mathcal{X}.$$

We have $\mathrm{ad}_{fX}^{k-1}GV = f^{k-1}G\mathrm{ad}_X^{k-1}V \mod \mathrm{ad}_X^{k-2}\mathcal{V}$, and hence

$$K_{k-1}^{fX} = f^2GK_X^{k-1}G^{-1} - c_kS^X(f)\mathrm{Id}.$$

□

Remark. Operator S^X defined by formula (6) is called *Schwartzian*. It is justified by the following reasoning. Let $\gamma\colon t\mapsto\gamma(t)$ be a curve representing X. It means that $X(g)\circ\gamma = \frac{d}{dt}(g\circ\gamma)$ for any function g. Let us consider reparametrization $\varphi\colon s\mapsto\varphi(s)$, such that $\gamma\circ\varphi$ represents vector field fX. Then $f(\gamma\circ\varphi)=\varphi'$ and

$$\begin{aligned} S^X(f)(\gamma\circ\varphi) &= 2\varphi'\Big(\frac{1}{\varphi'}\frac{d}{ds}\Big)^2(\varphi') - \Big(\frac{1}{\varphi'}\frac{d}{ds}\varphi'\Big)^2 \\ &= 2\frac{d}{ds}\Big(\frac{\varphi''}{\varphi'}\Big) - \Big(\frac{\varphi''}{\varphi'}\Big)^2 \\ &= 2\frac{\varphi'''}{\varphi'} - 3\Big(\frac{\varphi''}{\varphi'}\Big)^2.\end{aligned}$$

The last term is equal to the Schwartz derivative of φ.

Definition. X is called *projective vector field* if $\mathrm{tr}K_X^{k-1}=0$.

Proposition 2.2 implies that projective vector fields form a 2-parameter family on any integral curve of $\mathcal{X}$. Non-constant solutions to the equation $S^X(f)=0$ are of the form

$$f(x)=a(x+b)^2,$$

where x is a parameter on an integral line of X.

Definition. Let $\mathcal{P}$ be the family of projective vector fields. *Characteristic cones* of $(\mathcal{X},\mathcal{V})$ at $x\in M$ are the following sets

$$C^i(\mathcal{X},\mathcal{V})(x) \;=\; \mathrm{cl}\bigcup_{X\in\mathcal{P}}\mathcal{V}_X^i(x),$$

where $i=0,\ldots,k$. $C^k(\mathcal{X},\mathcal{V})(x)=\mathrm{cl}\bigcup_{X\in\mathcal{P}}\mathcal{H}_X(x)$ is also called *connection cone* at x.

The field of connection cones will be the most important object in this paper.

Lemma 2.3 *Let X and fX be projective vector fields. Assume that V and GV are corresponding normal vector fields. Then for $i=0,\ldots,k$*

$$\mathrm{ad}_{fX}^iGV \;=\; \sum_{j=0}^{i}c_{ij}f^jX(f)^{i-j}G\mathrm{ad}_X^jV \mod \mathcal{X},$$

where $c_{ij}=(-1)^{i+j}\frac{(k-j)\cdots(k-i+1)}{2^{i-j}}\binom{i}{j}$. Moreover $\mathrm{ad}_{fX}^{k+1}GV=f^{k+1}G\mathrm{ad}_X^{k+1}V \mod \mathcal{X}$.

Proof. Let us recall that $S^X(f)=2fX^2(f)-X(f)^2=0$ and $2fX(G)=-kX(f)G$ (equations (5) and (8)). Then

$$\begin{aligned} fX(f^{j+1}X(f)^{i-j-1}G) \;&=\; (j+1)f^{j+1}X(f)^{i-j}G+(i-j-1)f^{j+2}X(f)^{i-j-2}X^2(f)G \\ &\quad + f^{j+2}X(f)^{i-j-1}X(G) \\ &=\; \frac{i+j+1-k}{2}f^{j+1}X(f)^{i-j}G. \end{aligned} \tag{10}$$

Let d_{ij} denotes the coefficient at ad_X^jV in the expansion of ad_{fX}^iGV. We shall prove that $d_{ij}=c_{ij}f^jX(f)^{i-j}G$. We compute d_{i0} by induction. For $i=0$ this coefficient equals G. Then $d_{i+10}=fX(d_{i0})$. We use (10) with $j=-1$ and obtain

$$d_{i+10}=fX((-1)^i\frac{k!}{2^i(k-i)!}X(f)^iG)=(-1)^{i+1}\frac{k!}{2^{i+1}(k-i-1)!}X(f)^{i+1}G,$$

as required. The rest of coefficients d_{ij} is also computed by induction

$$\begin{aligned}
d_{i+1j+1} &= f d_{ij} + fX(d_{ij+1}) \\
&= (-1)^{i+j}\frac{(k-j)\cdots(k-i+1)}{2^{i-j}}\binom{i}{j} f^{j+1}X(f)^{i-j}G \\
&\quad + (-1)^{i+j+1}\frac{(k-j-1)\cdots(k-i+1)}{2^{i-j-1}}\binom{i}{j+1} fX\big(f^{j+1}X(f)^{i-j-1}G\big) \\
&= (-1)^{i+j}\frac{(k-j)\cdots(k-i+1)}{2^{i-j}}\binom{i}{j} f^{j+1}X(f)^{i-j}G \\
&\quad + (-1)^{i+j+1}\frac{(k-j-1)\cdots(k-i+1)}{2^{i-j-1}}\binom{i}{j+1}\frac{i+j+1-k}{2} f^{j+1}X(f)^{i-j}G \\
&= (-1)^{i+j}\frac{(k-j-1)\cdots(k-i)}{2^{i-j}}\binom{i+1}{j+1} f^{j+1}X(f)^{i-j}G.
\end{aligned}$$

Expression $\mathrm{ad}_{fX}^{k+1}GV = f^{k+1}G\mathrm{ad}_X^{k+1}V$ is proved by differentiation of $\mathrm{ad}_{fX}^{k}GV$. $\square$

From Lemma 2.3 we get the following description of characteristic cones $C^i(\mathcal{X},\mathcal{V})$.

Proposition 2.4 *Let X be a projective vector field and V be a corresponding normal vector field then*

$$\begin{aligned}
C^i(\mathcal{X},\mathcal{V})(x) &= \{c_{i0}a^iGV(x) + c_{i1}a^{i-1}bG\mathrm{ad}_XV(x) + \cdots + c_{ii}b^iG\mathrm{ad}_X^iV(x) \\
&\quad \mid a,b\in\mathbb{R},\ G\in\mathbb{R}^{m\times m}\} \mod \mathcal{X}. \qquad (11)
\end{aligned}$$

Let us notice that $C^i(\mathcal{X},\mathcal{V})(x)$ can be considered as an algebraic curve of order i in Grassmann manifold $\mathrm{Gr}_m(\mathcal{V}^i(x))$ of m dimensional subspaces in $(i+1)m$ dimensional vector space $\mathcal{V}^i(x) = (\mathrm{ad}_X^i\mathcal{V}(x) \mod \mathcal{X})$. Since all $\mathrm{ad}_X^iV_j$ are independent, this curve has no self intersections.

We will analyze geometry of $C^i(\mathcal{X},\mathcal{V})(x)$ in more detail in the future work. In particular we will show how they define conformal tensors. Let us only notice here an obvious fact that if $m=1$ then $C^2(\mathcal{X},\mathcal{V})(x)$ defines conformal Lorentzian metric on bundle $\mathcal{V}^2$, which is of rank 3.

2.2 Wilczynski Invariants

Let $A\in\mathrm{End}(\mathcal{V})$ and X be a section of $\mathcal{X}$. Lie derivative $L_X(A)$ is defined as

$$L_X(A) = \pi_{\mathcal{V}_0}^X\circ\mathrm{ad}_X\circ A - A\circ\pi_{\mathcal{V}_0}^X\circ\mathrm{ad}_X.$$

From Lemma 2.3 one can deduce the following theorem of E. J. Wilczynski (see [6], and also [4, 5]).

Theorem 2.5 (E. J. Wilczynski) *Let $\mathcal{P}$ be the family of projective vector fields. There exist constants $w_{ij}\in\mathbb{Q}$ such that the following mapping $\mathcal{P}\ni X\to W^i(X)\in$* End$(\mathcal{V})$

$$W^i(X) = K_X^i + \sum_{j=1}^{k-i-1} w_{ij}L_X^j(K_X^{i+j})$$

is homogeneous of order $k-i+1$, i.e. if $X, fX \in \mathcal{P}$ are projective, then $W^i(fX) = f^{k-i+1}W^i(X)$.

Definition. W^i is called i-th Wilczynski invariant.

If A is a $m \times m$ matrix then let $\sigma_i(A)$ denote coefficient at t^{m-i} in characteristic polynomial $\det(A - t\mathrm{Id})$. For instance, $\sigma_1 = \mathrm{tr}$ and $\sigma_m = \det$. We get that $W^i_j = \sigma_j W^i$ are homogeneous functions $\mathcal{P} \to \mathbb{R}$ of order $j(k-i+1)$. If some of them is non-degenerate then it defines a length element on the integral lines of $\mathcal{X}$. Thus, it defined *canonical sections* of $\mathcal{X}$: we call X *canonical* if $W^i_j(X) \equiv \pm 1$.

Definition. Let X be a canonical section of $\mathcal{X}$ defined by W^i_j. Curvature operator K^l_X is called (ijl)-*canonical curvature operator*, and is denoted K^{il}_j.

We have the following determinacy result.

Proposition 2.6 *If Wilczynski invariant W^i_j of $(\mathcal{X}, \mathcal{V})$ is non-degenerate along certain integral curve of $\mathcal{X}$ then corresponding canonical curvature operators determine $\mathcal{V}$ uniquely modulo $\mathcal{X}$ along this curve.*

Proof. It follows directly from equation (4) with $K^l_X = K^{il}_j$. This equation describes an evolution of $\mathcal{V}$ modulo $\mathcal{X}$ along integral curves of $\mathcal{X}$. □

From Proposition 2.6 we get that canonical curvature operators corresponding to one Wilczynski invariant determine all other canonical curvature operators (corresponding to different Wilczynski's invariants). Hence, in general, it is sufficient to know only one Wilczynski invariant. If $K^{k-1}_X \equiv \cdots \equiv K^{i+1}_X \equiv 0$ then the definition implies $W^i(X) = K^i_X$, so the formula for the first non-degenerate Wilczynski invariant is relatively simple.

3 Applications to Systems of ODE

3.1 Preliminary Remarks

Let us consider the following system (F)

$$\begin{aligned}
x_1^{(k+1)} &= F_1(t, x_1, \ldots, x_m, x_1', \ldots, x_m', \ldots, x_1^{(k)}, \ldots, x_m^{(k)}) \\
x_2^{(k+1)} &= F_2(t, x_1, \ldots, x_m, x_1', \ldots, x_m', \ldots, x_1^{(k)}, \ldots, x_m^{(k)}) \\
&\vdots \\
x_m^{(k+1)} &= F_m(t, x_1, \ldots, x_m, x_1', \ldots, x_m', \ldots, x_1^{(k)}, \ldots, x_m^{(k)}),
\end{aligned}$$

where $F_1, \ldots, F_m$ are smooth functions and $k > 1$. Let $J^k(1, m)$ denote the space of k-jets of functions $\mathbb{R} \to \mathbb{R}^m$. Then $(t, x_1, \ldots, x_m, x_1', \ldots, x_m', \ldots, x_1^{(k)}, \ldots, x_m^{(k)})$

constitutes a system of coordinates on $J^k(1,m)$. We recall that Cartan distributions $\mathcal{C}_i$ are defined as

$$\mathcal{C}_i = \bigcap_{j=1,\ldots,i} \ker \omega_j^1 \cap \ldots \cap \ker \omega_j^m.$$

where $\omega_r^i = dx_r^{(i)} - x_r^{(i+1)}dt$, for $r = 1,\ldots,m$ and $i = 0,\ldots,k-1$. We denote $\mathcal{X}_F$ a line bundle spanned by total derivative

$$\begin{aligned} X_F &= \partial_t + x_1'\partial_{x_1} + \cdots + x_m'\partial_{x_m} + \cdots + \\ &\quad x_1^{(k-1)}\partial_{x_1^{(k-2)}} + \cdots + x_m^{(k-1)}\partial_{x_m^{(k-2)}} + \\ &\quad F_1\partial_{x_1^{(k-1)}} + \cdots + F_m\partial_{x_m^{(k-1)}}. \end{aligned}$$

Integral lines of $\mathcal{X}_F$ correspond to solutions of (F). Let $\mathcal{V}_F$ be Cauchy characteristic of $\mathcal{C}_{k-2}$, i.e.

$$\mathcal{V}_F = \mathrm{Ch}(C_{k-2}) = \mathrm{span}\{\partial_{x_1^{(k-1)}}, \ldots, \partial_{x_m^{(k-1)}}\}.$$

The following lemma is obvious.

Lemma 3.1 *For any system (F) pair $(\mathcal{X}_F, \mathcal{V}_F)$ is good.*

We say that systems (F) and $(\tilde{F})$ are equivalent (by contact transformation of coordinates) if there exists diffeomorphism $\Psi\colon J^k(1,m) \to J^k(1,m)$ such that

$$\Psi_*\mathcal{X}_F = \mathcal{X}_{\tilde{F}}$$

and

$$\Psi_*\mathcal{C}_i = \mathcal{C}_i.$$

In particular

$$\Psi_*\mathcal{V}_F = \mathcal{V}_{\tilde{F}}.$$

In fact it is sufficient to assume that $\Psi_*\mathcal{X}_F = \mathcal{X}_{\tilde{F}}$ and $\Psi_*\mathcal{V}_F = \mathcal{V}_{\tilde{F}}$, since Cartan distributions are generated by Lie brackets of $\mathcal{X}_F$ and $\mathcal{V}_F$.

Definition. *Characteristic cones* of (F) at $x \in J^k(1,m)$ are the following sets

$$C_F^i(x) = C^i(\mathcal{X}_F, \mathcal{V}_F),$$

where $i = 0,\ldots,k$. $C_F^k(x)$ is also called *connection cone* of (F).

By definition, characteristic cones of (F) are preserved by contact transformations. In this way we proved the first part of Theorem 1.1.

We conclude also that Wilczynski invariants are assigned to any system (F). If they are non-degenerate then the canonical curvature operators are defined too. All these objects are invariantly assigned to (F). Note that canonical curvature operators are functional invariants. In the case of linear systems they determine (F) uniquely.

Proposition 3.2 *Let (F) be a linear system. If there is a Wilczynski invariant which does not vanish then the corresponding canonical curvature operators determine (F) uniquely up to contact transformations.*

Proof. It is implied by Proposition 2.6. Namely, it is straightforward that linear system are determined by $\mathcal{V}_F$ along one integral curve of $\mathcal{X}_F$. By Proposition 2.6 canonical curvature operators determine $\mathcal{V}_F$ uniquely modulo $\mathcal{X}_F$. Additionally $\mathcal{V}_F$ is determined uniquely in the direction of $\mathcal{X}_F$ by condition that it is integrable. $\square$

3.2 Generalized Wuenschmann Condition

Let us consider first the case of one equation of order 3

$$x''' = F(t, x, x', x'').$$

It is the classical case of S-S. Chern [1]. In this situation $\mathcal{V}_F$ is a line bundle and $J^2(1,1)$ is of dimension 4. Thus, the only possibly non-trivial Wilczynski invariant is W^0. The following fact is well known (see e.g. [2]); (symbol $f_{,i}$ denotes partial derivative $\partial_{x^{(i)}} f$).

Proposition 3.3 $W_0 \equiv 0$ *if and only if Wuenschmann condition holds*

$$A = F_{,0} + X_F(B) - \frac{2}{3}F_{,2}B = 0,$$

where

$$B = \frac{1}{6}X_F(F_{,2}) - \frac{1}{9}F_{,2}^2 - \frac{1}{2}F_{,1}.$$

In fact $W^0 \equiv A$.

It was proved by S-S. Chern [1] that if Wuenschmann condition holds then there is defined a conformal metric of signature $(+,+,-)$ on the solutions space. Let us denote the space of solutions of (F) by S_F, and let

$$q\colon\, J^2(1,1) \to S_F$$

be the quotient map. Wuenschmann condition implies that connection cone C_F^2 is preserved via the flow of $\mathcal{X}_F$ (see Lemma 3.5 below). Hence, its projection to S_F is well defined. We call it *quotient connection cone* and denote $q_*C_F^2$.

Proposition 3.4 *The quotient connection cone of equation* $x''' = F(t, x, x', x'')$ *coincides with the cone of null directions of Chern conformal metric*

$$[g] = 2dx \cdot (dx'' - \frac{1}{3}F_{,2}dx' + Bdx) - dx' \cdot dx',$$

where as before $B = \frac{1}{6}X(F_{,2}) - \frac{1}{9}F_{,2}^2 - \frac{1}{2}F_{,1}$.

Proof. Let $X = X_F = \partial_t + x'\partial_x + x''\partial_{x'} + F\partial_{x''}$ be a total derivative and let $V = \partial_{x''}$. Assume that fX is a projective vector field, and GV is normal. We have the following relations modulo $\mathcal{X}_F$

$$\begin{aligned}
\mathrm{ad}_{fX}GV &= fG\mathrm{ad}_XV + fX(G)V,\\
\mathrm{ad}_{fX}^2GV &= fX(fG)\mathrm{ad}_XV + f^G\mathrm{ad}_X^2V + f^2X(G)\mathrm{ad}_XV + fX(f(XG))V,\\
\mathrm{ad}_{fX}^3GV &= f^3G\mathrm{ad}_X^3V + fX(f^2G)\mathrm{ad}_X^2V + f^2X(fG)\mathrm{ad}_X^2V + fX(fX(fG))\mathrm{ad}_XV\\
&\quad + f^3X(G)\mathrm{ad}_X^2V + fX(f^2X(G))\mathrm{ad}_XV + f^2X(fX(G))\mathrm{ad}_XV\\
&\quad + fX(fX(fX(G)))V. \qquad (12)
\end{aligned}$$

Moreover, we compute directly that

$$\mathrm{ad}_X^3 V = H_2 \mathrm{ad}_X^2 V + H_1 \mathrm{ad}_X V + H_0 V,$$

where $H_0 = -F_{,0} + X(F_{,1}) - X^2(F_{,2})$, $H_1 = (F_{,1} - 2X(F_{,2}))$ and $H_2 = -F_{,2}$. From normality of GV and (12) it follows that

$$f^3 G H_2 + fX(f^2 G) + f^2 X(fG) + f^3 X(G) = 0. \tag{13}$$

Let $x \in J^2(1,1)$ be fixed. We can assume that $f(x) = 1$, $G(x) = 1$ and $f'(x) = 0$. Then (13) yields

$$H_2(x) + 3X(G)(x) = 0. \tag{14}$$

By differentiation of (13) we get

$$H_2 X(G)(x) + X(H_2)(x) + 3X^2(f)(x) + 3X^2(G)(x) = 0. \tag{15}$$

The fact that fX is projective and (12) imply

$$H_1(x) + X^2(f)(x) + 3X^2(G)(x) = 0. \tag{16}$$

Then, from (14), (15) and (16) we compute that at point x

$$\begin{aligned} X(G) &= -\frac{1}{3} H_2, \\ X^2(G) &= -\frac{1}{2} H_1 - \frac{1}{18} H_2^2 + \frac{1}{6} X(H_2). \end{aligned}$$

From above we see (we substitute $X(G)$, $X^2(G)$ and H_i to (12))

$$\begin{aligned} \mathrm{ad}_{fX} GV(x) &= -\partial_{x'} - \frac{2}{3} F_{,2}(x) \partial_{x''}, \\ \mathrm{ad}_{fX}^2 GV(x) &= \partial_x + \frac{1}{3} F_{,2}(x) \partial_{x'} + \Big(\frac{1}{2} F_{,1}(x) + \frac{5}{18} F_{,2}(x)^2 - \frac{1}{6} X(F_{,2})(x) \Big) \partial_{x''}. \end{aligned}$$

Hence

$$\begin{aligned} [g](q_* \mathrm{ad}_{fX}^2 GV, q_* \mathrm{ad}_{fX}^2 GV) &= 2\Big(\frac{1}{2} F_{,1} + \frac{5}{18} F_{,2}^2 - \frac{1}{6} X(F_{,2}) - \frac{1}{9} F_{,2}^2 \\ &\quad + \frac{1}{6} X(F_{,2}) - \frac{1}{9} F_{,2}^2 - \frac{1}{2} F_{,1} \Big) - \frac{1}{9} F_{,2}^2 \\ &= 0. \end{aligned}$$

So, for the vector field $Y = fX$ at point $q(x)$ the connection $q_* \mathcal{H}^Y$ is a null direction of $[g]$. Let us denote $W = GV$. In the rest of the proof we will need the following relations which hold at x and which can be easily deduced from above

$$[g](q_* W, q_* W) = 0, \qquad [g](q_* \mathrm{ad}_Y W, q_* \mathrm{ad}_Y W) = -1,$$

$$[g](q_* \mathrm{ad}_Y W, q_* W) = 0, \quad [g](q_* \mathrm{ad}_Y^2 W, q_* W) = 1, \quad [g](q_* \mathrm{ad}_Y^2 W, q_* \mathrm{ad}_Y W) = 0.$$

Let us consider an arbitrary projective vector field. It is of the form fY for a certain f such that $S^Y(f) = 0$ (it is now a different f than the one we had before).

We have $fY^2(f) = \frac{1}{2}Y(f)^2$, and if GW is normal for fY, then $fY(G) = -GY(f)$. Without the lost of generality we assume $G(x) = 1$ and compute

$$\mathrm{ad}_{fY}^2 GW(x) = f^2(x)\mathrm{ad}_Y^2 W(x) - f(x)Y(f)(x)\mathrm{ad}_Y W(x) + \frac{1}{2}Y(f)(x)^2 W(x).$$

Then

$$\begin{aligned}
[g](q_*\mathrm{ad}_{fY}^2 GW, q_*\mathrm{ad}_{fY}^2 GW) &= \frac{1}{4}[g](q_*W, q_*W) + f^2X(f)^2[g](q_*\mathrm{ad}_Y W, q_*\mathrm{ad}_Y W) \\
&\quad + f^4[g](q_*\mathrm{ad}_Y^2 W, q_*\mathrm{ad}_Y^2 W) - 2\frac{1}{2}fX(f)^3[g](q_*\mathrm{ad}_Y W, q_*W) \\
&\quad + 2\frac{1}{2}f^2X(f)^2[g](q_*\mathrm{ad}_Y^2 W, q_*W) - 2f^3X(f)[g](q_*\mathrm{ad}_Y^2 W, q_*\mathrm{ad}_Y W) \\
&= 0.
\end{aligned}$$

□

We return now to the general system of ODE.

Definition. We say that *generalized Wuenschmann condition* holds for (F) if all Wilczynski invariants vanish.

This condition was considered by B. Doubrov [2] as the necessary condition for contact trivialization of one equation. As in the case of Chern we denote by

$$q\colon J^k(1,m) \to S_F$$

the quotient map to the solution space S_F.

Lemma 3.5 *Let X be a projective vector field. If generalized Wuenschmann condition holds then*

$$[X, \mathcal{V}_X^k] = \mathcal{V}_X^k \mod \mathcal{X}.$$

Hence projection $q_*C_F^k$ is well defined.

Proof. Wuenschmann condition is equivalent to vanishing of all curvatures K_X^i (it is a direct consequence of the definition). Hence $[X, \mathrm{ad}_X^k V] = \mathrm{ad}_X^{k+1} V = 0$ for any normal sections V (equation (4)). Thus lemma follows, because $\mathcal{V}_X^k = \mathrm{span}\{\mathrm{ad}_X^k V\}$.
□

Definition. Assume that generalized Wuenschmann condition holds for (F) and let $x \in J^k(1,m)$. *Quotient connection cone* of (F) at $q(x) \in S_F$ is the following set

$$q_*C_F^k(x) = q_*(C^k(\mathcal{X}_F, \mathcal{V}_F)).$$

In this way we proved the second part of Theorem 1.1. The rest of the paper is devoted to its third part - determinacy result. We assume that Wuenschmann condition holds and we treat connection cones as curves in Grassmann manifold. We denote "grassmannization" of $q_*C_F'^k$ by G_F. Then (11) implies that $q_*\mathcal{V}_F(x) \in G_F(q(x))$ for any $x \in J^k(1,m)$.

Lemma 3.6 *The mapping*

$$\Phi_F\colon J^k(1,s) \ni x \mapsto q_*\mathcal{V}_F(x) \in G_F$$

is a local diffeomorphism.

Proof. Φ_F is obviously differentiable and Φ_{F*} has maximal rank on spaces transversal to $\mathcal{X}$. Let us assume that in a certain coordinate system in a neighborhood of x a vector field $X = \partial_0$ is projective. Then Wuenschmann condition and equation (4) imply that an evolution of $\mathcal{V}_F$ is described modulo $\mathcal{X}_F$ along a certain integral curve of X by the equation

$$\partial_0^{(n+1)}u = 0.$$

Hence

$$\mathcal{V}_F(\exp_{t\partial_0}(x)) = \operatorname{span}\{V(x) + \frac{t}{1!}\operatorname{ad}_X V(x) + \cdots + \frac{t^n}{n!}\operatorname{ad}_X^n V(x)\} \mod \mathcal{X},$$

and $q_*\mathcal{V}_F(\exp_{t\partial_0}(x)) \neq q_*\mathcal{V}_F(x)$ for $t \neq 0$. Moreover, one can see that $\Phi_{F*}(\partial_0) \neq 0$. □

Now we are in position to prove the following result, which clarify the last part of Theorem 1.1.

Theorem 3.7 *Assume that generalized Wuenschmann condition holds for* (F) *and* $(\tilde{F})$. *Let* $x_0, \tilde{x}_0 \in J^k(1,m)$. *There exists a local contact transformation of* (F) *and* $(\tilde{F})$ *from a neighborhood of* x_0 *onto a neighborhood of* $\tilde{x}_0$ *iff there exists a local diffeomorphism* $\psi\colon S_F \to S_{\tilde{F}}$ *such that*

$$\psi_*(\Phi_F(x_0)) = \Phi_{\tilde{F}}(\tilde{x}_0)$$

and

$$\psi_* q_* C_F = q_* C_{\tilde{F}}.$$

Proof. If (F) and $(\tilde{F})$ are locally equivalent then obviously the hypothesis is true. For a proof in the opposite direction let us observe that by Lemma 3.6 the following mapping

$$\Psi = \Phi_{\tilde{F}}^{-1} \circ \psi_* \circ \Phi_F$$

is a diffeomorphism in the neighborhood of x_0. We shall show that it is a contact transformation. The relation $\Psi_*\mathcal{X}_F = \mathcal{X}_{\tilde{F}}$, is the consequence of the fact that Ψ transforms fibers of $q\colon J^k(1,m) \to S_F$ onto fibers of $q\colon J^k(1,m) \to S_{\tilde{F}}$. So it is sufficient to prove that $\Psi_*\mathcal{V}_F = \mathcal{V}_{\tilde{F}} \mod \mathcal{X}_{\tilde{F}}$. It follows from the construction. Indeed, if we denote projection $G_F \to S_F$ by π, we get

$$\pi_* \circ \Phi_{F*}\mathcal{V}_F(x) = q_*\mathcal{V}_F(x) = \Phi_F(x),$$

for any $x \in J^k(1,s)$, and analogously for $(\tilde{F})$

$$\pi_* \circ \Phi_{\tilde{F}*}\mathcal{V}_{\tilde{F}}(x) = q_*\mathcal{V}_{\tilde{F}}(x) = \Phi_{\tilde{F}}(x).$$

A composition of these relations gives the desired relation. $\square$

Note that G_F (i.e. the "grassmannization" of the quotient connection cone) can be treated as the model space for equation (F). We understand this in the following way. Let $\mathcal{F}_G$ be the class of equations (F) such that Wuenschmann condition is fulfilled and G_F is equivalent to G. For any $(F) \in \mathcal{F}_G$ we have the mapping $\Phi_F \colon J^k(1,m) \to G$. If $(\tilde{F}) \in \mathcal{F}_G$ is another system and the composition $\Phi_{\tilde{F}}^{-1} \circ \Phi_F$ exists on some domain then, as in the proof of Theorem 3.7, it is a local contact transformation. Therefore one can treat (F) and $(\tilde{F})$ as two parts of one equation. But it is of course possible that two mappings Φ_F and $\Phi_{\tilde{F}}$ have disjoint images. Namely, one can take one equation and cut domain t into small intervals. Each of these intervals can be spread onto $\mathbb{R}$ so that new equations are obtained. Generically (if G has small symmetry group) two different resulting equations will be non equivalent.

There exists the canonical contact structure on G (i.e. sequence of distributions locally diffeomorphic to Cartan distributions on the jets space). It can be defined in two equivalent ways. One possibility is to push it forward from $J^k(1,m)$ via all Φ_F such that $(F) \in \mathcal{F}_G$. But it can be also defined intrinsically. If $\pi \colon G \to S$ is the projection to solution space then we define $\mathcal{X}_G = \ker \pi_*$. Moreover, there is a distribution D on G which comes from grassmannian structure, i.e.

$$D(y) = \{Y \in T_y G \mid \pi_* Y \in y\}$$

and one can define $\mathcal{V}_G = \mathrm{Ch}([\mathcal{X}_G, D])$. Lie brackets of $\mathcal{V}_G$ and X_G defines the desired contact structure. We conclude that

$$\pi_* C^k(\mathcal{X}_G, \mathcal{V}_G) = q_* C^k_F = G$$

for any $(F) \in \mathcal{F}_G$ - this follows immediately from Theorem 3.7.

References

[1] S-S. Chern, *The Geometry of the Differential Equation* $y''' = F(x, y, y', y'')$, Sci Rep. Nat. Tsing Hua Univ., 4 (1940), pp. 97-111.

[2] B. Doubrov, *Contact trivialization of ordinary differential equations*, Differential Geometry and Its Applications, Proc. Conf., Opava, 2001, 73-84.

[3] P. Nurowski, *Differential equations and conformal structures*, Journal of Geometry and Physics, Vol.55, Issue 1, September 2005, pp. 19-49.

[4] Y. Se-ashi, *On differential invariants of integrable finite type linear differential equations*, Hokkaido Math. J., 17 (1988), pp. 151-195.

[5] Y. Se-ashi, *A geometric Construction of Laguerre-Forsyth's Canonical Forms of Linear Ordinary Differential Equations*, Adv.Studies in Pure Math., 22 (1993), pp. 265-297.

[6] E. J. Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces*, Teubner, Lepzing, 1905.